REMARK. We consider here the so called transportation problem which is a special case of linear programming problems. The exercises below introduce the very problem as well as explain the algorithm so they are to be studied systematically, in the right order.

Transportation Problem (TP)

Some good (water, electricity, Toyota cars. . .) is provided at m places offering $a_1, \ldots, a_m$ quantity of that product, respectively. There are n recipients of that good; the recipient number i wants to receive the quantity b_i. We assume that the total demand $\sum_{j=1}^n b_j$ is equal to the total supply $\sum_{i=1}^m a_i$ so the only problem is to plan the distribution to minimize the cost of transportation — we are given the matrix of costs (c_{ij}), where c_{ij} is the cost of sending the unit of the good from the ith provider to the jth recipient. We denote by x_{ij} the quantity to be send from i to j. The problem is then

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij} \\ \text{subject to} \quad & \sum_{i=1}^m x_{ij} = b_j, \qquad j = 1, \ldots, n, \\ & \sum_{j=1}^n x_{ij} = a_i, \qquad i = 1, \ldots, m, \\ & x_{ij} \geqslant 0, \qquad \forall i, j. \end{aligned}$$

1. Note that TP can be written in the standard form $\min c \cdot x$ subject to $Ax = b$, $x \geqslant 0$ but then the matrix A has a large size; how many rows and columns are needed?

Instead, we can visualize the whole problem by a table, as in the following example

3	7	6	4	5
2	4	3	2	2
4	3	8	5	3
3	3	2	2	

Here there are 3 suppliers offering $5, 2, 3$ and four recipients wanting $3, 3, 2, 2$. The values in cells denote the corresponding cost.

2. Bicycles are produced in 3 factories and sold in 4 shops. Each factory produces 1000 items per year; the shops demand $700, 1000, 800, 500$, respectively. The distance from the ith factory to the jth shop is $c_{ij} = (i-j)^2 + i + 3j$. Visualize the corresponding TP as above.

Basic solutions of TP

3. Check that the equality constraints in TP are linearly dependent but there are $m+n-1$ independent conditions among them.

4. Let $X = \{x_{ij}\}$ be a basic feasible solution of TP. Prove that there are at most $m+n-1$ cells (i, j) such that $x_{ij} > 0$.

5. Given a basic solution $X = \{x_{ij}\}$, let $B = \{(i,j) : x_{ij} > 0\}$. Prove that B cannot contain a cycle. Here a cycle is a sequence of different cells of the form

$$(i_1, j_1), (i_1, j_2), (i_2, j_2), \ldots, (i_k, j_{k-1}), (i_k, j_1).$$

Note for chessplayers: On a chessboard, a cycle denotes a closed loop marking the moves of the rook.

HINT: Suppose that B contains a cycle C. Consider X^*, where $x^*_{ij} = x_{ij}$ for $(i,j) \notin C$, and

$$x^*_{i_1,j_1} = x_{i_1,j_1} - \delta, \quad x^*_{i_1,j_2} = x_{i_1,j_2} + \delta, \quad x^*_{i_2,j_2} = x_{i_2,j_2} - \delta, \ldots,$$
$$x^*_{i_k,j_{k-1}} = x_{i_k,j_{k-1}} - \delta, \quad x^*_{i_k,j_1} = x_{i_k,j_1} + \delta,$$

for small δ. Such X^* (considered for various δ) will show that X is not an extreme point of the polyhedron of all solutions.

6. For X's given below, check whether X is a basic solution (of the corresponding TP); empty cells have value zero. Note that by saying that X is a solution we uniquely determine parameters a_i and b_j.

1	2			1		
	3		2			
				4		1
		3	2	1	5	
	1					4

1	1				1
	1	1			
		1	1		
			1	1	
				1	1

1	1				1
	1	1			
		1	1		
			1	1	
				1	

7. Suppose that B is the set of cells in the table $m \times n$ containing more than $m + n - 1$ elements. Prove that B contains a cycle.

HINT: Think inductively.

8. Prove that if B is the set of $m + n - 1$ cells in the table $m \times n$ and B contains no cycles then one can construct a basic solution X of TP such that $x_{ij} = 0$ for cells (i,j) outside B. (Such a solution is feasible or not.)

For that reason, a collection B of $m + n - 1$ cells without a cycle is called a basis.

HINT: there must be a line (row or column) containing only one cell from B; then the value in this cell is uniquely determined; think inductively.

Reducing costs in TP

9. Given TP as above, consider another TP* with a new matrix of costs c^* where $c^*_{ij} = c_{ij} + u_i + v_j$ for some $u_i, v_j \in \mathbb{R}$. Prove that TP and TP* have the same optimal solutions.

10. Let B be a basis in $m \times n$ table. Prove that there are u_i, v_j such that $c_{ij} + u_i + v_j = 0$ whenever $(i,j) \in B$. To see the general idea, do it first for the following example (where

B consists of the cells with given values).

$*$	3	$*$	$*$	$*$	2
$*$	$*$	$*$	1	3	$*$
2	$*$	6	$*$	$*$	$*$
$*$	$*$	$*$	7	$*$	5
$*$	$*$	2	$*$	1	$*$

11. We can now formulate the test for optimality of a given solution of TP. Let X be a basic feasible solution of TP connected with the given basis B of cells. Find u_j, v_j such that the matrix of reduced costs $C(B)_{ij} = c_{ij} + u_i + v_j$. has value zero for all the cells in B. Prove that

Theorem. *If $C(B)_{ij} \geqslant 0$ for all i, j then X is the optimal solution of TP.*

Constructing basic feasible solutions

The general scheme goes as follows (here by a line we mean either row or column).

1. From the remaining lines choose a basic cell, according to some criterion.
2. Set the value in that cell so that you will use up all the remaining demand or supply (depending on which is smaller).
3. Eliminate that line (but only one).
4. If there is only one remaining line then fill in all the values (they are already determined)

There are many criteria that may be used in (1) above. The simplest is the (*North-West angle method*). It ignores the costs completely and goes like ths:

(i) Start for the cell $(1, 1)$.

(ii) If (i, j) has been chosen then choose $(i, j + 1)$ if there is still some undistributed quantity at the row i; otherwise move to $(i + 1, j)$.

Other methods want to be more clever. Vogel's approximation method:

(i) For each remaining lines calculate its penalty; this is the absolute value of the difference between the smallest cost in the line and the next one.

(ii) Find the line with the biggest penalty. In that line choose the cell with the smallest cost.

12. Use the following examples to see how those algorithms work.

3	7	6	4	5
2	4	3	2	2
4	3	8	5	3
3	3	2	2	

3	2	1	2	3	1
5	4	3	1	1	6
0	2	3	4	5	7
3	3	3	2	3	

The simplex method for TP

1. Find the initial feasible basic solution connected with the basis B of $m+n-1$ cells without cycles.
2. Calculate the reduced costs $C(B)_{ij} = c_{ij} + u_i + v_j$ so that $C(B)_{ij} = 0$ for $(i,j) \in B$.
3. If $C(B)_{ij} \geqslant 0$ for all other cells (i,j) then the solution is optimal; STOP.
4. Otherwise choose a cell (i_0, j_0) with the negative reduced cost. Then $B \cup \{(i_0, j_0)\}$ is a set of $m+n$ cells so it contains a cycle C. Find the biggest $\delta \geqslant 0$ such that one can put $x_{i_0 j_0} = \delta$ and change the values in C so that we get a new feasible solution $\overline{x}$. By maximality of δ, $\overline{x}_{rs} = 0$ for some cell $(r,s) \in C$ and we have a new basis $\overline{B} = B \setminus \{(r,s)| \cup \{i_0, j_0\}$.
5. $B := \overline{B}$; Goto (2).

Running the algorithm, we have to keep track of the current data as in the example below: the basic solution (values in circles), the basis and reduced costs (values in corners):

0; 5	2	0; 2	0; 1	8
1	0; 4	0; 9	3	13
0; 4	-5	2	8	4
9	4	11	1	

Remark: We may come across degenerate solutions, having zero values also in some cells from the basis. Remember that the basis should always consists of $m+n-1$ cells. Some iterations may give $\delta = 0$ and in such a case we do not change the solution but the basis gets changed.

13. Perform one iteration of the example given above and check if the new solution is optimal. Exercise using some previously given examples of TP.

14. Observe the following virtue of the algorithm: it gives integer-valued solutions whenever a_i, b_j are positive integers (this is crucial if we are planning the distribution of cars).

Modifications of TP

15. Consider TP for the case where total supply is greater than total demand, that is $\sum_i a_i > \sum_j b_j$:

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\ \text{subject to} \quad & \sum_{i=1}^{m} x_{ij} = b_j, \qquad j = 1, \ldots, n, \\ & \sum_{j=1}^{n} x_{ij} \leqslant a_i, \qquad i = 1, \ldots, m, \\ & x_{ij} \geqslant 0, \qquad \forall i, j. \end{aligned}$$

We can reduce such a problem to the previous form by introducing the phantom recipient that wants to receive $\sum_i a_i - \sum_j b_j$. Note that by solving such an extended problem we also solve the original one.

Analyse what to do in case total demand is greater than total supply.

16. Metro Water District supplies four towns Berdoo, Los Devils, San Go and Hollyglass with water. The water comes from three rivers: Colombo, Sacron and Calorie; however, there is no pipe connecting Calorie with Hollyglas. The costs, demand and supply are as follows (in some units):

	Berdoo	Los Devils	San Go	Hollyglas	Supply
Colombo	16	13	22	17	50
Sacron	14	13	19	15	60
Calorie	19	20	23	–	50
Min. demand	30	70	0	10	
Demand	50	70	30	∞	

Here Berdoo has signed a contract for getting 30 but they are willing to get up to 50 and so on. Check that, despite some complications, one can formulate a pure TP as follows. Replace $-$ by M, a parameter that is bigger than anything else. Note that Hollyglas can get 60 at most so we can replace ∞ by 60. Then the total demand is by 50 greater than the total supply. Introduce the river Phantom giving 50. Divide Berdoo into Berdoo 1 demanding 30 and Berdoo 2 demanding 20. Decide all the costs of the river Phantom so that the extended problem will automatically solve the real one (playing with M and 0).