1. Complete the suggested proof of the fact that every closed $A \subseteq 2^{\mathbb{N}}$ is a retract of $2^{\mathbb{N}}$:

We consider the metric ρ on $2^{\mathbb{N}}$, where $\rho(x,y) = \sum_{n=1}^{\infty} 2|x(n) - y(n)|/3^n$; for any $x \in 2^{\mathbb{N}}$ we let $r(x) = a$, where $\rho(x,a) = \rho(x,A)$. Then r is a well-defined continuous retraction.

2. For $A \subseteq 2^{\mathbb{N}}$ and $I \subseteq \mathbb{N}$ we write $A \sim I$ to say that A is determined by coordinates in I, that is

$$(\forall a \in A)(\forall x \in 2^{\mathbb{N}}) \quad a|I = x|I \Rightarrow x \in A.$$

Note that this is equivalent to saying that $A = \pi_I^{-1}\pi_I[A]$, where $\pi_I : 2^{\mathbb{N}} \to 2^I$ is the projection. Check that

(i) if $A \sim I$ and $I \subseteq J$ then $A \sim J$;

(ii) if $A, B \sim I$ then $A \cup B, A \setminus B \sim I$;

(iii) if $A \sim I$ and $A \sim J$ then $A \sim I \cap J$.

3. Check that if $A, B \in \mathrm{clop}(2^{\mathbb{N}})$ are determined by disjoint sets of coordinates then $\nu(A \cap B) = \nu(A) \cdot \nu(B)$. Prove that the same holds for Borel sets A, B, approximating them by clopens.

4. Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in 2^n$. Prove that the set $A(\varepsilon)$ of those $x \in 2^{\mathbb{N}}$, in which ε pops up at some moments satisfies $\nu(A(\varepsilon)) = 1$.

5. A set $A \subseteq 2^{\mathbb{N}}$ is a *tail set* if A is determined by coordinates in $\{n, n+1, \ldots\}$ for every n. Note that the set $B(\varepsilon)$ of those $x \in 2^{\mathbb{N}}$ in which a given finite sequence ε pops up infinitely many times is a tail set.

Note that by **(2)**, for a given set $A \subseteq 2^{\mathbb{N}}$, the family $\{I \subseteq \mathbb{N} : A \sim I\}$ is a filter (see **(8)**). The example of $B(\varepsilon)$ shows that such a filter may have empty intersection.

6. Check that $\nu(x \oplus A) = \nu(C)$ for every $x \in 2^{\mathbb{N}}$ and every $A \in \mathrm{clop}(2^{\mathbb{N}})$. Conclude that ν is a invariant measure on the group $2^{\mathbb{N}}$, that is the same holds for any Borel set A.

7. One can identify $2^{\mathbb{N}}$ with the power set $\mathcal{P}(\mathbb{N})$ by $\xi_A \leftrightarrow A$. Note that then $\oplus$ becomes the symmetric difference $\triangle$.

8. **Ultrafilters on** $\mathbb{N}$. Recall that $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ is a filter if $\emptyset \notin \mathcal{F}$, $\mathbb{N} \in \mathcal{F}$, and $\mathcal{F}$ is closed under taking intersections and supersets. An ultrafilter is a filter which is maximal, i.e. for every filter $\mathcal{F}' \supseteq \mathcal{F}$ we have $\mathcal{F}' = \mathcal{F}$.

Recall that a filter $\mathcal{F}$ is an ultrafilter iff for every $A \subseteq \mathbb{N}$ either $A \in \mathcal{F}$ or $\mathbb{N} \setminus A \in \mathcal{F}$. Recall also that there are ultrafilters $\mathcal{F}$ on $\mathbb{N}$ such that $\bigcap \mathcal{F} = \emptyset$; they are called free or nonprincipial ones.

9. Let $\mathcal{F}$ be a nonprincipial ultrafilter on $\mathbb{N}$. Note that, using **(7)**, we can think that $\mathcal{F}$ is a subset of $2^{\mathbb{N}}$.

Prove that $\mathcal{F}$ is not ν-measurable. In fact, one can check that $\nu_*(\mathcal{F}) = 0$ and $\nu^*(\mathcal{F}) = 1$.

HINT: $\mathcal{F}$ is a tail set so, if measurable, it should have measure either 0 or 1. On the other hand, it should have measure $1/2$ for some reason.

10. Consider $f : 2^{\mathbb{N}} \to [0,1]$, where $f(x) = \sum_{n=1}^{\infty} \frac{x(n)}{2^n}$. Check that $f[\nu] = \lambda$, that is $\lambda(B) = \nu(f^{-1}[B])$ for every $B \in Bor[0,1]$.

11. Consider a probability measure space (X, Σ, μ) which is nonatomic (i.e. for every $A \in \Sigma$, if $\mu(A) > 0$ then there is $B \in \Sigma$, $B \subseteq A$, such that $0 < \mu(B) < \mu(A)$).

Prove that there is a measurable function $f : X \to 2^{\mathbb{N}}$ such that $f[\mu] = \nu$.

HINT: A nonatomic measure has 'the Darboux property'; by induction choose a dyadic system $A_\varepsilon \in \Sigma$, $\varepsilon \in 2^n$, such that $A_{\varepsilon 0} \cup A_{\varepsilon 1} = A_\varepsilon$; $\nu(A_\varepsilon) = 2^{-n}$ for $\varepsilon \in 2^n$. For any $\tau \in 2^{\mathbb{N}}$ set $f(x) = \tau$ whenever $x \in \bigcap_n A_{\tau|n}$.

SUPLEMENT: POLISH SPACES

[En] R. Engelking, *General topology*
[Ke] A. Kechris, *Classical descriptive set theory*

By a *Polish space* we mean a separable topological space which is metrizable by a complete metric.

A. If X is a Polish space without isolated points then X contains a topological copy of $2^{\mathbb{N}}$.
HINT: Choose a dyadic system of sets A_ε, $\varepsilon \in 2^n$, $n \in \mathbb{N}$ such that A_ε are closed sets with nonempty interior, $A_{\varepsilon 0}, A_{\varepsilon 1} \subseteq A_\varepsilon$, $A_{\varepsilon 0} \cap A_{\varepsilon 1} = \emptyset$, the diameter of A_ε is $< 1/n$ for $\varepsilon \in 2^n$. Set $h : 2^{\mathbb{N}} \to X$, so that $h(\tau)$ is the unique point in $\bigcap_n A_{\tau|n}$ for $\tau \in 2^{\mathbb{N}}$.

B. Recall that a G_δ subset in a topological space is one which is an intersection of countably many open sets.
Theorem. A subspace Y of a Polish space X is itself Polish if and only if Y is a G_δ subspace of X.
HINT: Ke], 3.11, [En], 4.3.

C. It follows from (B) that $\mathbb{R} \setminus \mathbb{Q}$ is a Polish space. Prove that $\mathbb{R} \setminus \mathbb{Q}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$.