On Efimov spaces and Radon measures

Mirna Džamonja * & Grzegorz Plebanek †

Abstract

We give a construction under CH of an infinite Hausdorff compact space having no converging sequences and carrying no Radon measure of uncountable type. Under $\diamondsuit$ we obtain another example of a compact space with no convergent sequences, which in addition has the stronger property that every nonatomic Radon measure on it is uniformly regular. This example refutes a conjecture of Mercourakis from 1996 stating that if every measure on a compact space K is uniformly regular then K is necessarily sequentially compact.

1. Introduction.

Efimov's problem is a long standing open question asking if every infinite compact Hausdorff space contains either a nontrivial converging sequence or a copy of $\beta\omega$ (note that a compact space K contains $\beta\omega$ if and only if K can be continuously mapped onto $[0,1]^{\mathfrak{c}}$). Nyikos [18] gives an account of the present status of this and related questions and many further references. As in Dow [4], we shall say that an infinite compact space K is an *Efimov space* if K neither contains a converging sequence nor a copy of $\beta\omega$ (here by a space we mean an infinite Hausdorff space; a converging sequence is a sequence $(x_n)_n$ consisting of distinct points and converging to a point in the space).

Efimov spaces do exist under various set-theoretic assumptions; this was first shown by Fedorčuk [8] and later by various authors including Talagrand [23] and Džamonja and Kunen [6], and more recently Dow [4] and Brech [3]. Fedorčuk's construction was done under $\diamondsuit$ and later generalised to work under the so called Partition Hypothesis, see [4] for more discussion. Dow [4] constructed an Efimov space assuming $2^{\mathfrak{s}} < 2^{\mathfrak{c}}$ and $\mathrm{cof}([\mathfrak{s}]^\omega, \subseteq) = \mathfrak{s}$, where $\mathfrak{s}$ is the splitting number. These assumptions are weaker than

*School of Mathematics, University of East Anglia, Norwich NR47TJ, UK, h020@uea.ac.uk

†Institute of Mathematics, Wrocław University, 2/4 pl. Grunwaldzki, 50–384 Wrocław, Poland, grzes@math.uni.wroc.pl

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the Partition Hypothesis. Dow & Fremlin [5] show that there is an Efimov space in the random real models.

Still in connection with Fedorčuk's construction, Džamonja and Kunen [6] also used $\diamondsuit$ and obtained an Efimov space with various additional properties, for example it carries a nonseparable Radon probability measure and, like the original space of Fedorčuk, it is an S-space. Talagrand's construction uses only CH. It is quite involved but it gives several additional properties of the underlying Banach space $C(K)$, in particular the space $C(K)$ has the property of Grothendieck (this means that every weak* convergent sequence of Radon measures on K is in fact weakly convergent). The space constructed by Brech [3] is also a Grothendieck space. By a result of Haydon in [12] this implies that the space carries a nonseparable Radon measure.

It is not known if the statement that there are no Efimov spaces is consistent. Although this problem seems to be very hard and interesting and has become what many would now call 'the Efimov problem', there is another, perhaps more natural interpretation of the original question of Efimov. Namely, are there two properties S and L such that S corresponds to 'small' infinite compact spaces, such as $\omega+1$, L corresponds to 'large' ones, such as $[0,1]^{\mathfrak{c}}$, and every infinite compact space has at least one of these properties? In this paper we suggest a reasonable possibility for the pair (S,L), where S is still 'having a convergent sequence' and L is strictly weaker than being able to map onto $[0,1]^{\mathfrak{c}}$, and we show that CH violates even this weaker dichotomy:

Every compact space either contains a converging sequence or carries a Radon measure of uncountable Maharam type.

This is indeed a weakening of Efimov's dichotomy, as for any infinite cardinal κ, if there is a continuous surjection from K onto $[0,1]^\kappa$ then K carries a Radon measure of type κ. For $\kappa=\aleph_1$ the converse is not true under CH, as is shown by Talagrand's space [23] and an example in Džamonja-Kunen [6]. (In fact, the connection between mapping onto $[0,1]^\kappa$ and carrying a Radon measure of type κ was the subject of the well known Haydon problem, see Fremlin [9] and Plebanek [20] for the answers and further references).

We show that the suggested dichotomy is still not true under CH by giving a construction of an infinite compact space with no nontrivial convergent sequences in which every Radon measure is separable. We then give another example, constructed under $\diamondsuit$ which shows that even a weaker interpretation of L does not suffice. Namely, a stronger property of a measure than just being separable is that it is uniformly regular (uniform regularity implies separability and not vice versa, see §2 for definitions). Hence one might attempt to define L as the property 'having a Radon measure that is not uniformly regular'. Our second example shows that we still cannot expect a ZFC dichotomy with this interpretation of L. This example also strongly refutes a conjecture by Mercourakis [17], stating that if every measure on a compact space K is uniformly regular then K is necessarily sequentially compact. In fact, we show that any compact space which is constructed from the Cantor space in an inverse limit of length ω_1 using simple extensions will only support uniformly regular measures. Then the construction has to assure that there are also no convergent sequences. It is well known how to obtain such constructions under $\diamondsuit$, see [4] for a discussion. For completeness we shall sketch here a construction based on [6].

There remains the question of the necessity of CH or $\diamondsuit$ to refute these weakened Efimov dichotomies. Haydon [10] constructed in ZFC a compactification K of ω such that K admits only Radon measures of countable type but ω has no converging subsequence. This space however has other converging sequences. Dow's Fedorčuk's-style [4] does not map onto $[0,1]^{\mathfrak{c}}$ because its cardinality is smaller than $2^{\mathfrak{c}}$. However $\mathfrak{c} > \omega_1$ in this model (as $2^{\mathfrak{s}} < 2^{\mathfrak{c}}$ and $\mathfrak{s} \geq \omega_1$) and nothing in the construction prevents the space from mapping onto $[0,1]^{\omega_1}$ and certainly not from carrying a nonseparable measure.

2. Preliminaries.

If K is a compact space then we write $P(K)$ for the set of all probability Radon measures on K. We say that $\mu \in P(K)$ is *of countable type* or *separable* if its measure algebra is separable in the Frechet–Nikodym metric. Recall that $P(K)$ is itself a compact space when equipped with the *weak** topology inherited from $C(K)^*$.

Definition 2.1 *A Radon measure μ defined on a compact space K is called* uniformly regular *if there is a continuous surjection g from K onto a compact metric space such that $\mu(g^{-1}(g(F))) = \mu(F)$ for every compact $F \subseteq K$.*

Uniformly regular measures are also called strongly countably determined, see Pol [22].

Given two families $\mathcal{A}$ and $\mathcal{B}$ of μ–measurable sets, we shall say that $\mathcal{A}$ *approximates* $\mathcal{B}$ *from below* if for every $\varepsilon > 0$ and every $B \in \mathcal{B}$ there exists $A \in \mathcal{A}$ such that $A \subseteq B$ and $\mu(B \setminus A) < \varepsilon$. We recall a standard lemma concerning uniformly regular measures (see Babiker [1]).

Lemma 2.2 *(1) The following are equivalent for a Radon measure μ defined on a compact space K.*

(a) μ is uniformly regular;

(b) there is a countable family of zero subsets of K approximating all open sets from below;

(c) there is a countable family of cozero subsets of K approximating all open sets from below.

(2) If K is compact and zerodimensional then $\mu \in P(K)$ is uniformly regular if and only if there is a countable family of clopen sets which approximates all clopen sets from below.

For the sake of completeness we mention some further facts concerning uniform regularity, all of which are standard and can be easily checked.

i. Every uniformly regular measure is of countable type and has a separable support.

ii. Every Radon measure on a metrizable K is uniformly regular.

iii. If λ is the Lebesgue measure on $[0,1]$ then the corresponding measure $\widehat{\lambda}$ on the Stone space of the measure algebra of λ has countable type but is not uniformly regular.

iv. If $x \in K$ then the Dirac measure δ_x is uniformly regular if and only if x is a G_δ point in K.

v. If $\mu \in P(K)$ is uniformly regular then μ is a G_δ point in $P(K)$ (see Pol [22]).

vi. It is relatively consistent that if K is a first-countable compact space then every $\mu \in P(K)$ is uniformly regular, see Plebanek [21]; it is an open problem posed by Fremlin if this holds under $MA + \neg CH$. On the other hand, under CH there is a compact first–countable space carrying a measure of uncountable type, so a measure which is not uniformly regular, see Haydon [11] and Kunen [16]; cf. Džamonja & Kunen [6].

vii. If $\mu \in P(K)$ is uniformly regular then μ has a uniformly distributed sequence $(x_n)_n$, see Mercourakis [17].

Recall that for a given $\mu \in P(K)$, a sequence $(x_n)_n \subseteq K$ is said to be *uniformly distributed* (with respect to μ) if for every real–valued continuous function f defined on K one has

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(x_k) = \int_K f \, \mathrm{d}\mu$$

in the topology of $P(K)$. In the zerodimensional case this simply means that

$$\frac{|\{k \le n : \ x_k \in C\}|}{n} \to \mu(C),$$

for every clopen set $C \subseteq K$.

Finally, let us also give some notation on compact subspaces of 2^α. We shall use the usual identification of the compact space 2^α with the space of functions ${}^\alpha 2$, for α an ordinal. For $\alpha \le \beta$ the natural projection π^β_α is the function $\pi^\beta_\alpha : {}^\beta 2 \to {}^\alpha 2$ defined by $\pi^\beta_\alpha(f) = f \restriction \alpha$. If β is a limit ordinal and for some $\gamma < \beta$ we are given a sequence $\bar{X} = \langle X_\alpha : \ \alpha \in [\gamma, \beta)\rangle$ satisfying $X_\alpha \subseteq 2^\alpha$ for each α, then the *inverse limit* of $\bar{X}$ is the set $\{f \in 2^\beta : \ (\forall \alpha \in [\gamma,\beta) f \restriction \alpha \in X_\alpha\}$, taken in the subspace topology. This is a specific instance of an inverse limit construction, in which the so called 'bonding maps' are the natural projections. For a general theory of inverse limits one may consult e.g. Engelking's book [7].

A set $U \subseteq 2^\alpha$ is said to be *determined by the coordinates in* $I \subseteq \alpha$ if for all $f \in X^\alpha$ and $g \in U$, if $f \restriction I = g \restriction I$ then $f \in U$. Hence clopen sets are determined by finitely many coordinates.

3. Measures on Boolean algebras and the CH example.

In this section we give the first of our two examples and so prove the following theorem:

Theorem 3.1 *Assuming CH there is an infinite compact zerodimensional space K such that*

(a) K contains no nontrivial converging sequences;

(b) every measure $\mu \in P(K)$ is of countable type.

Before we give the proof we shall need some facts about measures on Boolean algebras. Let $\mathfrak{A}$ be an algebra of subsets of some set T. We denote by $\mathrm{ba}_+(\mathfrak{A})$ the set of all nonnegative finite finitely additive measures on $\mathfrak{A}$. In this section we call elements of $\mathrm{ba}_+(\mathfrak{A})$ simply measures. For a given measure $\mu \in \mathrm{ba}_+(\mathfrak{A})$ and any $X \subseteq T$ we write

$$\mu^*(X) = \inf\{\mu(A) : \ X \subseteq A \ \& \ A \in \mathfrak{A}\}.$$

If $\mathfrak{A}$ is contained in some bigger algebra $\mathfrak{B}$ and $\nu \in \mathrm{ba}_+(\mathfrak{B})$ is an extension of $\mu \in \mathrm{ba}_+(\mathfrak{A})$ then $\nu(B) \le \mu^*(B)$ for every $B \in \mathfrak{B}$. We say that $\mathfrak{A}$ is ν*–dense* in $\mathfrak{B}$ if for every $B \in \mathfrak{B}$

$$\inf\{\nu(B \bigtriangleup A) : \ A \in \mathfrak{A}\} = 0.$$

Note that for $\mathfrak{A}$ to be ν-dense in $\mathfrak{B}$ it is sufficient that the above equation holds for all $B \in \mathcal{G}$ for some $\mathcal{G}$ which generates $\mathfrak{B}$.

It is well–known that for every $\mu \in \mathrm{ba}_+(\mathfrak{A})$ and any $\mathfrak{B} \supseteq \mathfrak{A}$, μ admits an extension to $\nu \in \mathrm{ba}_+(\mathfrak{B})$ such that $\mathfrak{A}$ is ν–dense in $\mathfrak{B}$. Indeed, if $\mathfrak{B}$ is generated by $\mathfrak{A}$ and one additional element b then we can write down a formula for ν satisfying $\nu(b) = \mu^*(b)$, which implies denseness by the previous remark; general case follows by transfinite induction. The reader may consult Plachky [19] for more details.

Definition 3.2 *A measure $\mu \in \mathrm{ba}_+(\mathfrak{A})$ is said to be nonatomic if for every $\varepsilon > 0$ the set T is the union of finitely many elements of $\mathfrak{A}$ each of μ-measure $< \varepsilon$.*

We say that μ is separable if there is a countable subalgebra $\mathfrak{A}_0 \subseteq \mathfrak{A}$ which is μ–dense in μ.

Lemma 3.3 *Suppose that $\mathfrak{A} \subseteq \mathfrak{B}$ are algebras of sets and $\mathfrak{A}$ is countable. Further suppose that μ is a measure on $\mathfrak{B}$ such that $\mathfrak{A}$ is μ-dense. Then the measure ν that μ induces on the Stone space of $\mathfrak{B}$ is separable.*

If moreover $\mathfrak{A}$ approximates $\mathfrak{B}$ from below (with respect to μ) then the measure ν that μ induces on the Stone space of $\mathfrak{B}$ is uniformly regular.

Proof. By the definition of $\mathfrak{A}$ being dense in μ, the measure algebra of ν has a dense subalgebra generated by the restriction of ν to the countable algebra $\mathfrak{A}$, so ν must be separable. In the second case $\mathfrak{A}$ gives a countable family of clopen sets in the Stone spaces of $\mathfrak{A}$ and $\mathfrak{B}$ and this family witnesses that ν is uniformly regular. ★

Lemma 3.4 *Let $\mathfrak{A}$ be an algebra such that $|\mathrm{ba}_+(\mathfrak{A})| \leq \mathfrak{c}$. Fix $\mu \in \mathrm{ba}_+(\mathfrak{A})$ and suppose that $\mathfrak{B}$ is the algebra generated by $\mathfrak{A}$ and some family $\{B_\xi : \xi < \mathfrak{c}\}$, such that $B_\xi \cap B_\eta \in \mathfrak{A}$ and $\mu(B_\xi \cap B_\eta) = 0$ for all $\xi \neq \eta$. Then μ has at most $\mathfrak{c}$ extensions to a measure on $\mathfrak{B}$.*

Proof. Note that if $\nu \in \mathrm{ba}_+(\mathfrak{B})$ extends μ then the set $I = \{\xi < \mathfrak{c} : \nu(B_\xi) > 0\}$ is countable. For a fixed $\xi \in I$ there are only $\mathfrak{c}$ possibilities to define ν on the family $\{A \cap B_\xi : A \in \mathfrak{A}\}$, so the lemma follows. ★

The following is the main lemma of the section.

Extension Lemma 3.5 *Let $\mathfrak{A}$ be an algebra of subsets of an infinite set T such that $|\mathrm{ba}_+(\mathfrak{A})| \leq \mathfrak{c}$ and every $\mu \in \mathrm{ba}_+(\mathfrak{A})$ is separable. Suppose that we are given*

- *(a) a family $\{\mu_k : k < \omega\}$, where every μ_k is a nonatomic measure defined on some subalgebra of $\mathfrak{A}$;*
- *(b) a subalgebra $\mathfrak{A}_0$ of $\mathfrak{A}$ and a sequence $(\mathcal{H}_n)_n$ of distinct ultrafilters from $\mathrm{ULT}(\mathfrak{A}_0)$ converging to $\mathcal{H} \in \mathrm{ULT}(\mathfrak{A}_0)$.*

Then there is an algebra $\mathfrak{B} \supseteq \mathfrak{A}$ with the following properties

- *(i) $\mathfrak{A}$ is μ'_k-dense in $\mathfrak{B}$ for every k and every $\mu'_k \in \mathrm{ba}_+(\mathfrak{A})$ extending μ_k;*
- *(ii) if $\mathcal{H}'_n \in \mathrm{ULT}(\mathfrak{B})$ extends $\mathcal{H}_n$ for every n then the sequence $(\mathcal{H}'_n)_n$ is not converging;*
- *(iii) $|\mathrm{ba}_+(\mathfrak{B})| \leq \mathfrak{c}$ and every $\nu \in \mathrm{ba}_+(\mathfrak{B})$ is separable.*

Proof. We can choose a decreasing sequence $F_n \in \mathcal{H}$ such that $F_n \in \mathcal{H}_n$ and $F_{n+1} \notin \mathcal{H}_n$ for every n; this follows easily since the sequence $(\mathcal{H}_n)_n$ is converging (passing to a subsequence if necessary).

Let $R_n = F_n \setminus F_{n+1}$ for every n, so $R_n \in \mathcal{H}_n$ and the sets R_n's are pairwise disjoint. Let us also fix a set $N \subseteq \omega$ such that both N and $\omega \setminus N$ are infinite.

Let Γ be a family of $\mathfrak{c}$ many increasing functions $g : \omega \to N$ such that

- whenever $g_1, g_2 \in \Gamma$, $g_1 \neq g_2$ then $g_1[\omega]$ and $g_2[\omega]$ are almost disjoint;

- for any $h : \omega \to N$ there is $g \in \Gamma$ such that $h(n) \le g(n)$ for almost all n.

Note that such a family Γ can be easily defined from an almost disjoint family in $[N]^\omega$ of size $\mathfrak{c}$. Namely let $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ be an almost disjoint family in $[N]^\omega$, and let $\{f_\alpha : \alpha < \mathfrak{c}\}$ be a dominating family in ${}^\omega N$. Let $g_\alpha(n) = \min(A_\alpha \setminus \max\{f_\alpha(n), g_\alpha(m) : m < n\})$.

Since every μ_k is nonatomic we can find for every n a finite partition $\mathcal{P}_n$ of T into elements of $\mathfrak{A}$ such that

- $\mu_k^*(A) \le \frac{1}{n+1}$ whenever $k \le n$ and $A \in \mathcal{P}_n$;
- $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ for every n.

Denote by Φ the set of all functions $\varphi : \omega \to \mathfrak{A}$ such that $\varphi(n) \in \mathcal{P}_n$ and $\varphi(n+1) \subseteq \varphi(n)$ for every n. For any $\varphi \in \Phi$ and $g \in \Gamma$ we put

$$T(\varphi, g) = \bigcup_{n<\omega} [R_{g(n)} \cap \varphi(n)].$$

We shall now check that the algebra $\mathfrak{B}$ generated by $\mathfrak{A}$ and all the sets $T(\varphi, g)$, $\varphi \in \Phi$, $g \in \Gamma$ has the required properties.

To verify (i) fix k and an extension μ_k' of μ_k. For a given $\varphi \in \Phi$, $g \in \Gamma$ and $n \ge k$ we have $\mu_k'(\varphi(n)) \le \mu_k^*(\varphi(n)) \le 1/(n+1)$ and $T(\varphi, g) \setminus \varphi(n) \in \mathfrak{A}$ which implies that $\mathfrak{A}$ approximates $T(\varphi, g)$; hence $\mathfrak{A}$ is μ_k–dense in $\mathfrak{B}$.

For (ii), let $\mathcal{H}_n' \in \mathrm{ULT}(\mathfrak{B})$ extend $\mathcal{H}_n$ for every n. Fix $\mathcal{H}' \in \mathrm{ULT}(\mathfrak{B})$ extending $\mathcal{H}$; then there is $\varphi \in \Phi$ such that $\varphi(n) \in \mathcal{H}'$ for all n. If we suppose that $\mathcal{H}_n'$ converge to $\mathcal{H}'$ then for every n there is $h(n)$ such that for all $k \ge h(n)$ we have $\varphi(n) \in \mathcal{H}_k'$. For such a function h there is $g \in \Gamma$ that eventually dominates h. But if $g(n) \ge h(n)$ then $\varphi(n) \in \mathcal{H}_{g(n)}'$; since also $R_{g(n)} \in \mathcal{H}_{g(n)}'$ we see that $T(\varphi, g)$ is an element of almost all $\mathcal{H}_{g(n)}'$. On the other hand, $T(\varphi, g) \notin \mathcal{H}_k'$ whenever $k \in \omega \setminus N$, because for such k we have $R_k \in \mathcal{H}_k \subseteq \mathcal{H}_k'$ and

$$R_k \cap T(\varphi, g) \subseteq R_k \cap \bigcup_{n<\omega} R_{g(n)} = \emptyset.$$

To verify (iii) consider first $\mu \in \mathrm{ba}_+(\mathfrak{A})$ with

$$(*) \qquad \inf\{\mu(F_n) : n < \omega\} = 0.$$

Suppose that $\nu \in \mathrm{ba}_+(\mathfrak{B})$ is an extension of μ. Since $\mu(F_n) \to 0$ we must have that $\nu((T(\varphi, g)) = \lim_{k\to\infty} \mu(\bigcup_{n<k} R_{g(n)} \cap \varphi(n))$, which is uniquely determined by μ. Hence there is only one extension of μ to $\nu \in \mathrm{ba}_+(\mathfrak{B})$ and ν is clearly separable.

Let now μ satisfy

$$(**) \qquad \inf\{\mu(F_n) : n < \omega\} = \mu(T),$$

and consider an extension ν of μ with $\nu \in \mathrm{ba}_+(\mathfrak{B})$. Since for all n we have $\mu(F_n) = \mu(T)$, then for any n we have $\mu(R_n) = \mu(F_n \setminus F_{n+1}) = 0$. It follows that $\{T(\varphi, g) : (\varphi, g) \in \Phi \times \Gamma\}$ is a ν-pairwise disjoint family. Indeed, if $(\varphi_1, g_1) \neq (\varphi_2, g_2)$ then the set $T(\varphi_1, g_1) \cap T(\varphi_2, g_2)$ is contained in a union of finite number of R_n's, since

either $g_1 \neq g_2$ and g_1, g_2 has almost disjoint ranges, or

$\varphi_1 \neq \varphi_2$ which means that $\varphi_1(n) \cap \varphi_2(n) = \emptyset$ for almost all n.

By Lemma 3.4, there can only be $\mathfrak{c}$ extensions of μ to a measure on $\mathfrak{B}$. Coming back to a fixed extension ν, we see that at most countably many of the sets $T(\varphi, g)$ can have positive measure. Hence the measure algebra induced by $(\mathfrak{B}, \nu)$ is separable, because the one induced by $(\mathfrak{A}, \mu)$ is.

In the general case, any $\mu \in \mathrm{ba}_+(\mathfrak{A})$ can be written as $\mu = \mu_1 + \mu_2$, where μ_1 satisfies $(*)$ and μ_2 satisfies $(**)$, so every extension of μ onto $\mathfrak{B}$ is separable. By Lemma 3.4 μ_2 has $\mathfrak{c}$ such extensions (while μ_1 only one) so it follows that $|\mathrm{ba}_+(\mathfrak{B})| = \mathfrak{c}$. ★

Remarks 3.6 We note that the assumptions of the Extension Lemma also imply that both $\mathfrak{A}$ and $\mathfrak{B}$ have $\leq \mathfrak{c}$ ultrafilters, since every ultrafilter is in particular a measure.

Proof of Theorem 3.1 By induction on $\alpha < \omega_1$ we construct an increasing sequence $\langle \mathfrak{A}_\alpha : \alpha < \omega_1 \rangle$ of algebras of subsets of ω, and at the end we shall let $\mathfrak{B} = \bigcup_{\alpha < \omega_1} \mathfrak{A}_\alpha$. The compact space K will be the Stone space of $\mathfrak{B}$.

At the outset we fix a partition $\langle I_i : i < \omega_1 \rangle$ of ω_1 into disjoint sets of size $\aleph_1$. We start with a countable algebra $\mathfrak{A}_0$ and for each $1 < \alpha < \omega_1$ we construct the algebra $\mathfrak{A}_\alpha$ to have size $\leq \mathfrak{c}$. Each $\mathfrak{A}_\alpha$ will have the property that $|\mathrm{ba}_+(\mathfrak{A}_\alpha)| \leq \mathfrak{c}$ and that every measure on $\mathfrak{A}_\alpha$ is separable. Using this and the assumption of CH we can with every α inductively fix an enumeration $\{\mu^\alpha_\xi : \xi < \mathfrak{c}\}$ of all nonatomic measures on $\mathfrak{A}_\alpha$. In a similar manner we enumerate all nontrivial converging sequences $\bar{\mathcal{H}}$ that appear in $\mathfrak{A}_\alpha$ as $\{\bar{H}^\alpha_\xi : \xi < \mathfrak{c}\}$, where each sequence appears unboundedly often.

The choice of $\mathfrak{A}_0$ has already been described and for $0 < \delta < \omega_1$ limit we let $\mathfrak{A}_\delta = \bigcup_{\alpha < \delta} \mathfrak{A}_\alpha$. Note that such a choice and the inductive hypothesis guarantees that $|\mathrm{ba}_+(\mathfrak{A}_\delta)| \leq \mathfrak{c}$, since every measure μ on $\mathfrak{A}_\delta$ is uniquely determined by the sequence $\langle \mu \restriction \mathfrak{A}_\alpha : \alpha < \delta \rangle$, for which there are at most $|{}^\delta \mathfrak{c}| = \mathfrak{c}$ choices. We can similarly see that every measure on $\mathfrak{A}_\delta$ is separable.

At the stage $\alpha + 1$ we let $i = i(\alpha)$ be such that $\alpha \in I_i$. If $i \leq \alpha$ then let $\bar{\mathcal{S}}^\alpha = \langle \mathcal{H}_n : n < \omega \rangle$ be the first among $\bar{\mathcal{H}}^i_\xi$ which does not appear as any $\bar{\mathcal{S}}^\beta$ for $\beta < \alpha$. If $i > \alpha$, then let $\bar{\mathcal{S}}^\alpha$ be any converging sequence of ultrafilters on a subalgebra of $\mathfrak{A}_\alpha$, which certainly exists as we can take such a sequence in the countable algebra $\mathfrak{A}_0$. Now we apply the Extension Lemma 3.5 to $\mathfrak{A}_\alpha$ in place of $\mathfrak{A}$, $\bar{\mathcal{S}}_\alpha$ and its limit $\mathcal{H}$ in the role of a convergent sequence of ultrafilters, and an enumeration $\{\mu_k : k < \omega\}$ of $\{\mu^\beta_\eta : \beta, \eta < \alpha\}$ in the role of the sequence of nonatomic measures. (Note that the lemma applies as in it the sequence of measures is allowed to have distinct domains, which is likely to be the case in the construction).

Now we claim that the space K described by the above construction has the required properties. Let us first see that each nonatomic Radon measure on it is separable. Such a measure uniquely corresponds to a nonatomic measure μ on $\mathfrak{B}$. Then we have that $\mu \restriction \mathfrak{A}_\alpha$ is already nonatomic for some $\alpha < \omega_1$, and it appears as μ_ξ^α for some ξ. Then for $\beta = \max\{\xi, \alpha\}$ we have that $\mathfrak{A}_\beta$ is μ-dense in $\mathfrak{B}$, and hence μ is separable because $\mu \restriction \mathfrak{A}_\beta$ is separable. To see that K has no nontrivial convergent sequences, we note that each such sequence corresponds to a nontrivial convergent sequence $\langle \mathcal{H}'_n : n < \omega \rangle$ of ultrafilters on $\mathfrak{B}$. Then there is α such that with $\mathcal{H}_n = \mathcal{H}' \restriction \mathfrak{A}_\alpha$ for each n, we obtain a convergent sequence of distinct ultrafilters on $\mathfrak{A}_\alpha$. Let i be such that $\alpha \in I_i$. Then there must be $\beta \geq \alpha$ in I_i such that $\bar{\mathcal{S}}_\beta = \langle \mathcal{H}_n : n < \omega \rangle$, and hence $\langle \mathcal{H}'_n \restriction \mathfrak{A}_{\beta+1} : n < \omega \rangle$ cannot converge, a contradiction. ★

One may wonder if the above construction could be modified to give an example of a space in which there are no nontrivial convergent sequences and in which all Radon measures are uniformly regular (we give a construction of such a space under $\diamondsuit$ in the next section). We were not able to do so, and we give the appropriate version of the Extension Lemma to show the limitations (so the reader is asked to compare Lemma 3.5 and Lemma 3.7).

Lemma 3.7 *Let $\mathfrak{A}$ be an algebra of subsets of an infinite set T, and let $\langle \mu_n : n < \omega \rangle$ be a family of measures such that every μ_n is defined and nonatomic on some subalgebra of $\mathfrak{A}$. Consider a fixed converging sequence $(\mathcal{H}_n)_n$ in $\mathrm{ULT}(\mathfrak{A})$ consisting of distinct ultrafilters.*

Then there is a set $X \subseteq T$ such that the algebra $\mathfrak{B} = \mathfrak{A}(X)$ (generated by $\mathfrak{A}$ and $\{X\}$) has the following properties:

(i) whenever we extend every $\mathcal{H}_n$ to an $\mathcal{H}'_n \in \mathrm{ULT}(\mathfrak{B})$ then the sequence $(\mathcal{H}'_n)_n$ does not converge in $\mathrm{ULT}(\mathfrak{B})$;

(ii) for any k and any $\nu_k \in \mathrm{ba}_+(\mathfrak{B})$ extending μ_k, $\mathfrak{A}$ approximates $\mathfrak{B}$ from below with respect to ν_k.

Proof. Let $\mathcal{H}_n \to \mathcal{H} \in \mathrm{ULT}(\mathfrak{A})$. Note that for any nonatomic μ defined on some algebra $\mathfrak{A}' \subseteq \mathfrak{A}$ and $\varepsilon > 0$ there is $A \in \mathfrak{A}' \cap \mathcal{H}$ such that $\mu(A) < \varepsilon$. Since $\mathcal{H}$ is a filter, this easily implies that for every k there is $A_k \in \mathcal{H}$ such that

$$\mu_i^*(A_k) < \frac{1}{k} \text{ for } i \leq k.$$

For any k, since $\mathcal{H}_n \to \mathcal{H}$ and $A_k \in \mathcal{H}$ we have $A_k \in \mathcal{H}_n$ eventually. Thus, passing to a subsequence of $(\mathcal{H}_n)_n$ if necessary, we may assume that $A_k \in \mathcal{H}_n$ for $n \geq k$. Further, as $\mathcal{H}_n \to \mathcal{H}$ and $\mathcal{H}_n \neq \mathcal{H}_m$ for $n \neq m$, there are pairwise disjoint $B_n \in \mathfrak{A}$ with the following properties

$B_n \in \mathcal{H}_n$ for every n;

$B_n \notin \mathcal{H}_m$ for $m \neq n$;

$B_n \subseteq A_k$ whenever $n \geq k$.

Now we shall check that the set

$$X = \bigcup_{n\in\omega} B_{2n}$$

is as required. Indeed, (i) holds since $X \in \mathcal{H}'_n$ iff n is even (for n even $B_n \subseteq X$ so $X \in \mathcal{H}'_n$ for any $\mathcal{H}'_n \in \mathrm{ULT}(\mathfrak{B})$ extending $\mathcal{H}_n$, and if n is odd then $B_n \cap X = \emptyset$ and $B_n \in \mathcal{H}_n$).

Now fix k and let $\nu_k \in \mathrm{ba}_+(\mathfrak{B})$ extend μ_k. Write $E_n = B_2 \cup \ldots B_{2n}$. Then $E_n \subseteq X \subseteq E_n \cup A_{2n}$. Moreover $\nu_k(A_{2n}) \leq \mu_k^*(A_{2n}) < 1/(2n)$ whenever $k \leq n$. This shows that $\mathfrak{A}$ approximates X and X^c from below with respect to ν_k. It follows easily that $\mathfrak{A}$ approximates $\mathfrak{B}$ from below, and the proof is complete. ★

4. A simple extension.

In this section we give a construction under $\diamondsuit$ using simple extensions, defined below. This construction refutes the conjecture of Mercourakis [17] which states that if every measure on a compact space K is uniformly regular then K is necessarily sequentially compact. In fact, as the reader will realise below, what we prove is simply that every inverse limit of an inverse system of length ω_1 which starts with 2^ω and in which each successive element $K_{\alpha+1} \subseteq 2^{\alpha+1}$ is a simple extension of the previous one, is a space that only supports uniformly regular measures. It remains to be seen that such an inverse system can be obtained so that the final space has no convergent sequences. It is however well known that under $\diamondsuit$ one can have such an inverse system (see [4] for a discussion). For completeness we give a sketch of one such construction, based on Džamonja-Kunen [6]. We follow the language of [6] and use inverse systems of compact spaces in place of increasing sequences of Boolean algebras, which is the language we followed in the last section.

We do not know the answer to Mercourakis's conjecture under CH alone.

Definition. *An inverse limit $\langle X_\alpha : \alpha \in [\gamma, \gamma^*)\rangle$ with bonding maps $\langle f^\beta_\alpha : \gamma \leq \alpha \leq \beta < \gamma^*\rangle$ is obtained by* simple extensions *if for each $\alpha \in [\gamma, \gamma^*)$ there is exactly one point $x_\alpha \in X_\alpha$ such that the set $(f^{\alpha+1}_\alpha)^{-1}(x)$ is a singleton for all $x \neq x_\alpha$, and consists of two points for $x = x_\alpha$.*

Koppelberg [13] proved that the inverse limit of an inverse system $\langle X_\alpha : \alpha \in [\gamma, \gamma^*)\rangle$ obtained by simple extensions has the property that it maps onto 2^{ω_1} iff X_γ does. Her proof is given in terms of Boolean algebras, where one uses the corresponding notion of a *minimally generated algebra*. A proof in terms of inverse systems is given by Dow in [4]. Simple extensions and minimally generated algebras have been used in various contexts in set–theoretic topology, many deep results are contained in Koszmider [14].

If one only considers inverse systems in which $X_\alpha \subseteq 2^\alpha$ and the bonding maps are the natural projections π^β_α then a measure-theoretic point of view leads to a very simple proof of a strong variant of Koppelberg's result, as we now show. This will be used to show that the space we construct in Theorem 4.2 has the property that all nonatomic Radon measures on it are uniformly regular.

Lemma 4.1 *Suppose that $\langle X_\alpha : \alpha \in [\gamma, \gamma^*)\rangle$ is an inverse system in which each $X_\alpha \subseteq 2^\alpha$, the bonding maps are the natural projections, $\gamma < \gamma^* \leq \omega_1$, and the system is obtained by simple extensions. Then all nonatomic Radon measures on the limit X of the system are uniformly regular.*

(Consequently, the space does not map onto $[0,1]^{\omega_1}$ and only supports separable Radon measures).

Proof. If $\gamma^* < \omega_1$ then X is a metric space so the conclusion follows trivially. Hence let us assume that $\gamma^* = \omega_1$. The assumptions imply that for all $\alpha \in [\gamma, \omega_1)$ we may find closed sets A_α, B_α such that $A_\alpha \cup B_\alpha = X_\alpha$ and $A_\alpha \cap B_\alpha = \{s_\alpha\}$ for some point $s_\alpha \in X_\alpha$, and then $X_{\alpha+1} = (A_\alpha \times \{0\}) \cup (B_\alpha \times \{1\})$. For every Radon measure μ on X and $\alpha \in [\gamma, \omega_1)$ we denote by μ_α the measure induced by μ on X_α by projection, namely $\mu_\alpha(U) = \mu(U)$ for basic clopen sets $U \subseteq X_\alpha$.

Let μ be a nonatomic finite Radon measure on X, meaning that $\mu(\{x\}) = 0$ for all points $x \in X$. We observe that there must be $\alpha < \omega_1$ such that μ_α is nonatomic. Otherwise, there are uncountably many α for which there is a point x_α with $\mu_\alpha(\{x_\alpha\}) > 0$. There can be at most countably many pairwise disjoint sets among $(\pi^{\omega_1}_\alpha)^{-1}(\{x_\alpha\})$, so there must be $x \in X$ such that unboundedly often we have $x \restriction \alpha = x_\alpha$. Then $\mu(\{x\}) > 0$, a contradiction.

Let us fix α such that μ_α is nonatomic. We claim that the countable (since $\alpha < \omega_1$) family $\mathcal{F}_\alpha$ of basic clopen subsets of X_α satisfies that the set $\{(\pi^{\omega_1}_\alpha)^{-1}(U) : U \in \mathcal{F}_\alpha\}$ approximates all clopen sets from below. Clearly it suffices to deal with basic clopen sets, so let U be such a set, $\varepsilon > 0$ given and let $\{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\}$ be the increasing enumeration of a set I such that U is determined by the coordinates in I. If $n = 0$ or $\alpha_{n-1} < \alpha$ then $U \in \mathcal{F}_\alpha$ and we are done. Suppose then that this is not the case and let k be the first such that $\alpha_k \geq \alpha$. For simplicity we shall assume that $k = n-1$, as the proof is similar in other cases. Without loss of generality we may assume that $f(\alpha_{n-1}) = 0$ for all $f \in U$. Now the proof proceeds by induction on α_{n-1}. Let $D = (\pi^{\alpha+1}_\alpha)(U)$. If $x \in D \cap B_\alpha$ we must have $x = s_\beta$, so $D \subseteq A_\alpha$. Since $\mu_\alpha(\{s_\alpha\}) = 0$ we may find a closed set $F \subseteq D \setminus B_\alpha$ such that $\mu_\alpha(F) > \mu_\alpha(D) + \varepsilon$. We may without loss of generality assume that F is clopen, hence it belongs to $\mathcal{F}_\alpha$ and we are done. Other cases of the induction are handled similarly. ★

Note that atomic measures need not be uniformly regular even on spaces X as in the lemma above (since the Dirac measure δ_x is uniformly regular only when x is a G_δ-point). An interesting twist to Lemma 4.1 is provided by considering inverse limits of length $> \omega_1$ which are obtained by simple extensions. One can easily see that the limit of such a system still cannot support a nonseparable measure, but it is no longer true that every measure is uniformly regular. It was however proved by Borodulin-Nadzieja [2] that every such measure is uniformly regular on its support.

Theorem 4.2 *Assuming $\diamondsuit$ there is an Efimov space in which all nonatomic Radon measures are uniformly regular.*

Proof. The main construction of the theorem is due to Fedorčuk [8] who essentially showed that assuming $\diamondsuit$, there is an inverse limit system $\langle X_\alpha : \omega \le \alpha \le \omega_1\rangle$ of compact spaces satisfying $X_\alpha \subseteq 2^\alpha$ and with bonding maps the natural projections $\langle \pi^\beta_\alpha : \omega \le \alpha \le \beta \le \omega_1\rangle$, which satisfies

(i) each $X_{\alpha+1}$ is obtained as a simple extension of X_α,

(ii) there are no isolated points in X (and every point in $X \stackrel{\text{def}}{=} X_{\omega_1}$ has character ω_1) and

(iii) there are no convergent sequences in X.

By Lemma 4.1 we conclude that X is an Efimov space in which all nonatomic Radon measures are uniformly regular.

For completeness of the argument we sketch the construction of X. The sketch is based on [6].

The inverse limit is constructed by building X_α inductively so to satisfy various requirements. These requirements are to start with:

R1 Each X_α is a closed subset of 2^α, $X_\omega = 2^\omega$ and for all $\omega \le \alpha < \omega_1$ we have $X_\alpha = A_\alpha \cup B_\alpha$ where A_α, B_α are closed and $A_\alpha \cap B_\alpha = \{s_\alpha\}$ is a singleton. Also, A_α, B_α have no isolated points. Then we have $X_{\alpha+1} = A_\alpha \times \{0\} \cup B_\alpha \times \{1\}$.

R2 For all $\alpha < \omega_1$ and $y \in X_\alpha$, there is $\beta \in [\alpha, \omega_1)$ with $s_\beta \in (\pi^\beta_\alpha)^{-1}(\{y\})$.

For the next requirement, we need the notion of a strong limit point.

Definition. *Suppose that $\bar Y = \langle Y_n : n < \omega\rangle$ is a sequence of disjoint closed sets. A point x is* a strong limit point *of $\bar Y$ iff $(\forall U$ open $\ni x)(\exists n)\,[Y_n \subseteq U \setminus \{x\}]$.*

The assumption $\diamondsuit$ is used in the following equivalent form of it: there is a sequence $\langle \bar x_\alpha = \langle x^\alpha_n : n < \omega\rangle : \omega \le \alpha < \omega_1\rangle$ such that for all α and n we have that $x^\alpha_n \in 2^\alpha$, and for all sequences $\langle x_n : n < \omega\rangle$ in 2^{ω_1} there are stationarily many α such that for every n we have $x_n \restriction \alpha = x^\alpha_n$.

The next requirement is supposed to hold for each $\beta \in [\omega, \omega_1)$:

R3'(β) Suppose $\alpha \le \beta$, $y \in X_\beta$ and $\bar x_\alpha$ is a sequence of distinct points in X_α with a limit point $y \restriction \alpha$. Then y is a strong limit point of $\langle (\pi^\beta_\alpha)^{-1}(\{x^\alpha_n\}) : n < \omega\rangle$.

In order to be able to preserve R3' inductively, we in fact need to replace it by a different requirement, R3(β):

R3(β) Suppose that $\alpha \leq \beta$ and $\bar{x}_\alpha$ is a sequence of distinct points in X_α, s_β is a strong limit point of $\langle (\pi^\beta_\alpha)^{-1}(\{x^\alpha_n\}) : n < \omega \rangle$, and U is an open neigbourhood of s_β. Then there are infinitely many n such that $(\pi^\beta_\alpha)^{-1}(\{x^\alpha_n\}) \subseteq U \setminus A_\beta$, and infinitely many n such that $(\pi^\beta_\alpha)^{-1}(\{x^\alpha_n\}) \subseteq U \setminus B_\beta$.

We hence perform an inductive construction to satisfy R1, R2 and R3(β) for all $\beta \in [\omega, \omega_1)$. By CH we enumerate $\bigcup_{\alpha \in [\omega,\omega_1)} 2^\alpha$ as $\langle p_\beta : \beta \in [\omega, \omega_1) \rangle$ so that if $p_\beta \in {}^\alpha 2$ then $\alpha \leq \beta$ and so that each point appears unboundedly often. The main point of the construction the successor stage $\beta + 1$ of the construction.

We first choose s_β so that if $p_\beta \in X_{\mathrm{dom}(p_\beta)}$ then s_β is any element of $(\pi^\beta_{\mathrm{dom}(p_\beta)})^{-1}(\{p_\beta\})$. Otherwise we choose s_β to be any singleton from X_β. Our choice of A_β and B_β will be made to guarantee R3(β). Since $\beta < \omega_1$ we can fix a strictly decreasing sequence $\langle V_n : n < \omega \rangle$ of clopen sets such that $V_0 = X_\beta$ and $\cap_{n<\omega} V_n = \{s_\beta\}$.

Claim 4.3 *Suppose that $\varphi : \omega \to \omega$ is strictly increasing with $\varphi(0) = 0$. Further suppose that $A_\beta = \{s_\beta\} \cup \bigcup_{n<\omega}(V_{\varphi(2n)} \setminus V_{\varphi(2n+1)})$, and $B_\beta = \{s_\beta\} \cup \bigcup_{n<\omega}(V_{\varphi(2n+1)} \setminus V_{\varphi(2n+2)})$. Then $A_\beta \cap B_\beta = \{s_\beta\}$, $A_\beta \cup B_\beta = X_\beta$ and A_β, B_β are closed.*

Proof. The statement is very easy to prove, for example, the complement of A_β in X_β is $\bigcup_{n<\omega}(V_{\varphi(2n+1)} \setminus V_{\varphi(2n+2)})$, which is open, hence A_β is closed. ★

One then needs to show that one can choose a function φ as above so that the resulting choices of A_β and B_β satisfy R3(β). ★

We note that if X is the space from the theorem above then the space $P(X)$ has the following properties. Since X contains no converging sequences, no sequence of distinct Dirac measures δ_{x_n} is weak* convergent; in particular no δ_x is a G_δ point in $P(X)$. On the other hand, every nonatomic $\mu \in P(X)$ is uniformly regular and so a G_δ point in $P(X)$ (by a result due to Pol [22]); moreover such μ is a limit of a sequence of purely atomic measures $(1/n)\sum_{i \leq n} \delta_{x_i}$ for some $x_i \in X$ (see section 2).

References

[1] A.G. Babiker, *On uniformly regular topological measure spaces*, Duke Math. J. 43 (1976), 775–789.

[2] P. Borodulin-Nadzieja, *Measures on minimally generated Boolean algebras*, preprint 2005.

[3] C. Brech, *On the density of Banach $C(K)$ spaces with the Grothendieck property*, to appear in the Proceedings of AMS.

[4] A. Dow, *Efimov spaces and the splitting number*, to appear in Topology Proceedings.

[5] A. Dow, D.H. Fremlin *Compact sets without converging sequences in the random real model*, preprint (2005).

[6] M. Džamonja, K. Kunen, *Measures on compact HS spaces*, Fund. Math. 143 (1993), 41–54.

[7] R. Engelking, *General topology*, 2nd edition, Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin 1980, Translated from the Polish by the author.

[8] В. В. Федорчук, *О мощности наследственно цепарабельных бикомпактов*, Доклады Академии Наук СССР 222 (1975) 302–305.

[9] D.H. Fremlin, *On compact spaces carrying Radon measures of uncountable Maharam type*, Fund. Math. 154 (1997), 295–304.

[10] R. Haydon, *On Banach spaces which contain l^τ and types of measures on compact spaces*, Israel J. Math. 28 (1977), 313–324.

[11] R. Haydon, *On dual L^1–spaces and injective bidual Banach spaces*, Israel J. Math. 31 (1978) 142–152.

[12] R. Haydon, *An unconditional result about Groethendick spaces*, Proceedings of AMS 100 no. 3 (1987), 511-516.

[13] S. Koppelberg, *Minimally generated Boolean algebras*, Order 5 (1989), no. 4, 393–406.

[14] P. Koszmider, *Forcing minimal extensions of Boolean algebras*, Trans. Amer. Math. Soc. 351 (1999), no. 8, 3073–3117.

[15] K. Kunen, *Set Theory: an introduction to independence proofs*, North Holland 1980.

[16] K. Kunen, *A compact L–space under CH*, Topology Appl. 12 (1981), 283–287.

[17] S. Mercourakis, *Some remarks on countably determined measures and uniform distribution of sequences*, Monats. Math. 121 (1996), 79–101.

[18] P.J. Nyikos, *Classic problems*, in **Problems from Topology Proceedings**, E. Pearl (edt.), 69–89.

[19] D. Plachky, *Extremal and monogenic additive set functions*, Proc. Amer. Math. Soc. 53 (1976), 193–196.

[20] G. Plebanek, *Nonseparable Radon measures and small compact spaces*, Fund. Math. 153 (1997), 25–40.

[21] G. Plebanek, *Approximating Radon measures on first–countable compact spaces*, Colloquium Mathematicum 86 (2000), 15–23.

[22] R. Pol, *Note on the spaces of regular probability measures whose topology is determined by countable subsets*, Pacific J. Math. 100 (1982), 185–201.

[23] M. Talagrand, *Un nouveau $C(K)$ qui possede la propriété de Grothendieck*, Israel J. Math. 37 (1980), 181–191.