1. The complex version of the Stone-Weierstrass theorem says that if $W \subseteq C(K, \mathbb{C})$ is a subset of complex-valued continuous functions on a compactum K, W contains constant functions, W is closed under addition and multiplications, if $f \in W$ then $\overline{f} \in W$, then W is uniformly dense in $C(K, \mathbb{C})$ provided it distinguishes points of K.

Applying the theorem to linear combinations of functions $x \to e^{itx}$, $t \in \mathbb{R}$, conclude (directly for (ii)) that

(i) for $\mu, \nu \in P(\mathbb{R})$, if $\widehat{\mu} = \widehat{\nu}$ then $\mu = \nu$;

(ii) for $\mu_n, \mu \in P([a, b])$, if $\widehat{\mu_n}$ converge pointwise to $\widehat{\mu}$ then $\mu_n \longrightarrow \mu$ weakly (here $a < b$ are fixed).

2. Concerning Theorem 9.1: Every two uncountable Polish space X, Y are Borel-isomorphic, that is there is a bijection $f : X \to Y$ such that f, f^{-1} are Borel maps; this is Theorem 15.6 in [Kechris], and we may take it for granted.

Therefore, to prove that if $\mu \in P(X)$ and $\nu \in P(Y)$ are two continuous measures then one can be transferred into the other, it is enough to consider the case $X = Y = [0, 1]$. The argument is given in [Kechris, Theorem 17.41], and you might enjoy reading it.

3. Coming back to uniformly distributed sequences (see L5/P10). The Weyl criterion can be extended to prove that if $\theta_1, \theta_2 \in \mathbb{R} \setminus \mathbb{Q}$ are linearly independent over $\mathbb{Q}$ then the sequence

$$p_n = (n \cdot \theta_1 - [n \cdot \theta_1], n \cdot \theta_2 - [n \cdot \theta_2]),$$

is uniformly distributed with respect to the planar Lebesgue measure on $[0, 1]^2$.

This fact can be generalized to $[0, 1]^d$ and $[0, 1]^{\mathbb{N}}$.

4. Every measure $\mu \in P(X)$ (where X is separable and metrizable) has a uniformly distributed sequence. If you know the law of large numbers (LoLN) then you can prove it as follows:

Write ν for the infinite product measure $\bigoplus_n \mu$ on $X^{\mathbb{N}}$. Given $g \in C_b(X)$, check that for ν-almost all $x = (x_n)_n \in X^{\mathbb{N}}$ we have $1/n \sum_{k=1}^{n} g(x_k) \to \int_X g \, \mathrm{d}\mu$ by using LoLN to independent random variables $g \circ \pi_n$. Then use the fact that there is a countable family of $g_k \in C_b(X)$ testing weak convergence of measures.

5. For the next lecture we need to know that a nonatomic measure μ on a σ-algebra Σ has the Darboux property: if $A \in \Sigma$ and $t \in [0, \mu(A)]$ then there is $A \supseteq B \in \Sigma$ such that $\mu(B) = t$.

Try to prove the following: If μ and ν are two nonatomic probability measures on Σ then for every $t \in [0, 1]$ there is $A \in \Sigma$ such that $\mu(A) = \nu(A) = t$. Once it is done, prove that the set $\{(\mu(A), \nu(A)) : A \in \Sigma\}$ is closed and convex in the unit square.

This is the Lyapunov convexity theorem — it holds in every finite dimension and has interesting connections with finite combinatorics[1].

[1] see e.g. T. Vilmos, *A Tale of Two Integrals*, The American Mathematical Monthly, 106:3 (1999), 227–240