**1.** For completeness, two general useful facts. Let $(X, \Sigma, \mu)$ be any probability space.

(a) Every family $\mathcal{A} \subseteq \Sigma$ of pairwise disjoint sets of positive measure is countable.

(b) If $g : X \longrightarrow Y$ is $\Sigma$–$\Theta$ measurable or some $\sigma$-algebra $\theta$ in $Y$ then the image measure $\nu = f[\mu]$ defined on $\Theta$ satisfies $\int_Y h \, \mathrm{d}\nu = \int_X h \circ f \, \mathrm{d}\mu$, whenever the integrals make sense.

**2.** The support of a measure $\mu \in P(X)$ is the smallest closed set $F \subseteq X$ of measure 1. Prove that for separable metrizable $X$ such a support exists for every $\mu$; such $F$ satisfies $\mu(F \cap V) > 0$ whenever $V$ is open and $V \cap F \neq \emptyset$.

**3.** A measure $\mu \in P(X)$ is **discrete** if $\mu(A) = 1$ for some countable $A \subseteq X$. Note that a discrete measure is of the form $\mu = \sum_n c_n \delta_{x_n}$. A measure $\mu \in P(X)$ is continuous if $\mu(\{x\}) = 0$ for every $x \in X$. Note that every measure $\mu \in X$ is a sum of a discrete measure and a continuous one.

**4.** Prove that for a (as always, separable metrizable) $X$, a measure $\mu \in P(X)$ is continuous if and only if $\mu$ is nonatomic i.e. for $B \in Bor(X), \mu(B) > 0$ there is a Borel set $A \subseteq B$ such that $0 < \mu(A) < \mu(B)$.

**5.** Prove, using L2/A, that on every uncountable Polish space there is a continuous probability measure. We shall later see that this need not hold without Polishness.

**6.** **Limits along ultrafilters**. Let $\mathcal{U}$ be a nonprincipial ultafilter on $\mathbb{N}$. For a sequence of points $x_n$ in a topological space $X$, we say that $x = \lim_{n \to \mathcal{U}} x_n$ if $\{n \in \mathbb{N} : x_n \in V\} \in \mathcal{U}$ for every open neighbourhood $V \ni x$.

Prove that if $\mathcal{U}$ is a nonprincipial ultrafilter on $\mathbb{N}$ and $(a_n)_n$ is a bounded sequence of reals then $\lim_{n \to \mathcal{U}} a_n$ exists and is uniquely defined for every $\mathcal{U}$. Actually, the same holds for any sequence of $a_n$ in a compact topological space.

**7.** Define $\varphi : C_b(\mathbb{R}) \to \mathbb{R}$ as $\varphi(g) = \lim_{n \to \mathcal{U}} g(n)$ (see above). Check that $\varphi$ is a positive norm-one functional which is **not** represented by any measure on $\mathbb{R}$ (so the Riesz representation theorem fails in this case). If you know the number of ultrafilters on $\mathbb{N}$ you can see that the space of functionals on $C_b(\mathbb{R})$ is really huge, it has $2^{\mathfrak{c}}$ elements; note that there are only $\mathfrak{c}$ functionals on $C([0, 1])$.

**8.** Try to prove directly the Riesz theorem for $C([0, 1])$. For instance, check if the following works:

Given $\varphi$ a positive norm-one functional, define $D(t)$ to be the supremum of $\varphi(g)$ for continuous $g$ sarisfying $0 \leqslant g \leqslant \chi_{(0,t)}$. Then $D$ is the distribution function of some measure.

**9.** Check the following facts on convergence of measures in $P[0, 1]$:

(a) $\mu_n \longrightarrow \lambda$ iff $\mu_n([0, t]) \to t$ for every $t \in [0, 1]$.

(b) $\mu_n \longrightarrow \mu$ iff $\int_0^1 x^k \, \mathrm{d}\mu_n \to \int_0^1 x^k \, \mathrm{d}\mu$ for every $k \geqslant 1$.

Does $(b)$ hold also for measures on $(0, 1)$?