1. Check that c_0 and ℓ_p for $1 \leqslant p < \infty$ are separable while ℓ_∞ is not. In fact, there are $\mathfrak{c}$ many vectors in the unit ball of ℓ_∞ such that the distance bewteen every pair is 2.

2. Prove that all the norms on $\mathbb{R}^n$ are equivalent.

3. Prove that a finitely dimensional subspace of a Banach space is necessarily closed.

4. Prove that no infinite dimensional Banach space is a countable union of its finite dimensional subspaces.

5. Let $e_n = (0, \ldots, 0, 1, 0, \ldots)$. Note that e_n converge weakly in c_0 but not in ℓ_1.

6. A sequence (f_n) in $C[0,1]$ converges weakly to 0 if and only if f_n are uniformly bounded and converge pointwise to 0.

HINT: For uniform boundedness recall the Banach–Steinhaus theorem.

7. Prove that if f_n converge weakly to f in $C(K)$ then their some convex combinations converge uniformly to f.

HINT: Mazur's theorem says that if a subset A of a Banach space X is convex then the weak closure of A coincides with the norm closure of $.A$.

8. Check directly that c_0 is norm-closed and *weak** dense in ℓ_∞.

9. Prove that for any Banach space X and $x^*, x_1^*, \ldots, x_n^* \in X^*$, if $\bigcap_{i \leqslant n} \ker(x_i^*) \subseteq \ker(x^*)$ then x^* is a linear combination of x_i^*.

HINT: Consider $L : X \to \mathbb{R}^n$, $Lx = (x_1^*(x), \ldots, x_n^*(x))$ and $L[X] \subseteq \mathbb{R}^n$.

10. Prove that for $x_1^*, \ldots, x_n^* \in X^*$ and $x^{**} \in X^{**}$ there is $x \in X$ such that $x^{**}(x_i^*) = x_i^*(x)$ for $i \leqslant n$.

Consequently, X is *weak**–dense in X^{**} (see also 15).

11. Prove that no infinite dimensional Banach space is metrizable in its weak topology.

HINT: Using 9 and 4 check that $0 \in X$ has no countable base in the weak topology.

12. Start to appreciate the Eberlein-Smulyan theorem: Let $A = \{e_m + m e_n : 1 \leqslant m < n\} \subseteq \ell_2$. Check that 0 is in the weak closure of A but there is no seqeunce in A converging to 0 weakly.

13. Let X be infinite dimensional Banach space. Prove Riesz's lemma saying that for $\theta < 1$ there is a sequence $x_n \in B_X$ such that $\|x_n - x_m\| > \theta$ for $n \neq m$.

HINT: Suppose that $Y \subseteq X$ is finite dimensional subspace. Given $x \in X \setminus Y$, the distance $d = \inf\{\|x - y\| : y \in Y\}$ is positive by 3. Pick $z \in Y$ such that $\|z - x\| < d/\theta$; consider

$$x_\theta = \frac{x - z}{\|x - z\|},$$

which has norm one and its distance from Y is $> \theta$.

REMARK. See [Diestel], page 7 how to treat the case $\theta = 1$. For (some) $\theta > 1$ this is a deep theorem due to Elton and Odell (Chapter XIV in [Diestel]).

14. A comment on the previous problem: Let

$$X = \{x \in C[0,1] :\ x(0) = 0\}, \quad Y = \{x \in X :\ \int_0^1 x(t)\ \mathrm{d}t = 0\}.$$

Then Y is a closed subspace of X but there is no $x \in X$ with $\|x\| = 1$ and $\mathrm{dist}(x, Y) \geqslant 1$.

15. Consider $x_1^*, \ldots, x_n^* \in X^*$and $a_1, \ldots, a_n \in \mathbb{R}$. Helly's theorem says that the followinf are equivalent:

(i) for every $\varepsilon > 0$ there is $x \in X$ such that $||x|| \leqslant 1 + \varepsilon$ and $x_i^*(x) = a_i$ for $i = 1, \ldots, n$;

(ii) for any scalars $t_1, \ldots, t_n$ we have

$$\left|\sum_{i=1}^n t_i a_i\right| \leqslant \left\|\sum_{i=1}^n t_i x_i^*\right\|.$$

Once you prove Helly's theorem, you will know why B_X is *weak** dense in $B_{X^{**}}$.