G. Plebanek	*Measures on topological spaces*	no. 1

1. Show that a function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \lambda(A \cap (x + A))$ is continuous for any measurable set A of finite measure.

2. Generalize the Steinhaus theorem: if the sets $A, B \subseteq \mathbb{R}$ have positive measure then their algebraic difference $A - B$ contains some nontrivial interval.

3. Prove that if $A \subseteq \mathbb{R}$ is a set of positive measure then the complement of $\mathbb{Q} + A$ has measure zero.

4. Show that there is a set $Z \subseteq \mathbb{R}^2$ of zero planar measure such that $Z \cap (A \times B) \neq \emptyset$ for every Borel rectangle $A \times B$ of positive measure.

5. Construct a set $A \subseteq \mathbb{R}$ such that A has density $t \in (0, 1)$ at the point 0; likewise, find A such that the density of A at 0 is not defined.

6. Let $A = \{(x, y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant x^2\}$. Calculate the density of A, that is

$$\lim_{\delta \to 0^+} \frac{\lambda_2(A \cap B(p, \delta))}{\pi\delta^2},$$

at various points p. (Here $B(p, \delta)$ denotes the metric ball on the plane, λ_2 is the planar Lebesgue measure.)

7. Check if the following conditions are equivalent for measurable $A \subseteq \mathbb{R}^2$ and $p = (p_1, p_2)$

$$(i) \quad \lim_{\delta \to 0^+} \frac{\lambda_2(A \cap B(p, \delta))}{\pi\delta^2} = 1, \quad (ii) \quad \lim_{\delta \to 0^+} \frac{\lambda_2(A \cap (p_1 - \delta, p_1 + \delta) \times (p_2 - \delta, p_2 + \delta)}{4\delta^2} = 1.$$

8. For a measurable set $A \subseteq \mathbb{R}$ denote by $\phi(A)$ the set of points at which the density of A equals 1. Check that

(i) $\phi(A \cap B) = \phi(A) \cap \phi(B)$;

(ii) if $A \subseteq B$ then $\phi(A) \subseteq \phi(B)$;

(iii) $\phi(A) \cup \phi(B) \subseteq \phi(A \cup B)$, the equality need not hold;

(iv) if $\lambda(A \triangle B) = 0$ then $\phi(A) = \phi(B)$.

(v) $\phi(V) \supseteq V$ whenever V is open.

9. Prove that the union of an arbitrary family of sets satisfying $A \subseteq \phi(A)$ (see above) is measurable.

Hint: Given $\phi(A_t), t \in T$, choose a countable subfamily with the union of maximal measure.

10. * Let $\mathcal{T}$ be an arbitrary family of nondegenerate closed triangles on the plane. Prove that the set $\bigcup \mathcal{T}$ is measurable (but does not have to be Borel).

Hint: The Vitali covering or Lebesgue density theorems.

11. Let ν be the usual measure on the Cantor set. Find analogues of the Steinhaus, Vitali and Lebesgue theorems for ν.

12. * Let $f : [0, 1] \times [0, 1] \longrightarrow \mathbb{R}$ be a function which is continuous in each variable (separately). Another theorem due to Lebesgue: there is a sequence of continuous functions f_n converging to f pointwise (in particular, the function f is Borel).