Complemented subspaces

1. Note that for every Banach space X, X^* is complemented in X^{***}.

Hint: Consider $X^\perp = \{x^{***} : x^{***}|X \equiv 0\}$.

2. Let K be a compact space containing a convergent sequence of distinct $t_n \in K$. Prove that $C(K)$ contains a complemented copy of c_0.

Hint: Such a copy is spanned by a sequence of norm-one functions $g_n \in C(K)$ such that $g_n \cdot g_k = 0$ for $n \neq k$.

3. Prove that if $T : X \to \ell_1$ is a surjective operator then X contains a complemented copy of ℓ_1 as follows. Pick $x_n \in X$ such that $Tx_n = e_n \in \ell_1$ so that $\|x_n\| \leqslant C$ for some $C > 0$. Then $S : \ell_1 \to X$ defined by $Se_n = x_n$ is a bounded operator and TS is the identity. Consider the operator ST.

Conjugate operators

4. Recall that if $T : X \to Y$ is an embedding then $T^* : Y^* \to X^*$ is surjective. Check that if T is only one-to-one then $T^*[Y^*]$ is *weak** dense in X^*.

5. Let $g : K \to L$ be a continuous surjection between compact spaces.

(i) Check that $T : C(L) \to C(K)$, $Tf = f \circ g$ is an isometric embedding.

(ii) Check that $T^* : C(K)^* \to C(L)^*$ is then $T^*\mu = g[\mu]$ (where $g[\mu]$ is the image measure).

(iii) Conclude that for every $\nu \in C(L)^*$ there is $\mu \in C(K)^*$ such that $g[\mu] = \nu$.

ℓ_1-sequences

6. We say that a bounded sequence $(x_n)_n$ in a Banach space X is an ℓ_1-sequence (or that it is equivalent to the standard basis of ℓ_1) if there is $\theta > 0$ such that $\| \sum_{i \leqslant n} c_i \cdot x_i\| \geqslant \theta \sum_{i \leqslant n} |c_i|$ for any scalars c_i and every n. Check that the following are ℓ_1 sequences:

(i) norm-one $f_n \in L_1[0,1]$ such that $f_n \cdot f_k = 0$ for $n \neq k$.

(ii) projections $\pi_n \in C(2^\omega)$.

(iii) $f_n(t) = \cos(2\pi n t) \in C[0,1]$.

(Weak) Banach-Saks Property

7. A Banach space X has the Banach-Saks Property (Weak Banach-Saks Property) if every bounded sequence in X (weakly converging sequence, respectively) has a subsequence $(x_n)_n$ with norm-converging averages $1/n \cdot (x_1 + \ldots x_n)$. Prove that

(i) c_0 and ℓ_1 have WBSP;

(ii) ℓ_1 does not have BSP.

What about ℓ_p for $p > 1$?

8. **Schreier's family.** Let $\mathcal{S}$ be a family of those finite sets $S \subseteq \mathbb{N}$ for which $|S| \leqslant \min S$. Note the following properties of $\mathcal{S}$:

(i) every finite $A \subseteq \mathbb{N}$ contains $S \in \mathcal{S}$ such that $|S| \geqslant (1/2) \cdot |A|$;

(ii) there is no infinite $\mathcal{S}$-homogeneous set X.

Remark: It is an open problem if there is a family $\mathcal{S}$ on an uncountable set satisfiyng (i) and (ii).

9. Let K be a subspace of $2^{\mathbb{N}}$ consisting of χ_S where $S \in \mathcal{S}$ for the Shreier family $\mathcal{S}$. Check that K is countable and compact.

Prove that $C(K)$ does not have the Weak Banach-Saks property.

HINT: Consider projections $\pi_n : 2^{\mathbb{N}} \to 2$ restricted to K.

10. Note that, by Mazur's theorem, a weakly convergent sequence in a Banach space must have *some* convex combinations converging in norm.

ROSENTHAL ℓ_1-THEOREM

11. Prove that every Banach space X having the Banach-Saks property is reflexive.

HINT: X cannot contain ℓ_1 so every bounded sequence in X has a subsequence that is weakly Cauchy. What happens if additionally $1/n(x_1 + \ldots + x_n)$ converge in norm?

12. Prove that if infinite dimensional X has the Schur property (i.e. every weakly convergent sequence converges in norm) then X ontains ℓ_1.