1. For a set $A \subseteq \mathbb{N}$,

$$d(A) = \lim_{n\to\infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n},$$

is called the asymptotic density of A, provided the limit above exists. Wride $\mathcal{D}$ for the family of those subsets of $\mathbb{N}$ for which d is defined. Note that d is finitely additive but $\mathcal{D}$ is not an algebra of sets: there are $A, B \in \mathcal{D}$ such that $A \cap B \notin \mathcal{D}$.

2. Note that if we take a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, then the formula

$$\mu(A) = \lim_{n\to\mathcal{U}} \frac{|A \cap \{1, 2, \ldots, n\}|}{n},$$

extends d defined in **1.** to a finitely additive measures defined for all subsets of $\mathbb{N}$. Check also that such μ may be treated as a translation invariant finitely additive measure on the additive group $\mathbb{Z}$ of integers.

3. Write $\mathcal{Z} = \{A \subseteq \mathbb{N} : d(A) = 0\}$. Then $\mathcal{Z}$ is an ideal of subsets of $\mathbb{N}$ and we may form the quotient algebra $\mathfrak{A} = \mathcal{P}(\mathbb{N})/\mathcal{Z}$.

Prove that the formula $\rho(A/\mathcal{Z}, B/\mathcal{Z}) = d^*(A \triangle B)$ defines a metric on this algebra; here d^* is the upper density defined as

$$d^*(A) = \limsup_{n\to\infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n}.$$

4. Prove that the metric space $(\mathfrak{A}, \rho)$ defined above is complete.

HINT. This requires some effort, see 491I in Fremlin's Measure Theory vol 4

5. Prove that there is a Boolean homomorphism $h : \mathrm{clop}(2^{\mathbb{N}}) \to \mathcal{P}(\mathbb{N})/\mathcal{Z}$ such that $\nu(C) = d(h(C))$ for every clopen set C (here ν denotes the usual measure on the Cantor set.

HINT. The measure ν has a uniformly distributed sequence.

6. Prove that the measure algebra $Bor(2^{\mathbb{N}})/\mathcal{N}(\nu)$ can be isomorphically embedded into $\mathcal{P}(\mathbb{N})/\mathcal{Z}$ (here ν is again the usual measure on $2^{\mathbb{N}}$) so that the measure is transferred into the density.

HINT. Use **4.**, **5.** and the argument from the proof of Theorem 10.4.

REMARK. The result, saying that, in a sense, the measure is the asymptotic density, is due to Fremlin (see 491P in his Measure Theory linked above). The problem if the measure algebra embeds into $\mathcal{P}(\mathbb{N})/\mathcal{I}$, where $\mathcal{I}$ is the ideal of finite sets is more delicate — it cannot be decided by the usual axioms of set theory.

7. Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra of sets and let μ be a finitely additive probability measure on $\mathcal{A}$. We define the outer and inner measure of any $Y \subseteq X$ by

$$\mu^*(Y) = \inf\{\mu(A) :\ Y \subseteq A \in \mathcal{A}\}, \quad \mu_*(Y) = \sup\{\mu(A) : Y \supseteq A \in \mathcal{A}\}.$$

For a given $Z \subseteq X$ consider

$$\overline{\mathcal{A}} = \{(A \cap Z) \cup (B \cap Z^c) : A, B \in \mathcal{A}\}.$$

(a) Note that $\overline{\mathcal{A}}$ is the smallest algebra containing $\mathcal{A} \cup \{Z\}$.

(b) Check that

$$\overline{\mu}((A \cap Z) \cup (B \cap Z^c) = \mu^*(A \cap Z) + \mu_*(B \cap Z^c),$$

defines an extension of μ to a finitely additive measure on $\overline{\mathcal{A}}$.

(c) Prove that for every t such that $\mu_*(Z) \leqslant t \leqslant \mu^*(Z)$ there is an extension ν of μ to a finitely additive measure on $\overline{\mathcal{A}}$ such that $\nu(Z) = t$.

8. Prove that every finitely additive measure on an algebra $\mathcal{A} \subseteq \mathcal{P}(X)$ can be extended to a finitely additive measure on $\mathcal{P}(X)$.

HINT. Use **7.** and the Kuratowski-Zorn lemma.

9. It is good to realize that **7.** has an obvious σ-additive version whereas the argument for **8.** does not work in the σ-additive setting.