## Flows in networks minimizing cost

We now think that $V = \{1, 2, \ldots, n\}$.

**The network.** Given a directed graph $G = (V, A)$ together with

- a function $V \ni i \to b_i \in \mathbb{R}$ defining an external supply; we assume $\sum_i b_i = 0$;
- a function $c : A \to \mathbb{R}_+$, $c_{ij}$ is the cost in the arc $(i, j) \in A$

**Definition.** A flow $f = (f(i, j))_{(i,j)\in A}$ is **feasible** if

(i) $0 \leqslant f(i, j)$ for every arc $(i, j) \in A$;
(ii) $\sum_{j\in \mathrm{In}(i)} f(j, i) + b_i = \sum_{j\in \mathrm{Out}(i)} f(i, j)$ for every vertex $i$.

**General problem.** Minimize

$$\sum_{(i,j)\in A} c_{ij} f(i, j),$$

over all feasible flows.

## Adapt the general simplex method to network flows

**Theorem.** Every feasible flow that is a BFS flows through some spanning tree of the graph.

**Theorem.** Every spanning tree of the graph define a basic solution (feasible or not).

## Reducing the costs

**Theorem.** If $p : V \to \mathbb{R}$ is any function and we reduce the costs by the formula

$$\overline{c_{ij}} = c_{ij} - (p_i - p_j),$$

then we get an equivalent problem.

*Proof.*

$$\sum_{(i,j)\in A} \overline{c_{ij}} f(i,j) - \sum_{(i,j)\in A} c_{ij} f(i,j) = -\sum_{(i,j)\in A} (p_i - p_j) f(i,j) =$$
$$= -\sum_i p_i \sum_{j\in \mathrm{Out}(i)} f(i,j) + \sum_j p_j \sum_{i\in \mathrm{In}(j)} f(i,j) =$$
$$= -\sum_i p_i \sum_{j\in \mathrm{Out}(i)} f(i,j) + \sum_i p_i \sum_{j\in \mathrm{In}(i)} f(j,i) = \sum_i p_i b_i.$$

**Theorem.** If $f$ is a feasible flow connected with some spanning tree $T$ then there are $p_i$ such that the reduced costs $\overline{c_{ij}} = c_{ij} - (p_i - p_j)$ is zero for every arc $(i,j)$ from that tree.

## Changing BFS.

Suppose that $f$ is a feasible flow on some spanning tree $T$. We have reduced the costs and have $\overline{c_{ij}} < 0$ for some edge $(i,j)$ outside $T$.

Adding $(i,j)$ to the edges of $T$ we get a unique cycle $C$. Let $B$ denote te arcs in that cycle that are 'backward'; $F = C \setminus B$.

If $B = \emptyset$ then the optimal costs is $-\infty$.

Otherwise, we take $\theta = \min\{f(e) : e \in B\}$ and modify the flow:

$$\widehat{f}(x,y) = \begin{cases} f(x,y) + \theta & \text{when } (x,y) \in F \\ f(x,y) - \theta & \text{when } (x,y) \in B \\ f(x,y) & \text{otherwise} \end{cases}$$

## Summary

Simplex for network flows.

(1) Find some spanning tree and the unique flows through that tree. Assume it is feasible.
(2) Reduce the costs; if they are nonnegative then the flows is optimal.
(3) Otherwise, modify the flow incorporating 'negative' edge to the tree.; GoTo (2).

## Duality

We now think that $V = \{1, 2, \ldots, n\}$.

Consider a problem (D)

$$\max \sum_{i \leqslant n} b_i y_i \quad \text{subject to}$$

$$y_i - y_j \leqslant c_{ij} \text{ for all } (i,j) \in A.$$

**Theorem (weak duality).** If $y$ is a feasible solution of (D) then

$$b \cdot y = \sum_{i \in V} b_i y_i \leqslant \sum_{(i,j) \in A} c_{ij} f(i,j),$$

for every feasible flow $f$.

*Proof.*

$$\sum_{(i,j)\in A} c_{ij} f(i,j) \geqslant \sum_{(i,j)\in A} (y_i - y_j) f(i,j) =$$

$$= \sum_{(i,j)\in A} y_i f(i,j) - \sum_{(i,j)\in A} y_j f(i,j) =$$

$$= \sum_i y_i \sum_{j \in \mathrm{Out}(i)} f(i,j) - \sum_j y_j \sum_{i \in \mathrm{In}(j)} f(i,j) =$$

$$= \sum_i y_i \sum_{j \in \mathrm{Out}(i)} f(i,j) - \sum_i y_i \sum_{j \in \mathrm{In}(i)} f(j,i) =$$

$$= \sum_i y_j \left( \sum_{j \in \mathrm{Out}(i)} f(i.j) - \sum_{j \in \mathrm{In}(i)} f(i,j) \right) = \sum_i y_i b_i.$$

**Complementarity.** If we find feasible $f$ and $y$ such that

$$c_{ij} f(i,j) = (y_i - y_j) f(i,j) \text{ for every } (i,j) \in A,$$

then $f$ and $y$ are optimal.

## Approach to the dual problem

$$\max \sum_{i \leqslant n} b_i y_i \quad \text{subject to}$$

$$y_i - y_j \leqslant c_{ij} \text{ for all } (i,j) \in A.$$

**Observe that:** $y = 0$ is a feasible solution and for every feasible $y$, $y + a$ is feasible too.

**Jargon.** An arc $(i,j)$ is **saturated** (with respect to $y$) if $c_{ij} = y_i - y_j$.
A set $S \subseteq V$ is **balanced** if threre is no saturated arc leaving $S$.

**Lemma.** If $y$ is feasible, $S$ is balanced then $y^* = y + \theta\chi_S$ is also feasible, where

$$\theta = \min\{(c_{ij} - (y_i - y_j)) : (i,j) \in A, i \in S, j \notin S\} > 0.$$

## Outline of the algorithm for the dual problem

**When $y^*$ is better:** If and only if

$$\sum_{i \in V} b_i y_i^* - \sum_{i \in V} b_i y_i = \theta \sum_{i \in S} b_i > 0.$$

**Idea**

(1) Start from some feasible $y$.
(2) Look for balanced $S \subseteq V$ such that $\sum_{i \in S} b_i > 0$. If there are no such $S$ then STOP — $y$ is optimal.
(3) Change $y$ to $y^*$ and repeat.

## Primar-dual algorithm

**Outline**

(1) Start from some $y$ feasible for the dual problem.

(2) Use FF to check how much can flow from the source vertices $V^+$ to target vertices $V^-$ through the graph consisting only of saturated arcs.

(3) If the volume of the flow is equal to $\sum_{i \in V^+} b_i$ then STOP (we have an optomal flow).

(4) Otherwise LA finds a balanced $S$ with $\sum_{i \in S} b_i$ while checking that the flow is maximal.

(5) Change $y$ to $y^* = y + \theta \chi_S$; GoTo (2).