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Diamonds are a Topologist's Best Friend

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Introduction

The \diamond -principle, introduced by Jensen in 1972 ([Jen72]), became one of the most used auxiliary axioms to ZFC. It has proven to be a powerful tool for many recursive constructions. One can go as far as to say that almost anything can be constructed under \diamond . In particular, \diamond implies the Continuum Hypothesis, the existence of an Aronszajn tree [Jen72] and an Ostaszewski Space [Ost76]. Its implications, however, reach beyond the scope of set theory and topology – for example, it was used in algebra to solve the Whitehead Problem [She74].

This apparent strength of the principle can, however, become its weakness. More subtle, detailed problems fail under \diamond . Is there an Ostaszewski space of size smaller than \mathfrak{c} ? Can one exist in the absence of an Aronszajn tree? A subtler, tailored tool is required for answering such questions.

In 2004 a team of J.T.Moore, M.Hrušák and M.Džamonja devised a tool called parametrized \diamond -principles [MHD04]. They used a general form of a cardinal invariant to construct a schema of axioms, all implied by \diamond , but usually compatible with $\neg\text{CH}$.

In this work, we study the usage of parametrized \diamond -principles in topological problems. We begin by showcasing a proof using parametrized \diamond -principles on a known result that $\diamond(\mathfrak{s}^\omega)$ implies the existence of an Ostaszewski space. We repeat the proof from [MHD04] with additional details and insights.

Then we answer, in positive, a question of A. Avilés and P. Koszmider posed in [AK13]:

Question 0.1. *Consistently, is there a Radon-Nikodým-compact space of weight less than \mathfrak{c} with a continuous image which is not Radon-Nikodým?*

It is worth mentioning that, by [Avi05], any Radon-Nikodým-compact space with a non-Radon-Nikodým-compact continuous image has a weight of at least \mathfrak{b} .

In [AK13], such a space is constructed with weight \mathfrak{c} , both in ZFC and using \diamond . Here, we strengthen the second of those constructions using a \diamond -principle parametrized by a particular invariant equal to $\text{cof}(\mathcal{M})$. Finally we study possible improvements to our construction and find a suitable Radon-Nikodým-compact space of weight ω_1 in many models of ZFC with $\text{non}(\mathcal{M}) = \omega_1$.

1 Preliminaries

1.1 Cardinal invariants

Recall that \forall^∞ means "for all but finitely many" and \exists^∞ "there are infinitely many".

Definition 1.1. If $\sim \subseteq A \times B$ is any relation and I is an infinite set, then \sim^* , $\sim^\infty \in A^I \times B^I$ are defined as

$$\begin{aligned} f \sim^* g &\iff (\forall^\infty i \in I) f(i) = g(i) \\ f \sim^\infty g &\iff (\exists^\infty i \in I) f(i) = g(i) \end{aligned}$$

For the purpose of this paper we restrict our interest to the following definition.

Definition 1.2. A triple (A, B, \sim) where A, B are arbitrary sets and $\sim \subseteq A \times B$ is called an *invariant*. We put

$$\|A, B, \sim\| = \min\{|\mathcal{B}| : \mathcal{B} \subseteq B, (\forall a \in A) (\exists b \in \mathcal{B}) a \sim b\}$$

as the evaluation of such invariant.

This definition encompasses most of the classical invariants with the notable omission of \mathfrak{a} , \mathfrak{t} and \mathfrak{u} . For example

- $\mathfrak{b} = \|\omega^\omega, \omega^\omega, \not\leq^*\|$,
- $\mathfrak{d} = \|\omega^\omega, \omega^\omega, \leq^*\|$,
- $\mathfrak{non}(\mathcal{M}) = \|\mathcal{M}, \mathbb{R}, \not\leq\|$,
- $\mathfrak{cof}(\mathcal{M}) = \|\mathcal{M}, \mathcal{M}, \subseteq\|$.

From now on, we will slightly abuse the notation and write (A, B, \sim) for both the invariant and its evaluation. It is worth noting that many different triplets may have evaluations that are always equal. In particular the classical ideal-based invariants, such as $\mathfrak{non}(\mathcal{M})$ are better represented using a basis of \mathcal{M} consisting of F_σ sets. This allows us to represent such invariants in a Borel manner.

Definition 1.3. An invariant (A, B, \sim) is **Borel** if there are Polish spaces X, Y such that A, B, \sim are Borel subsets of X, Y and $X \times Y$, respectively.

We will skip verification of being Borel for the invariants used in this paper, as it is straightforward for each of them.

The following notion can be used to express the inequalities between invariants.

Definition 1.4. A *morphism* φ from (A, B, \sim) to (A', B', \sim') is a pair of functions $\varphi_1 : A' \rightarrow A, \varphi_2 : B \rightarrow B'$ such that

$$(\forall a' \in A') (\forall b \in B) \text{ if } \varphi_1(a') \sim b \text{ then } a \sim' \varphi_2(b).$$

Fact 1.5. If there is a morphism φ from (A, B, \sim) to (A', B', \sim') then $\|A, B, \sim\| \geq \|A', B', \sim'\|$.

Proof. Take $X \subset B$ meeting every $a \in A$ via \sim . Then for any $a' \in A'$ we have $x \in X$ with $\varphi_1(a') \sim x$. Thus $\varphi_2[X]$ meets every $a' \in A'$ via \sim' , thus witnessing the proclaimed inequality. \square

More details on these notions can be found, for example, in [Bla10].

1.2 Diamonds

Definition 1.6. A set $C \subseteq \omega_1$ is a **club** if it is

- **Closed:** for any countable $A \subseteq C$, we have $\sup A \in C$.
- **Unbounded:** for any $\alpha \in \omega_1$ there is $\beta \in C$ with $\alpha < \beta$. Equivalently, since $\text{cf}(\omega_1) = \omega_1$, C is uncountable.

Fact 1.7. An intersection of countably many clubs is a club. In particular clubs form a basis of the filter

$$\mathcal{F}_\clubsuit = \{A \subseteq \omega_1 : A \text{ contains a club}\}.$$

Proof. Let $\langle C_n : n \in \omega \rangle$ be a collection of clubs and let $C_\omega = \bigcap_{n \in \omega} C_n$. If $A \subseteq C_\omega$, then $A \subseteq C_n$ for each n and so $\sup A \in C_n$ and $\sup A \in C$.

Take any $\alpha \in \omega_1$ and let $f : \omega \rightarrow \omega$ be such that each $f^{-1}[n]$ is infinite. There is β_0 in $C_{f(0)}$ with $\beta_0 > \alpha$ (this is possible as $C_{f(0)}$ is unbounded). Recursively construct an increasing sequence $\langle \beta_k : k \in \omega \rangle$ with $\beta_k \in C_{f(k)}$. Let $\gamma = \sup\{\beta_k : k \in \omega\}$. Then for each n the set $\{\beta_k : k \in f^{-1}[n]\}$ is a subset of C_n and has supremum of γ . Thus $\gamma \in C_n$ for each n and $\gamma \in C_\omega$ (and $\gamma > \alpha$). \square

Definition 1.8. A set is **stationary** if it is positive according to \mathcal{F}_\clubsuit . Equivalently, a set is stationary if it intersects every club.

Note that stationary sets are unbounded, as for any $\alpha < \omega_1$ if C is a Club then so is $C \setminus \alpha$.

Definition 1.9. The \diamond principle is the following statement:

There exists a \diamond -sequence, i.e. $\langle A_\alpha : \alpha < \omega_1 \rangle$ with $A_\alpha \subseteq \alpha$ s.t. for any $A \subseteq \omega_1$ the set $\{\alpha : A \cap \alpha = A_\alpha\}$ is stationary.

Fact 1.10. \diamond implies CH.

Proof. Let $A \subseteq \omega \subset \omega_1$. Then for some $\alpha > \omega$ there is $A_\alpha = A$. Hence every \diamond -sequence enumerates all subsets of ω and CH follows. \square

In [DS78], the author introduces the following axiom called the weak \diamond -principle, which follows from $2^\omega < 2^{\omega_1}$ so, in particular, from \diamond .

For any $F : 2^{<\omega_1} \rightarrow 2$ there is $\langle A_\alpha : \alpha \in \omega_1 \rangle$ with $A_\alpha \subseteq \alpha$ such that for any $\delta \in 2^{\omega_1}$ the set $\{\alpha : F(\delta \upharpoonright \alpha) = A_\alpha\}$ is stationary.

The formulation of this axiom inspired the shape of parametrized \diamond principles of Moore, Hrušák and Džamonja ([MHD04]).

Definition 1.11. For a Borel invariant (A, B, \sim) the $\diamond(A, B, \sim)$ principle is the following axiom:

For each Borel $F : 2^{<\omega_1} \rightarrow A$ there is a $\diamond(A, B, \sim)$ -sequence, i.e. $g : \omega_1 \rightarrow B$ such that for any $\gamma \in 2^{\omega_1}$ the set $\{\alpha : F(\gamma \upharpoonright \alpha) \sim g(\alpha)\}$ is stationary.

Where $F : 2^{<\omega_1} \rightarrow A$ is said to be **Borel** if $F \upharpoonright_{2^\alpha}$ is Borel for each $\alpha < \omega_1$.

To properly introduce these axioms we present a few facts.

Fact 1.12. For any Borel (A, B, \sim) we have $\diamond(A, B, \sim)$ implies $\|A, B, \sim\| \leq \omega_1$.

Proof. Note that, since (A, B, \sim) is Borel, then $|A|, |B| \leq \mathfrak{c}$. Let $f : 2^\omega \rightarrow A$ be a Borel surjection. For t of finite length set $F(t)$ arbitrarily, and otherwise let $F(t) = f(t \upharpoonright \omega)$. If g is a $\diamond(A, B, \sim)$ -sequence for F , then $g[\omega_1]$ witnesses the fact that $\|A, B, \sim\| \leq \omega_1$. \square

Fact 1.13. \diamond implies $\diamond(A, B, \sim)$ for any Borel invariant (A, B, \sim) .

Proof. Take any Borel $F : 2^{<\omega_1} \rightarrow A$ and let $\langle A_\alpha : \alpha < \omega_1 \rangle$ be a \diamond sequence. Each A_α can be treated as an element of 2^α . Set $g(\alpha)$ to be any element $b \in B$ with $F(A_\alpha) \sim b$. Then for any $\gamma \in 2^{\omega_1}$ we have

$$\{\alpha : \gamma \upharpoonright \alpha = A_\alpha\} \subseteq \{\alpha : F(\gamma \upharpoonright \alpha) \sim g(\alpha)\},$$

implying that g is a $\diamond(A, B, \sim)$ -sequence. \square

Fact 1.14. Let $\mathfrak{c} = (\mathbb{R}, \mathbb{R}, =)$. Then $\diamond(\mathfrak{c})$ implies \diamond .

Proof. For each infinite $\alpha \in \omega_1$, fix a Borel bijection $h_\alpha : 2^\alpha \rightarrow \mathbb{R}$. For $t \in 2^\alpha$, let $F(t) = h_\alpha(t)$. Let g be the $\diamond(\mathfrak{c})$ sequence for F . Then $A_\alpha = h_\alpha^{-1}(g(\alpha))$ is a \diamond -sequence. \square

The following theorem is of great interest to us, as it provides many models for parametrized \diamond 's with \neg CH. Observe that multiple classical forcing notions – ex. Sacks, Laver, Miller – have the following homogeneity property: $\mathbb{P} \cong \mathcal{P}(2)^+ \times \mathbb{P}$, where $\mathcal{P}(2)$ is the 4-element boolean algebra with 2 atoms and $\mathcal{P}(2)^+ = \mathcal{P}(2) \setminus \{0\}$.

Theorem 1.15 ([MHD04]). If \mathbb{P} is a Borel, proper forcing with $\mathbb{P} \cong \mathcal{P}(2)^+ \times \mathbb{P}$ and \mathbb{P}_{ω_2} is the countable support iteration of \mathbb{P} of length ω_2 , then

$$\mathbb{P}_{\omega_2} \Vdash \diamond(A, B, \sim) \text{ iff } \mathbb{P}_{\omega_2} \Vdash \|A, B, \sim\| \leq \omega_1.$$

1.3 Example: the Ostaszewski Space

In this section we give an example of a proof using parametrized diamonds. This proof and many others can be found in [MHD04]. We provide more, if not overabundant, details here.

A convenient way of thinking about this technique is that of encoding and decoding. Due to some formalities, a typical construction seems to follow three steps:

- (1) defining a function F and obtaining a $\diamond(\cdot)$ -sequence,
- (2) using that sequence for a recursive construction of length ω_1 ,
- (3) proving the properties.

It might, however, be useful to first focus on steps (3) and (1). In them, one can find the encoding process – trying to pass information to the construction. Then the focus of (2) is to decode and use that information to ensure the desired properties.

This can be seen in the proof of Theorem 1.17. First, we establish some notions.

Definition 1.16. *An uncountable space is called an **Ostaszewski space** if it is normal, noncompact, countably compact, and all its open subsets are either countable or co-countable.*

A set $A \subseteq \omega$ **splits** $B \subseteq \omega$ if both $A \cap B$ and $B \setminus A$ are infinite. Let $(\omega)_\omega^\omega$ be the set of all infinite partitions of ω into infinite sets. We put

$$\mathfrak{s}^\omega = ([\omega]^\omega, (\omega)_\omega^\omega, \text{is split by every piece of}).$$

It is worth mentioning that $\mathfrak{s}^\omega \leq \text{non}(\mathcal{M})$ and $\mathfrak{s}^\omega \leq \text{non}(\mathcal{N})$ (see ex. in [MHD04]).

Theorem 1.17. *Under $\diamond(\mathfrak{s}^\omega)$, there is an Ostaszewski space.*

Proof. Begin by setting a bijection $e_\beta : \omega \rightarrow \beta$ for each $\beta < \omega_1$.

It is convenient to think about F as a function taking triplets (α, \mathcal{B}, D) as arguments, with an ordinal α no bigger than ω_1 , $\mathcal{B} = \langle U_\beta : \beta < \alpha, n \in \omega \rangle$, $\beta \in U_\beta \subseteq \beta + 1$ and $D \subseteq \alpha$ instead of a binary sequence of length α . One can achieve this by fixing an encoding of the desired domain to $2^{<\omega_1}$. As the construction can be assumed to be limited to a club set, we only specify the encoding on the limit ordinals. The encoding needs to support restrictions in the following sense:

If (α, \mathcal{B}, D) is a triple as above and $\beta < \alpha$ is a limit ordinal, then the sequence obtained by restricting encoding of (α, \mathcal{B}, D) to β encodes $(\beta, \mathcal{B} \upharpoonright_\beta, D \cap \beta)$.

This can be easily done for limit ordinals so that the function F defined below is still Borel when interpreted as a function from $2^{<\omega_1}$. For example one can use the following recipe:

Given (α, \mathcal{B}, D) as above, with α limit, for each limit $\beta < \alpha$ partition $[\beta, \beta + \omega)$ into countably many infinite sequences. Then pick $\sigma \in 2^\alpha$ so that

- σ on the first sequence encodes $D \cap [\beta, \beta + \omega)$ as a subset of $[\beta, \beta + \omega)$,
- σ on the $(n + 1)$ -st sequence encodes $U_{\beta+n} \setminus \{\beta + n\}$ through $e_{\beta+n}$.

Let $\mathcal{T}_{\mathcal{B}}$ be the topology on α given by taking \mathcal{B} as a clopen subbase. Construct F so that $F(\alpha, \mathcal{B}, D) = \omega$ unless

- (1) α is a limit ordinal,
- (2) U_β is compact in $\mathcal{T}_{\mathcal{B}}$ for each $\beta < \alpha$,
- (3) for $\beta < \alpha$, the closure of $[\beta, \beta + \omega)$ is $[\beta, \alpha)$ (in $(\alpha, \mathcal{T}_{\mathcal{B}})$),
- (4) the closure of D in $\mathcal{T}_{\mathcal{B}}$ is not compact.

In such case we create a partition of α into compact, open sets from \mathcal{B} . More precisely define V_n by setting $V_0 = U_{e_\alpha(0)}$ and

$$V_n = U_{e_\alpha(k)} \setminus \bigcup_{i < n} V_i,$$

for the minimal k for which such set is nonempty. Now set $F(\alpha, \mathcal{B}, D)$ to be the set of those n for which $D \cap V_n$ is nonempty. Note that, since D does not have a compact closure, we know that $F(\alpha, \mathcal{B}, D)$ is infinite.

Let g be a $\diamond(\mathfrak{s}^\omega)$ -sequence for F (we treat $g(\alpha)$ as a sequence of sets). Starting from $\alpha = \omega$ with $U_n = \{n\}$, we recursively define the topology $(\omega_1, \mathcal{T}_{\mathcal{B}})$ as follows: On the steps that are limits of limit ordinals put $\mathcal{B} \upharpoonright_\alpha = \bigcup_{\beta < \alpha} \mathcal{B} \upharpoonright_\beta$. Note that (1)-(3) are preserved by such union, since the construction ensures that for $\alpha < \beta$ the topology on α as a subspace of $(\beta, \mathcal{T}_{\mathcal{B} \upharpoonright_\beta})$ is exactly $(\alpha, \mathcal{T}_{\mathcal{B} \upharpoonright_\alpha})$.

Assume $\mathcal{B} \upharpoonright_\alpha$, satisfying the above conditions are already defined. We can alter $g(\alpha)$, so that $\bigcup_{n \in g(\alpha)(k)} V_n$ is cofinal in α for any k , without losing its splitting property. Define

$$U_{\alpha+k} = \bigcup_{n \in g(\alpha)(k)} V_n \cup \{\alpha + n\}.$$

We want to argue that if $(\alpha, \mathcal{B} \upharpoonright_\alpha)$ satisfies (1)-(3), then so does $(\alpha + \omega, \mathcal{B} \upharpoonright_{\alpha+\omega})$.

To see (2) consider a cover of U_β by some sets from the subbase $\{U_\gamma^s : \gamma < \alpha + \omega, s \in \{-1, 1\}\}$. Note that we only need to consider $\beta \geq \alpha$. Then (since $\beta \in U_\gamma \implies \gamma = \beta$) we can assume that there is some U_γ^c in the cover. If $\gamma < \alpha$ then U_γ is compact. Otherwise $U_\gamma \cap U_\beta = \emptyset$ since V_n are pairwise disjoint and so are $g(\alpha)(n)$.

For (3) consider $\beta < \alpha$ (for $\beta \geq \alpha$ it is trivial). As the topology on α was not altered, we know that the closure of $[\beta, \beta + \omega)$ contains $[\beta, \alpha)$. Since each set $U_\gamma \cap \alpha$ for $\gamma \geq \alpha$ is cofinal in α , we know that the closure of $[\beta, \alpha)$ is exactly $[\beta, \alpha + \omega)$ and (3) follows.

Since $(\alpha, \mathcal{T}_{\mathcal{B}\upharpoonright\alpha})$ is zero-dimensional, and so regular, then it also metrizable (being a second-countable space).

It remains to see that $(\omega_1, \mathcal{T}_{\mathcal{B}})$ is indeed an Ostaszewski space.

As initial segments are open in $(\omega_1, \mathcal{T}_{\mathcal{B}})$ we know that it is Hausdorff and not compact. Notice that all sets in \mathcal{B} are bounded, so $\mathcal{T}_{\mathcal{B}}$ has a base consisting of sets that are either bounded or co-bounded i.e. such that the component is bounded.

Take a set $D \subseteq \omega_1$ which is not compact. By 1.19, for some limit α we have both $D \cap \alpha$ having no compact closure in $(\alpha, \mathcal{T}_{\mathcal{B}\upharpoonright\alpha})$, and all elements of $g(\alpha)$ splitting $F(\alpha, \mathcal{B}\upharpoonright\alpha, D \cap \alpha)$. This means that for each k we have

$$D \cap \alpha \text{ is split by } \bigcup_{n \in g(\alpha)(k)} V_n.$$

Hence each $U_{\alpha+k}$ splits $D \cap \alpha$ and so $D \cap \alpha$ accumulates at $\alpha + k$. So the closure of $D \cap \alpha$ in $\alpha + \omega$ contains $[\alpha, \alpha + \omega)$. Inductively, from (3), we see that the closure of $D \cap \alpha$ is co-bounded and so is the closure of D .

So any closed set is either co-bounded or compact (and thus bounded). Now every countable open cover of $(\omega_1, \mathcal{T}_{\mathcal{B}})$ has a co-bounded set. Notice that its complement is closed and bounded so, by the observation above, it is also compact. Hence $(\omega_1, \mathcal{T}_{\mathcal{B}})$ is countably compact.

Take A, B closed disjoint sets. At least one of them is bounded – assume it is A . Let α be a bound of A and either B or B^c (whichever is bounded). If B is bounded then in the subspace topology on α , which is exactly $(\alpha, \mathcal{T}_{\mathcal{B}\upharpoonright\alpha})$, there are open sets separating A and B . These sets are still open in $(\omega_1, \mathcal{T}_{\mathcal{B}})$. Now assume that B^c is bounded by α . Let $\beta = \alpha + \omega$. Now A and $B \cap \beta$ can be separated in $(\beta, \mathcal{T}_{\mathcal{B}\upharpoonright\beta})$. Notice that no set in $\mathcal{B}\upharpoonright\beta$ contains $[\alpha, \beta)$. Hence, if $U \supseteq A$ with $U \cap (B \cap \beta) = \emptyset$ is open in $\mathcal{T}_{\mathcal{B}\upharpoonright\beta}$, then U is a union of sets from $\mathcal{B}\upharpoonright\beta$ (without complements of such sets). Since A is bounded, it is also compact in $\mathcal{T}_{\mathcal{B}}$. Thus A can be covered with finitely many sets from $\mathcal{B}\upharpoonright\beta$ which are all disjoint from B . The union of those sets is clopen in ω_1 , thus separating A from B .

□

Given an elementary submodel $M \prec H(\theta)$ for sufficiently large θ , we define $\Delta_M = \sup(\omega_1 \cap M)$.

Fact 1.18. *Given countable $A \subseteq H(\theta)$, the set $\{\Delta_M : A \subseteq M \prec H(\theta), M \text{ is countable}\}$ contains a club.*

Proof. We recursively construct an \prec -chain of submodels of $H(\theta)$ of length ω_1 . Start with any $M_0 \prec H(\theta)$ with $A \subseteq M_0$. Given M_α , let $M_{\alpha+1}$ be any countable elementary submodel of $H(\theta)$ with $M_\alpha \subseteq M_{\alpha+1}$ and $\Delta_{M_\alpha} \in M_{\alpha+1}$. Then $\Delta_{M_{\alpha+1}} > \Delta_{M_\alpha}$ and, inductively, $\Delta_{M_{\alpha+1}} \geq \alpha + 1$. Since $M_\alpha, M_{\alpha+1} \prec H(\theta)$ then $M_\alpha \prec M_{\alpha+1}$. For limit γ , notice that $\langle M_\alpha : \alpha < \gamma \rangle$ forms an \prec -chain, so $M_\gamma = \bigcup_{\alpha < \gamma} M_\alpha$ is an elementary submodel of $H(\theta)$ and $\Delta_{M_\gamma} = \sup\{\Delta_{M_\alpha} : \alpha < \gamma\}$ (and inductively $\Delta_{M_\gamma} \geq \gamma$). The set $\{\Delta_{M_\alpha} : \alpha \in \omega_1\}$ is a club. \square

Fact 1.19. *Let \mathcal{T} be a topology on ω_1 such that any limit α is open in \mathcal{T} . If $D \subseteq \omega_1$ is a closed, non-compact set, then the set of limit $\alpha < \omega_1$, such that $D \cap \alpha$ is not compact in α , contains a club.*

Proof. Notice that the notions of closure, compactness, being limit and a club set are all definable using only ω, ω_1 and \mathcal{T} as parameters, and thus so is the statement of the fact.

Let $M \prec H(\theta)$ for some sufficiently large θ , be a countable model with $\mathcal{T}, D \in M$. It is worth mentioning that, for any countable ordinal $\alpha \in M$, we have $\alpha \subseteq M$. Take a closed set D which is not compact. Then $M \models "D \text{ is not compact}"$. Let $\mathcal{U} = \langle U_\alpha : \alpha < \omega_1 \rangle \in M$ be an open cover of D such, that $M \models " \mathcal{U} \text{ has no finite subcover}"$. Notice that Δ_M is a limit ordinal and $\Delta_M \subseteq M$. Hence $\mathcal{U} \cap M = \langle U_\alpha : \alpha < \Delta_M \rangle$ and $D \cap \Delta_M$ is closed in Δ_M .

Assume, towards contradiction, that $D \cap \Delta_M$ is compact. Take any $x \in D \cap \Delta_M$. Such x is an element of M , so there is $\alpha < \Delta_M$ with $x \in U_\alpha$. Therefore $\langle U_\alpha : \alpha < \Delta_M \rangle$ is an open cover of $D \cap \Delta_M$, and there are $U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}$ covering $D \cap \Delta_M$ (with each $\alpha_i < \Delta_M$). Fix $d \in D \cap M$. Then $d \in D \cap \Delta_M$ and there is i with $d \in U_{\alpha_i}$. By elementarity $M \models d \in U_{\alpha_i}$, so $M \models \forall x \in D \bigvee_{i \leq n} x \in U_{\alpha_i}$ contradicting the definition of \mathcal{U} .

Applying fact 1.18 finishes the proof. \square

2 A continuous image of RN-compact space which is not RN-compact.

This section is heavily based on a paper of the same title by Antonio Avilés and Piotr Koszmider [AK13]. A Banach space X is said to have the Radon-Nikodým property if the Radon-Nikodým theorem holds for X -valued vector measures. It is a crucial property in the theory of vector measures, which are important tools in Banach space theory (see [DU77], or more recently [ÁG23]). Weak* compact subsets of Radon-Nikodým spaces were studied alongside them (ex. in [Ste81]). In [Rei82] the term Radon-Nikodým-compact (or RN-compact) is used to refer to a space homeomorphic to a weak* compact subset of a dual Banach space with the Radon-Nikodým property. In [Nam87], Isaac Namioka provides a characterisation of

RN-compact spaces, which we will use as the definition (see 2.2). A question posed in that same paper stood open up until [AK13].

Is the class of RN-compact spaces closed under continuous images?

Several partial results seemed to indicate a positive answer – the most relevant to us appear in [Avi05], where the author shows that the continuous images of all RN-compact spaces of weight $< \mathfrak{b}$ are also RN-compact. In [AK13], appears an example of an RN-compact space of weight \mathfrak{c} , that has a continuous image which is not RN-compact. In the next two sections we will construct a space of weight ω_1 assuming a \diamond -principle parametrized by a variant of $non(\mathcal{M})$.

2.1 More preliminaries

Definition 2.1. *Let K be a compact space and let $d : K^2 \rightarrow [0, \infty)$ be a metric (not necessarily related to the topology on K). Then*

- (1) d **fragments** K if for every $\varepsilon > 0$ and every closed $F \subseteq K$ there is an open $U \subseteq K$ with $U \cap F \neq \emptyset$ and

$$\text{diam}_d(U \cap F) = \sup\{d(x, y) : x, y \in U \cap F\} < \varepsilon,$$

- (2) d is **Reznichenko** if for any distinct $x, y \in K$ there are open $U \ni x, V \ni y$ with $d(U, V) > 0$,

- (3) d is **lower semicontinuous** (l.s.c.) if for any distinct $x, y \in K$ and $0 < \delta < d(x, y)$ there are open $U \ni x, V \ni y$ such that

$$d(U, V) = \inf\{d(a, b) : a \in U, b \in V\} > \delta.$$

Definition 2.2. *A compact space K is **quasi-RN-compact** if there is a Reznichenko metric $d : K^2 \rightarrow [0, \infty)$ that fragments K (by [Nam02]). K is **RN-compact** if there is a lower semicontinuous metric that fragments K .*

Note few facts about relationship between quasi-RN-compactness, RN-compactness and the problem of continuous images.

Fact 2.3. *Every RN-compact space is quasi-RN-compact.*

Theorem 2.4 ([Arv02]). *A zero-dimensional quasi-RN-compact space is RN-compact.*

Theorem 2.5 ([Arv02]). *A continuous image of an RN-compact space is quasi-RN-compact.*

Recall that the **weight** of a space X (denoted $w(X)$) is the smallest size of its basis. In [Avi05] it is shown that

Theorem 2.6. *If L is a continuous image of an RN-compact space and $w(L) < \mathfrak{b}$ then L is RN-compact.*

Coupled together with the fact that a continuous image of a compact space does not decrease the weight, we have that

Corollary 2.7. *If L_0 is RN-compact, but has a continuous image L_1 that is not RN-compact, then $w(L_0) \geq \mathfrak{b}$.*

Above facts show why the problem of finding an RN-compact space with a continuous image which is not RN-compact proves to be challenging.

Before the construction we will establish some notation.

Let \sqsubseteq be the end-extension of sequences in $2^{<\omega}$. By $i^{(k)}$ denote the sequence of k consecutive i 's.

Definition 2.8. *A set $D \subseteq 2^{<\omega}$ is **dense** if for any $t \in 2^{<\omega}$ there is $s \in D$ with $t \sqsubseteq s$. D is **open** if it is closed under \sqsubseteq . By \mathcal{D} we denote the collection of all open dense subsets of $2^{<\omega}$.*

An open dense set D naturally induces an open dense subset of 2^ω by taking

$$D' = \{\tau \in 2^\omega : \exists n \tau \upharpoonright_n \in D\}.$$

By potentially shrinking D , still to an open dense set, one can additionally assume that, if $s \frown 0, s \frown 1 \in D$, then $s \in D$ (this can be done by removing $s \frown 1^{(k)}$ over all $k \in \omega$, for any problematic s). Notice that, over such sets, the above-mentioned mapping is bijective and

$$s \in D \iff [s] \subseteq D'.$$

For $s, t \in 2^{<\omega}$ define $\Gamma_t^s : 2^\omega \rightarrow 2^\omega$ as

$$\Gamma_t^s(\sigma) = \begin{cases} t \frown (0, 0, \dots), & \text{if } \sigma <_{lex} s \frown (0, 0, \dots), \\ t \frown \lambda, & \text{if } \sigma = s \frown \lambda, \\ t \frown (1, 1, \dots), & \text{if } \sigma >_{lex} s \frown (1, 1, \dots). \end{cases}$$

Furthermore let $q : 2^\omega \rightarrow [0, 1]$ be the standard continuous surjection given by calculating the value of a sequence as a base 2 number, i.e. $q(\tau) = \sum_{i \in x^{-1}(\omega)} 2^{-(i+1)}$.

2.2 Construction

This construction follows steps from [AK13] with slight modifications. The most prominent change occurs in the following definition, which provides a basis for the construction.

Definition 2.9. *A **pre-basic** space is a compact Hausdorff space K of the form $K = \bigcup_{t \in 2^{<\omega}} A_t \cup B \cup \{c\}$ (with A_t, B infinite) where*

(1) All points of $A = \bigcup_{t \in 2^{<\omega}} A_t$ are isolated in K .

(2) For every $x \in B$ there is a countably infinite set $C_x \subseteq A$ such that $\text{cl}(C_x) = C_x \cup \{x\}$ and $\text{cl}(C_x)$ is open in K .

Furthermore, the collection

$$\{\{a\} : a \in A\} \cup \{\{x\} \cup C_x \setminus F : x \in B, F \subseteq A, F \text{ finite}\}$$

forms a basis of $A \cup B$ and $A \cup B \cup \{c\}$ is the one-point-compactification of $A \cup B$.

A pre-basic space is **basic*** if

(3) There is a function $f : B \rightarrow 2^{<\omega 2^{<\omega}}$ such that for every $D : A \rightarrow \mathcal{D}$, we can find $x \in B$ for which the set of t 's satisfying

$$\{a \in A_t \cap C_x : f(x)(t) \in D(a)\} \text{ is infinite}$$

contains infinitely many full levels of $2^{<\omega}$ (i.e. for infinitely many n 's the above is true for each $t \in 2^n$).

The idea behind the condition (3) is to get a witness – x – knowing which elements are "good" for each $a \in A$. For this witness and a specific t , we need a sequence within A_t that converges to x , for each element of which $f(x)(t)$ is "good"; and each sequence of distinct elements of C_x converges to x .

Fact 2.10. For given $A = \bigcup_{t \in 2^{<\omega}} A_t$ and B , the set $\{C_x : x \in B\}$ defines a pre-basic space if and only if the family $\{C_x \cap A_t : x \in B\}$ is almost disjoint.

Proof. First assume that $A \cup B \cup \{c\}$ is pre-basic with the topology defined using C_x 's. Take $x \neq y \in B$ and assume that $C_x \cap C_y$ is infinite. Then there are no basic sets $C_x \setminus F_1$ and $C_y \setminus F_2$ that are disjoint – contradicting the Hausdorff property of K .

Now assume that $\{C_x : x \in B\}$ is almost disjoint. Since K is Hausdorff, it is left to show that $\text{cl}(C_x) = C_x \cup \{x\}$. Take $y \neq x$ from B . Then $C_y \setminus C_x$ is open. Hence $C_x \cup \{x\}$ is closed in $A \cup B$. Any cover of $C_x \cup \{x\}$ contains a superset of $C_x \cup \{x\} \setminus F$ for some finite F . Thus $C_x \cup \{x\}$ is compact in $A \cap B$ and closed in K . \square

In the original paper, a stronger condition – simply called a basic space – is used instead. In such case, the cardinality of the space is at least \mathfrak{c} . In [AK13], it is shown that there is a basic (and thus also a basic*) space in ZFC.

Lemma 2.11. If K is a pre-basic space then $w(K) = |K|$.

Proof. The weight of a space can never be smaller than the weight of any of its subspaces. Since both A and B are discrete (in the subspace topology of A and B respectively) we get

$$w(K) \geq \max(w(A), w(B)) = \max(|A|, |B|) = |K|.$$

On the other hand, if \mathcal{U} is the basis of $A \cup B$ given in the definition, we can see that some subset of $\{U \cup \{c\} : U \in \mathcal{U}\}$ is a basis of neighbourhoods of c . Thus $w(K) \leq |\mathcal{U}| = |K|$. \square

We will now show that, given a basic* space, one can construct an RN-compact space with a non-RN-compact continuous image.

Let $K = A \cup B \cup \{c\}$ be a pre-basic space and let

$$L = (A \times 2^\omega) \cup B \cup \{c\}.$$

If U is a basic open set in 2^ω then we set $\{a\} \times U$ as a basic neighbourhood of (a, σ) in L (for any $\sigma \in U$). Similarly if U is a neighbourhood of $x \in B \cup \{c\}$ in K then $((U \cap A) \times 2^\omega) \cup U \setminus A$ is a neighbourhood of x in L . It is immediate that $w(L) = w(K)$.

If possible, take $f : B \rightarrow 2^{<\omega}$ satisfying the basic* condition for K , else let f be arbitrary. For $x \in B$ define $g_x : (L \setminus \{x\}) \rightarrow 2^\omega$ as follows

- $g_x(y, x) = (0, 0, 0, \dots)$ if $y \notin C_x \times 2^\omega, y \neq x$
- $g_x(a, \sigma) = \Gamma_t^{f(x)(t)}(\sigma)$ for $a \in A_t \cap C_x, \sigma \in 2^\omega$

and let $h_x : (L \setminus \{x\}) \rightarrow [0, 1]$ be defined by $h_x = q \circ g_x$.

Now let

$$\begin{aligned} \mathbb{L}_0 &= \{(u, v) \in L \times (2^\omega)^B : v(x) = g_x(u) \text{ for all } x \in B \setminus \{u\}\}, \\ \mathbb{L}_1 &= \{(u, v) \in L \times [0, 1]^B : v(x) = h_x(u) \text{ for all } x \in B \setminus \{u\}\}. \end{aligned}$$

\mathbb{L}_1 is a continuous image of \mathbb{L}_0 by the function $\pi(u, v) = (u, q(v(x)))_{x \in B}$. The following two facts are crucial to us. The first of them is indirectly proven in [AK13, Proposition 4.1]. It is easy to notice that that proof does not depend on the function f , thus being valid for any pre-basic space.

Proposition 2.12. *If K is a pre-basic space then \mathbb{L}_0 is a RN compact space.*

Proposition 2.13. *If K is a basic* space then \mathbb{L}_1 is not RN compact.*

Proof. Note that \mathbb{L}_1 is compact, as a continuous image of \mathbb{L}_0 . Assume that \mathbb{L}_1 is RN-compact with $\delta : \mathbb{L}_1 \times \mathbb{L}_1 \rightarrow \mathbb{R}$ being a lower semicontinuous submetric fragmenting it. For $a \in A$ and $z \in 2^\omega$ let $[a, z]$ denote a unique point in \mathbb{L}_1 with the first coordinate equal (a, z) .

We now define a function from A to a set of dense subsets of $2^{<\omega}$ that are closed under end-extensions. For $a \in A_t$ let $D(a)$ be the set of $s \in 2^{<\omega}$ such that

$$\text{diam}_\delta([a, [s]]) < \frac{1}{4^{|t|}}. \quad (*)$$

$D(a)$ is clearly closed under taking end-extensions in $2^{<\omega}$. Using fragmentability condition of δ for the set $[a, [\tau]]$ we can find s satisfying (*) and extending τ , hence witnessing density of $D(a)$.

By (3) we get $x \in B$ such that, for t on infinitely many levels, the set

$$\{a \in C_x \cap A_t : f(x)(t) \in D(a)\}$$

is infinite. For such t let $\langle a_n \rangle \subseteq C_x \cap A_t$ be an infinite sequence with $f(x)(t) \in D(a)$. Hence

$$\delta([a_n, f(x)(t) \frown (0, 0, \dots)], [a_n, f(x)(t) \frown (1, 1, \dots)]) < \frac{1}{4^{|t|}}. \quad (**)$$

Let $\tau^i = q(\tau \frown (i, i, \dots)) \in [0, 1]$ and $x \otimes \sigma$ be the only point $(x, \theta) \in \mathbb{L}_1$ with $\theta(x) = \sigma$. We now can see that

$$h_x(a_n, f(x)(t) \frown (i, i, \dots)) = q(\Gamma_t^{f(x)(t)}(f(x)(t) \frown (i, i, \dots))) = t^i,$$

and, since in K we have $a_n \rightarrow x$, we get $[a_n, f(x)(t) \frown (i, i, \dots)] \rightarrow x \otimes t^i$.

By lower semi-continuity of δ we can extend the (**) inequality to limits obtaining

$$\delta(x \otimes t^0, x \otimes t^1) \leq \frac{1}{4^{|t|}}.$$

Consider a level of $2^{<\omega}$ on which the above inequality holds. Since t^1 is equal to s^0 for certain s of length $|t|$, using the triangle inequality for δ we get

$$\delta(x \otimes (0, 0, \dots), x \otimes (1, 1, \dots)) \leq \frac{1}{4^{|t|}}.$$

But, by the property (3), we have infinitely many such levels. Hence $\delta(x \otimes (0, 0, \dots), x \otimes (1, 1, \dots)) = 0$ – a contradiction. □

Let us see that $w(\mathbb{L}_0) = w(\mathbb{L}_1) = |K|$. It is enough to show that $w((2^\omega)^B) = w([0, 1]^B) = |B|$. Typically for a product one considers a basis of the form

$$\left\{ \prod_{i \in I} U_i \times \prod_{i \notin I} X : I \subseteq B \text{ finite}, U_i \in \mathcal{U} \right\}$$

where \mathcal{U} is a basis of X , and X is respectively 2^ω and $[0, 1]$. Since I is finite in this definition, we can assume all U_i 's to be the same ($\bigcap_{i \in I} U_i$ is open) thus obtaining a basis of cardinality $\aleph_0 \cdot |\text{Fin}(B)| = |B|$.

3 Construction of a small basic* space

3.1 First construction – $\text{cof}(\mathcal{M})$

The first construction uses $\diamond(\widetilde{\text{non}}(\mathcal{M}) \times \mathfrak{d})$. It is well known that $\text{non}(\mathcal{M}) \times \mathfrak{d} = \max\{\text{non}(\mathcal{M}), \mathfrak{d}\} = \text{cof}(\mathcal{M})$ (see ex. in [Bla10], originally due to [Mil81]).

Using a fixed bijection, one can assume that the domain of the functions in \mathfrak{d} is $2^{<\omega} \times \omega$.

We also use a variant of $\text{non}(\mathcal{M})$. In [Bla10, Theorem 5.9], it is shown that:

$$\text{non}(\mathcal{M}) = \|\omega^\omega, \omega^\omega, =^\infty\|.$$

Let

$$\widetilde{\text{non}}(\mathcal{M}) = \left((2^{<\omega})^{2^{<\omega}}, (2^{<\omega})^{2^{<\omega}}, \text{equal on infinitely many levels} \right).$$

Lemma 3.1. $\text{non}(\mathcal{M}) = \|\widetilde{\text{non}}(\mathcal{M})\|$.

Proof. For each n a definable bijection $\pi_n : (2^{<\omega})^n \rightarrow \omega$. Furthermore for each n let $\langle s_k^n : k < 2^n \rangle$ be an enumeration of all sequences of length n as ordered by the lexicographic order.

Given $g \in (2^{<\omega})^{2^{<\omega}}$ let $\varphi_1(g)(n) = \pi_n(\langle g(s_k^n) : k < 2^n \rangle)$. Conversely, if $f \in \omega^\omega$, then let $\varphi_2(f)(s_k^n) = \pi_n^{-1}(f(n))_k$.

It is a simple check that φ_1 is the inverse of φ_2 , (φ_1, φ_2) is a morphism from $(\omega^\omega, \omega^\omega, =^\infty)$ to $\widetilde{\text{non}}(\mathcal{M})$ and (φ_2, φ_1) from $\widetilde{\text{non}}(\mathcal{M})$ to $(\omega^\omega, \omega^\omega, =^\infty)$. \square

The space $K = A \cup B \cup \{c\}$ will have B equal to the set of all limit ordinals in $\omega_1 \setminus \{0\}$ and each A_t will be a copy of ω_1 . Recall that we have set a bijection $e_\beta : \omega \rightarrow \beta$ for each infinite $\beta < \omega_1$ and, for $\beta \geq \omega^\omega$ (ordinal), additionally $\varepsilon_\beta : \omega \rightarrow B \cap \beta$.

The construction begins by defining $F = (F_1, F_2) : 2^{<\omega_1} \rightarrow (2^{<\omega})^{2^{<\omega}} \times \omega^\omega$. Choosing a suitable bijection we change the domain of F_1 and F_2 . Let F_1 be the identity function on ω^ω and let F_2 take as arguments a sequence of sets $\langle C_\gamma : \gamma \in B \cap \alpha \rangle$ and a collection of sets $\{G_t : t \in 2^{<\omega}\}$ with $C_\gamma \cap A_t \subseteq \gamma$ and $G_t \subseteq A_t \cap \alpha$ for all $t \in 2^{<\omega}$ and $\gamma \in B \cap \alpha$.

Set F_2 to have a value of the constant zero sequence unless

- (1) $\alpha \in B \setminus \omega^\omega$,
- (2) each G_t is unbounded in $A_t \cap \alpha$.

In such case, put

$$C'_n = C_{\varepsilon_\alpha(n)} \setminus \bigcup_{k < n} C_{\varepsilon_\alpha(k)}.$$

Now F_2 will be the smallest (in the sense of $e_{\varepsilon_\alpha(n)}$) element of $C'_n \cap G_t$. More precisely:

$$F_2(\langle C_\gamma \rangle, \{G_t\})(t, n) = \min\{e_{\varepsilon_\alpha(n)}^{-1}(a) : a \in G_t \cap C'_n\}.$$

Note that $G_t \subseteq A_t$ and A_t is a copy of ω_1 .

Let (g_1, g_2) be the $\diamond(\widetilde{\text{non}}(\mathcal{M}) \times \mathfrak{d})$ -sequence for $F = (F_1, F_2)$ and set $f(\alpha) = g_1(\alpha)$.

Now we recursively construct C_γ 's to satisfy the basic* property. Firstly, for any $\alpha \in B$ that is not a limit of limit ordinals, there is β – the biggest element of $B \cap \alpha$ – set $C_\alpha \cap A_t = \alpha \setminus \beta$.

Assume we have already defined C_β for all $\beta < \alpha$, and that α is a limit of limit ordinals. Define C'_n 's as above. Take C_α to be a collection of all elements of each C'_n that are not above $g(\alpha)(n, t)$ in the sense of $e_{\varepsilon_\alpha(n)}$, i.e. let

$$C_\alpha = \left\{ a \in C'_n \cap A_t : n \in \omega, t \in 2^{<\omega}, |t| \leq n, e_{\varepsilon_\alpha(n)}^{-1}(a) \leq g_2(\alpha)(t, n) \right\}.$$

Clearly C_α is almost disjoint with all the previous C_β 's.

To show the basic* property take $D : A \rightarrow \mathcal{D}$. For each $t \in 2^{<\omega}$ there is $s(t) \in 2^{<\omega}$ that is in uncountably many $D(a)$'s for $a \in A_t$. Let $G_t = \{a \in A_t : s(t) \in D(a)\}$. Consider the constructed sequence $\langle C_\gamma : \gamma \in B \rangle$ together with $\{G_t : t \in 2^{<\omega}\}$. The set of those $\alpha \in B$ for which each G_t is cofinal in α is a club (as a countable intersection of clubs). Hence there is $\alpha \in B$ for which the conditions (1-2) are met, and such that

$$s = F_1(s) \stackrel{\infty \text{ levels}}{=} g_1(\alpha) = f(\alpha)$$

as well as

$$F_2(\langle C_\gamma : \gamma \in B \cap \alpha \rangle, \{G_t \cap \alpha : t \in 2^{<\omega}\}) \leq^* g(\alpha).$$

We can also assume that α is a limit of limit ordinals. Note that $\{C_\beta : \beta \in B \cap \alpha\}$ covers α (by the choice of C_β for β that are not limit of limits).

Fix $t \in 2^{<\omega}$. We want to find infinitely many elements of G_t in C_α . Pick $m > |t|$. Since each $C_\gamma \cap A_t$ is bounded in α then so is $\bigcup_{k \leq m} C'_k \cap A_t$, and hence there is some $a \in G_t \setminus \bigcup_{k \leq m} C'_k$. This a is in $C'_n \cap A_t$ for some $n > m$, and so $g(\alpha)(n, t) \geq e_{\varepsilon_\alpha(n)}^{-1}(a)$ and $a \in C_\alpha$. As the choice of m was arbitrary, we can find infinitely many such n 's and a 's.

In other words, it is true that $\{a \in A_t \cap C_\alpha : s(t) \in D(a)\}$ is infinite for each t . Since $f(\alpha) \stackrel{\infty \text{ levels}}{=} s$, the basic* property is satisfied for D .

This proves the following

Theorem 3.2. *Under $\diamond(\widetilde{\text{non}}(\mathcal{M}) \times \mathfrak{d})$, there is basic* space of size ω_1 .*

Using Propositions 2.12, 2.13, together with Theorem 1.15 for the Sacks forcing, we obtain the following.

Corollary 3.3. *In the Sacks Model there is an RN-compact space of weight ω_1 with a continuous image that is not RN-compact, while $\mathfrak{c} = \omega_2$.*

3.2 Invariants once again

We will now study possible improvements to the invariants used in the construction. Invariants we use are different from classical ones when considered as triples (A, B, \sim) , thus producing diamonds that are formally different, more exotic statements. However, since evaluations of those invariants correlate with classical invariants, those diamonds allow us to reach similar conclusions.

In order to improve \mathfrak{d} , as the invariant encoding the sets, one may look at the following relation, known as the unbounded everywhere relation:

$$\mathfrak{b}^* = (\omega^\omega \times [\omega]^\omega, \omega^\omega, \not\leq^*_\uparrow),$$

where $(f, A) \not\leq^*_\uparrow g$ whenever $f \upharpoonright_A \not\leq^* g \upharpoonright_A$, which can be abbreviated as $f \not\leq^*_A g$.

This can be further extended to

$$(\mathfrak{b}^*)^\omega = ((\omega^\omega \times [\omega]^\omega)^\omega, (\omega^\omega)^\omega, \not\leq^*_\uparrow \text{ on each coordinate}).$$

Note that $(\omega^\omega)^\omega$ can be easily reduced to ω^ω by taking f' with $f'(n) = \max\{f_k(n) : k < n\}$, but the above is simpler to directly insert into the construction.

Fact 3.4. $(\mathfrak{b}^*)^\omega = \mathfrak{b}^* = \mathfrak{b}$.

Proof. Clearly $(\mathfrak{b}^*)^\omega \geq \mathfrak{b}^* \geq \mathfrak{b}$.

For a set $A \subseteq \omega$, let $Nxt_A(n)$ be the second element of A not smaller than n i.e.

$$Nxt_A(n) = \min(A \cap (\min A \setminus n, \infty)).$$

Furthermore, for any increasing function $g \in \omega^\omega$, let $\psi(g) \in \omega^\omega$ be given as

$$\psi(g)(n) = \max g[n, g(n)].$$

Let $\{g_\alpha : \alpha < \mathfrak{b}\}$ be an unbounded family of functions. We can assume that for $\alpha \leq \beta < \mathfrak{b}$ we have $g_\alpha \leq^* g_\beta$. To do this, start with any unbounded family $\{g'_\alpha : \alpha < \mathfrak{b}\}$ and, recursively for $\alpha < \mathfrak{b}$, let g_α be a \leq^* -bound of $\{g_\gamma : \gamma < \alpha\} \cup \{g'_\alpha\}$. Such a bound exists because $\alpha < \mathfrak{b}$. Furthermore we can assume that each g_α is increasing.

Take any $f \in \omega^\omega$ and $A \in [\omega]^\omega$, we can assume that f is increasing. For some α we have $g_\alpha \not\leq^* f$, hence there is $B \in [\omega]^\omega$ such that $g_\alpha \upharpoonright_B \geq f \upharpoonright_B$. Potentially shrinking B , we can assume that between each two elements of B there is an element of A i.e.

$$\forall b \in B [b, \min(B \cap (b, \infty))] \cap A \neq \emptyset.$$

Now let g_γ be such that $g_\gamma \not\leq^* Nxt_B$ and $\gamma > \alpha$. Then $g_\gamma \upharpoonright_B \geq^* f \upharpoonright_B$. Take any n such that $g_\gamma(n) \geq Nxt_B(n)$. There are $b_1 < b_2 \in B$ such that $b_1, b_2 \in [n, g_\gamma(n)]$, hence there is $a \in A \cap [n, g_\gamma(n)]$. For such a , we know that $g_\gamma(a) \geq g_\gamma(n) \geq b_2$ and so $\psi(g_\gamma)(a) \geq g_\gamma(b_2) \geq f(b_2) \geq f(a)$. Thus $\psi(g_\gamma) \not\leq^*_A f$ and $\{\psi(g_\alpha) : \alpha \in \mathfrak{b}\}$ witnesses $\mathfrak{b}^* \leq \mathfrak{b}$.

Let $\{g_\alpha : \alpha < \mathfrak{b}^*\}$ be a witness of \mathfrak{b}^* . Since $\mathfrak{b}^* = \mathfrak{b}$, we can assume that $\{g_\alpha : \alpha < \mathfrak{b}^*\}$ is \leq^* -increasing. Take any $\langle f_n, A_n : n \in \omega \rangle \subseteq \omega^\omega \times [\omega]^\omega$. For each f_n there is α_n so that $f_n \not\leq_{A_n}^* g_{\alpha_n}$. Let $\alpha = \sup_n \alpha_n$. Since $\mathfrak{b} > \omega$ and $cf(\mathfrak{b}) = \mathfrak{b}$ (by [Bla10, Theorem 2.4]) we have $\alpha < \mathfrak{b}$. Hence, $f_n \not\leq_{A_n}^* g_\alpha$ for each n , and $\{(g_\alpha)^\omega : \alpha \in \mathfrak{b}^*\}$ witnesses $(\mathfrak{b}^*)^\omega \leq \mathfrak{b}^*$. \square

Since $\mathfrak{b} \leq non(\mathcal{M})$, we have that $\widetilde{non}(\mathcal{M}) \times (\mathfrak{b}^*)^\omega = non(\mathcal{M})$.

Corollary 3.5. *Under $\diamond(\widetilde{non}(\mathcal{M}) \times (\mathfrak{b}^*)^\omega)$, there is a basic* space of size ω_1 .*

Proof. The changes to the construction above are only minor. In particular F_1 is the same, as $\widetilde{non}(\mathcal{M})$ was unchanged.

Fixing a bijection between ω and $2^{<\omega}$ we can assume that we work with $(\mathfrak{b}^*)^{2^{<\omega}}$. Now F_2 encodes function as previously, but additionally it should also encode a $2^{<\omega}$ indexed collection of subsets of ω . For clarity, we introduce \tilde{F}_2 for this purpose, leaving F_2 as before. Let

$$\tilde{F}_2(\langle C_\gamma \rangle, \{G_t\})(t) = \{n : n \geq |t|, G_t \cap C'_n \neq \emptyset\}$$

if the conditions (1-2) from the previous construction are met, and $\tilde{F}_2(\langle C_\gamma \rangle, \{G_t\})(t) = \omega$ otherwise.

If

$$F_2(\langle C_\gamma \rangle, \{G_t\})(t) \upharpoonright_{\tilde{F}_2(\mathcal{C}, \mathcal{G})(t)} \not\leq^* g(\alpha)(t) \upharpoonright_{\tilde{F}_2(\mathcal{C}, \mathcal{G})(t)},$$

then there are infinitely many $a \in G_t \cap C_\alpha$ and the proof proceeds as before. \square

Definition 3.6. *For any relation $\sim \subseteq A \times B$ define $\overset{\infty \text{ levels}}{\sim} \subseteq A^{2^{<\omega}} \times B^{2^{<\omega}}$ as*

$$f \overset{\infty \text{ levels}}{\sim} g \iff (\exists^\infty n \in \omega \forall t \in 2^n) f(t) \sim g(t).$$

First improvement over $non(\mathcal{M})$ that comes to mind is the invariant

$$\mathfrak{b}_{2^{<\omega}} = \left((2^{<\omega})^{2^{<\omega}}, (2^{<\omega})^{2^{<\omega}}, \overset{\infty \text{ levels}}{\sqsubseteq} \right).$$

However, one can see that

Fact 3.7. $\mathfrak{b}_{2^{<\omega}} = \|(2^{<\omega})^\omega, (2^{<\omega})^\omega, \sqsubseteq^\infty\| = non(\mathcal{M})$.

Proof. Notice that $\overset{\infty \text{ levels}}{\sqsubseteq}$ is weaker than $\overset{\infty \text{ levels}}{=}$ and, by Lemma 3.1, $non(\mathcal{M}) = \|\widetilde{non}(\mathcal{M})\| = \|(2^{<\omega})^{2^{<\omega}}, (2^{<\omega})^{2^{<\omega}}, \overset{\infty \text{ levels}}{=}\|$. Hence $\mathfrak{b}_{2^{<\omega}} \leq non(\mathcal{M})$.

As $\mathfrak{b}_{2^{<\omega}} \geq \|(2^{<\omega})^\omega, (2^{<\omega})^\omega, \sqsubseteq^\infty\|$ is immediate, we only have to show that

$$\|(2^{<\omega})^\omega, (2^{<\omega})^\omega, \sqsubseteq^\infty\| \geq non(\mathcal{M}).$$

Let $\Phi : \omega^\omega \rightarrow (2^{<\omega})^\omega$ be given by $\Phi(f)(n) = 1^{(f(n)) \frown 0}$ (where $c^{(k)}$ is the constant sequence of length k and value c). Φ can be reversed – let $\Psi(f)(n) = k$ if $f(n) = 1^{(k)} \frown 0 \frown t$ for some $t \in 2^{<\omega}$ or $\Psi(f)(n) = 0$ otherwise. Now if $\Phi(f) \sqsubseteq^\infty g$ then $f =^\infty \Psi(g)$. \square

Fact 3.8. For any $D : \omega_1 \rightarrow \mathcal{D}$ the set $D_{\omega_1} = \{t \in 2^{<\omega} : \{a \in \omega_1 : t \in D(a)\} \text{ is bounded in } \omega_1\}$ is open and dense.

Proof. Let $G_t = \{a \in \omega_1 : t \in D(a)\}$. Pick $s \in 2^{<\omega}$. Since the set $[s] \subseteq 2^{<\omega}$ is countable, and for each a we have $[s] \cap D(a) \neq \emptyset$ there has to be some $t \sqsupseteq s$ that repeats uncountably many times, witnessing density of D_{ω_1} . Being open is then immediate, by the properties of $D(a)$. \square

With the above fact in mind one can improve the above construction using the following invariant:

$$\mathfrak{b}_{\mathcal{D}} = \left(\mathcal{D}^{2^{<\omega}}, (2^{<\omega})^{2^{<\omega}}, \overset{\infty \text{ levels}}{\exists} \right).$$

This invariant is minimal in the following sense

Fact 3.9. Let $K = A \cup B \cup \{c\}$ be a basic* space. Then $|B| \geq \mathfrak{b}_{\mathcal{D}}$.

Proof. Assume that is not the case. Then there is $\langle D_t : t \in 2^{<\omega} \rangle \subseteq \mathcal{D}$ so that for no $x \in B$ the function $f(x)$ meets $\langle D_t \rangle$ on infinitely many levels of $2^{<\omega}$. Let $D : A \rightarrow \mathcal{D}$ be defined as

$$D(a) = D_t \text{ for } a \in A_t.$$

Then the basic* property is not met for D . \square

Using the fact that finite intersections of open dense sets are open dense, it is not hard to see that

$$\mathfrak{b}_{\mathcal{D}} = \|\mathcal{D}^\omega, (2^{<\omega})^\omega, \exists^\infty\|.$$

Also, one can see that $\mathfrak{b} \leq \mathfrak{b}_{\mathcal{D}} \leq \|(2^{<\omega})^\omega, (2^{<\omega})^\omega, \sqsubseteq^\infty\| = \text{non}(\mathcal{M})$. This leaves the following question:

Question 3.10. Is $\mathfrak{b}_{\mathcal{D}} < \text{non}(\mathcal{M})$ consistent? In particular does $\mathfrak{b}_{\mathcal{D}} = \mathfrak{b}$?

3.3 Final construction – $\text{non}(\mathcal{M})$ or less

Theorem 3.11. Under $\diamond(\mathfrak{b}_{\mathcal{D}} \times (\mathfrak{b}^*)^\omega)$, there is a basic* space of size ω_1 .

Proof. We again alter the proof of 3.2. As before, take $F = (F_1, F_2)$ with the addition of \tilde{F}_2 (as in 3.5). The functions F_2, \tilde{F}_2 will take as arguments $\langle C_\gamma : \gamma \in B \cap \alpha \rangle$ and $\{G_{s,t} : s, t \in 2^{<\omega}\}$ with $C_\gamma \cap A_t \subseteq \gamma$ and $G_{s,t} \subseteq A_t \cap \alpha$ for each $s, t \in 2^{<\omega}$ and $\gamma \in B \cap \alpha$. Using a fixed bijection, consider $(\mathfrak{b}^*)^{2^{<\omega} \times 2^{<\omega}}$ instead of $(\mathfrak{b}^*)^\omega$.

For $\alpha \notin B \setminus \omega^\omega$ we set F_2 to be constant 0, and \tilde{F}_2 to constant ω . Define $\langle C'_n : n \in \omega \rangle$ as before. Now, if $G_{s,t}$ is unbounded, set

$$F_2(\langle C_\gamma \rangle, \{G_{s,t}\})(s, t)(n) = \min\{e_{\varepsilon_\alpha(n)}^{-1}(a) : a \in G_{s,t} \cap C'_n\},$$

and

$$\tilde{F}_2(\langle C_\gamma \rangle, \{G_{s,t}\})(s, t) = \{n : n \geq |t|, G_{s,t} \cap C'_n \neq \emptyset\}.$$

F_1 remains the identity function, but now on $\mathcal{D}^{2^{<\omega}}$.

Constructing the space, we still set $f(\alpha) = g_1(\alpha)$ but now

$$C_\alpha = \{a \in C'_n \cap A_t : n \in \omega, t \in 2^{<\omega}, |t| \leq n, e_{\varepsilon_\alpha(n)}^{-1}(a) \leq g_2(\alpha)(f(\alpha)(t), t)(n)\}.$$

Given $D : A \rightarrow \mathcal{D}$, for each $t \in 2^{<\omega}$ define

$$d(t) = \{s \in 2^{<\omega} : \{a \in A_t : s \in D(a)\} \text{ is uncountable}\}$$

with $d(t) \in \mathcal{D}$ (by 3.8). Furthermore, for $s \in d(t)$ let

$$G_{s,t} = \{a \in A_t : s \in D(a)\}$$

and set $G_{s,t} = \omega_1$ otherwise. $\langle C_\gamma : \gamma \in B \rangle$ is defined as in the previous construction.

Then there is a limit α such that all $G_{s,t}$ are unbounded in respective $A_t \cap \alpha$, and such that:

- (a) $g_1(\alpha) = f(\alpha) \overset{\infty \text{ levels}}{\in} d$
- (b) for each $s, t \in 2^{<\omega}$ and $A = \{n : n \geq |t|, G_{s,t} \cap C'_n \neq \emptyset\}$ we have infinitely many $n \in \omega$ such that:

$$g_2(\alpha)(s, t)(n) \geq \min\{e_{\varepsilon_\alpha(n)}^{-1}(a) : a \in G_{s,t} \cap C'_n\}.$$

Note that $g_1(\alpha) \in (2^{<\omega})^{2^{<\omega}}$.

Take t so that $f(\alpha)(t) \in d(t)$. Then (b) is true for $s = f(\alpha)(t)$ and

$$\{n : G_{f(\alpha)(t), t} \cap C'_n \cap C_\alpha \neq \emptyset\}$$

is infinite and so is $\{a \in A_t \cap C_\alpha : f(\alpha)(t) \in D(a)\}$.

□

Finally we arrive at the following conclusions:

Corollary 3.12. *Under $\diamond(\mathfrak{b}_{\mathcal{D}} \times (\mathfrak{b}^*)^\omega)$, there is an RN-compact space of weight ω_1 with a continuous image which is not RN-compact.*

Corollary 3.13. *Let \mathbb{P} be a Borel, proper forcing with $\mathbb{P} \cong \mathcal{P}(2)^+ \times \mathbb{P}$ and let \mathbb{P}_{ω_2} denote the countable support iteration of \mathbb{P} of length ω_2 . If $V^{\mathbb{P}_{\omega_2}} \models \text{non}(\mathcal{M}) = \omega_1$, then in $V^{\mathbb{P}_{\omega_2}}$ there is an RN-compact space of weight ω_1 with a continuous image that is not RN-compact.*

The Miller model is obtained by taking a countable support iteration of the Miller forcing over a model of CH. Using well known facts about this model and the Cohen forcing (see [Bla10]), one can obtain the following corollary:

Corollary 3.14. *In the Miller model, as well as in the model obtained by taking a countable support iteration of the Cohen forcing over a model of CH, there is an RN-compact space of weight ω_1 with a continuous image which is not RN-compact, while $\text{cof}(\mathcal{M}) = \mathfrak{d} = \mathfrak{c} = \omega_2$.*

Let \mathcal{C}_λ be the Cohen algebra of length λ . The following theorem from [MHD04] might inspire us to conjecture further.

Theorem 3.15. \mathcal{C}_{ω_1} forces $\diamond(\text{non}(\mathcal{M}))$.

Tracing the proof, but for ω_2 and $\diamond(\widetilde{\text{non}}(\mathcal{M}) \times (\mathfrak{b}^*)^\omega)$, one should be able to show that:

Conjecture 3.16. \mathcal{C}_{ω_2} forces $\diamond(\widetilde{\text{non}}(\mathcal{M}) \times (\mathfrak{b}^*)^\omega)$. In particular, in the Cohen model – obtained by forcing with \mathcal{C}_{ω_2} over a model of CH – there is an RN-compact space of weight ω_1 with a continuous image that is not RN-compact, while $\text{cof}(\mathcal{M}) = \mathfrak{d} = \mathfrak{c} = \omega_2$.

In the light of the fact that the weight of an RN-compact space with a non-RN-compact continuous image is no less than \mathfrak{b} , the following question leaves little room for answers:

Question 3.17. *Consistently, is there an RN-compact space of weight smaller than $\text{non}(\mathcal{M})$ with a continuous image which is not RN-compact?*

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