1. Recall that if μ is a probability measure on some σ-algebra Σ of subsets of some X then every pairwise disjoint family $\mathcal{A} \subseteq \Sigma$ of sets A with $\mu(A) > 0$ is countable.

Prove that the same holds if we replace 'pairwise disjoint' by 'point-finite', meaning every $x \in X$ belongs to finitely many of them.

Remark. One can use CH to show that the fact does not hold for point-countable families.

2. Let X be a metrizable space and let the measure $\mu \in P(X)$ be strictly positive (that is, $\mu(U) > 0$ for every nonempty open U). Prove that X is separable.

Hint. Check that a metrizable space in which there is no uncountable many nonempty pairwise disjoint open sets is separable.

3. Assume CH; as we have seen, then there is a Lusin set $L \subseteq [0,1]$ of cardinality $\mathfrak{c}$, and it satisfies $\mu^*(L) = 0$ for every continuous $\mu \in P([0,1])$.

Show that in such a case there is a sequence of $Z_n \subseteq [0,1]$ such that the Lebesgue measure cannot be extended to a measure on a σ-algebra containing all Z_n.

Rectangle problem In the Scottish book Stanisław Ulam asked if

$$\mathcal{P}([0,1]^2) = \mathcal{P}([0,1]) \otimes \mathcal{P}([0,1]),$$

that is if every set $E \subseteq [0,1]^2$ can be obtained from rectangles $A \times B$ (where $A, B \subseteq [0,1]$ are arbitrary) using countable operations. There are several answers (depending on additional axioms of set theory), and they have interesting consequences even in functional analysis; see this easy reading.

4. Let $|X| \leqslant \mathfrak{c}$. Check that the graph of any function $f : Y \to X$, where $Y \subseteq X$ belongs to the product σ–algebra $\mathcal{P}(X) \otimes \mathcal{P}(X)$.

5. Prove that $\mathcal{P}(X) \otimes \mathcal{P}(X) = \mathcal{P}(X \times X)$ for any set X of cardinality ω_1.

Hint. We can take ω_1 itself. Consider first any subset of $\Delta = \{(\alpha, \beta) : \beta \leqslant \alpha < \omega_1\}$ and use **(4)**.

6. Thinking of Fubini, conclude from **(5)** that there is no universal measure on ω_1.

7. Let μ be a universal measure on X. Check that the product measure $\mu \otimes \mu$ extends to the universal measure π on $X \times X$ via the Fubini formula

$$\pi(E) = \int_X \mu(E_x)\, \mathrm{d}\mu.$$

Note that $\mu \otimes \mu$ has more than one extension to the power set of $X \times X$.

8. Using the Marczewski-Sikorski theorem prove that, under the absence of universal measures, for any probability space (T, Σ, μ) and a measurable function $f : T \to X$ into a metrizable space X there is a separable subspace $X_0 \subseteq X$ such that $f(t) \in X_0$ almost everywhere.