Some unique Boolean algebras

1. An algebra $\mathfrak{A}$ is nonatomic if every nonzero element can be split into two nonzero elements. Note that the algebra $\mathfrak{C}$ of clopen subsets of 2^ω is countable and nonatomic and, clearly, $\mathrm{ult}(\mathfrak{C})$ is 2^ω.

Prove that every countable nonatomic Boolean algebra is isomorphic to $\mathfrak{C}$.

2. Let $\mathfrak{B}$ be the measure algebra of the Lebesgue measure λ on $[0,1]$. Note that $\mathfrak{B}$ is separable in the metric $(a,b) \to \lambda(a \triangle b)$. Recall that the compact space $\mathrm{ult}(\mathfrak{B})$ is not separable:-)

Prove that this (Frechet-Nikodym) metric is complete.

Hint: Use the Riesz theorem, stating that a sequence of functions that is Cauchy in measure has a subsequence converging almost everywhere. Alternatively, take $a_n \in \mathfrak{B}$ such that $\lambda(a_n \triangle a_{n+1}) \leqslant 1/2^n$ and check that the sequence of a_n converges to to its upper limit.

3. Prove the following (Baby Maharam Theorem);

If (X, Σ, μ) is a nonatomic probability measure space and its measure algebra $\mathfrak{A}$ is separable in the Frechet-Nikodem metric then there is an isomorphism $\varphi : \mathfrak{A} \to \mathfrak{B}$ ($\mathfrak{B}$ as above) such that $\lambda(\varphi(a)) = \mu(a)$ for $a \in \mathfrak{A}$.

Hint: Define finite algebras $\mathfrak{A}_n \subseteq \mathfrak{A}$, $\mathfrak{B}_n \subseteq \mathfrak{B}$ and partial (consistent) isomorphisms $\varphi_n : \mathfrak{A}_n \to \mathfrak{B}_n$ such that $\mathfrak{A}' = \bigcup_n \mathfrak{A}_n$ is metrically dense in $\mathfrak{A}$ and $\mathfrak{B}' = \bigcup_n \mathfrak{B}_n$ is metrically dense in $\mathfrak{B}$. Then note that the isomorphism $\mathfrak{A}' \to \mathfrak{B}'$ extends uniquely to $\mathfrak{A} \to \mathfrak{B}$ because it is an isometry.

Measures on ω

4. Recall that $M(\mathcal{P}(\omega))$ is the space of all finitely additive signed measures on $\mathcal{P}(\omega)$ that have bounded variation. Recall also that μ has a bounded variation if and only if $\mu = \mu^+ - \mu^-$ where μ^+, μ^- are finite nonnegative measures.

Check that $M(\mathcal{P}(\omega))$ is a Banach space with the norm $\|\mu\| = |\mu|(\omega)$.

Note that every $\mu \in M(\mathcal{P}(\omega))$ has a unique decomposition $\mu = \mu_1 + \mu_2$ where μ is countably additive and μ_2 vanishes on finite sets. Moreover, $\|\mu\| = \|\mu_1\| + \|\mu_2\|$.

It follows that ℓ_1 is a complemented subspace of $(\ell_\infty)^*$.

5. Let $\mathcal{U}$ be a nonprincipial ultrafilter on ω. Then

$$\mu(A) = \lim_{n \to \mathcal{U}} \frac{|A \cap n|}{n}$$

defines an extension of the asymptotic density to an element of $M(\mathcal{P}(\omega))$ and $\mu \in \ell_\infty^* \setminus \ell_1$.

Check that μ is, moreover, translation invariant (so it witnesses that the additive group $\mathbb{Z}$ is amenable).

Dunford-Pettis Property (DPP)

6. A Banach space X has DPP if for every weakly null sequence $(x_n)_n$ and every weakly null sequence $(x_n^*)_n$ in X^* we have $\lim_n x_n^*(x_n) = 0$.

Check that c_0 and ℓ_1 have DPP for 'obvious' reasons.

7. Note that if X^* has DPP then so does X.

8. Not that if X has DPP and Y is a complemented subspace of X then Y also has DPP.

9. Prove that no reflexive infinitely dimensional space X has DPP.

Hint: Define $x_n \in B_X$ and $x_n^* \in X^*$ such that $x_n^*(x_n) = 1$ and $x_n^*(x_i) = 0$ for $i < n$.