Large families of mutually singular Radon measures

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Abstract

We investigate how many mutually singular measures one can define on a given space. In particular, we show that under Martin's axiom there is a compact space X of cardinality $\mathfrak{c}$ which carries $2^{\mathfrak{c}}$ mutually singular Radon probability measures.

1. Introduction. We present here a partial answer to the following problem posed by the first author (problem FR in [6]):

(FR) *Is there a compact Hausdorff space X with a family $(\mu_s)_{s\in S}$ of mutually singular Radon measures on X such that $\#(S) > \#(X)$?*

In section 2 we present a construction which shows that FR has a positive answer under some mild set–theoretic assumptions. In particular, Martin's axiom implies that there is a compact space of cardinality $\mathfrak{c}$ carrying $2^{\mathfrak{c}}$ mutually singular Radon probability measures (see Theorem 3A).

We hope that the reader will find problem FR to be interesting in itself. Let us note that, although spaces of measures have been investigated from various points of view, see e.g. [2], [11], [3] and the references therein, even basic questions concerning cardinalities in spaces of measures seem to be open (see section 4). Moreover, our partial solution to problem FR settles negatively a conjecture due to Haydon [10] on the isometric structure on conjugate L^1–spaces. We are grateful to R. Haydon for his remarks on that conjecture and problem FR (see section 5).

We now fix some terminology and notation. In the sequel, we consider only finite measures, so we say that μ is a Radon measure on a (Hausdorff topological) space X if μ is a finite complete measure defined on a σ–algebra containing all Borel subsets of X, and μ is inner regular with respect to compact sets.

For a cardinal number κ, we denote by μ_κ the completion of the usual product measure on the Cantor cube $\{0,1\}^\kappa$. Recall that Kakutani's theorem

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([8], 416T) implies that μ_κ is a Radon measure on $\{0,1\}^\kappa$. Basic properties of μ_κ can be found in [5] 1.15–1.16, [7], 254, [8], 416.
For a fixed κ, the uniformity of the measure μ_κ is defined as the least cardinality of a set $X \subseteq \{0,1\}^\kappa$ which has a positive outer measure. We shall use some results on the uniformity of μ_κ which can be found in [5], see Definition 6.1 and Theorem 6.13(d).

2. Construction. Let us suppose that for some cardinal numbers κ and λ there is a family $(N_\xi)_{\xi<\lambda}$ of subsets of $\{0,1\}^\kappa$ with the following properties:

(i) $\mu_\kappa(N_\xi) = 0$ for every $\xi < \lambda$;

(ii) $N_\eta \subseteq N_\xi$ for every $\eta < \xi < \lambda$;

(iii) $\bigcup_{\xi<\lambda} N_\xi = \{0,1\}^\kappa$.

For each $\xi < \lambda$ we choose a nondecreasing sequence $(F_{\xi n})_{n\in\mathbb{N}}$ of closed subsets of $\{0,1\}^\kappa \setminus N_\xi$, such that $\lim_{n\to\infty} \mu_\kappa F_{\xi n} = 1$.
Denoting $Y = \{0,1\}^\kappa \times \{0,1\}^{\lambda\times\mathbb{N}}$, we define a subset X of Y by

$$X = \bigcap_{\xi<\lambda, n\in\mathbb{N}} \{(t,u) \in Y : \ t \in F_{\xi n} \text{ or } u(\xi,n) = 0\}.$$

Clearly X is a compact space. Moreover, we have the following.

2A *If $2^\theta \le 2^\kappa$ for every $\theta < \lambda$ then $\#(X) \le 2^\kappa$.*

Indeed, for every $t \in \{0,1\}^\kappa$ there is a $\xi < \lambda$ such that $t \in N_\xi$. Then $t \notin F_{\eta n}$ for every $\eta \ge \xi$ and every $n \in \mathbb{N}$, so that $\#(\{u : (t,u) \in X\}) \le \#(\mathcal{P}(\xi \times \mathbb{N})) \le 2^\kappa$; accordingly $\#(X) \le 2^\kappa$. ∎

For every $\varepsilon \in \{0,1\}^\lambda$ we define a function $f_\varepsilon : \{0,1\}^\kappa \to X$ by setting

$$f_\varepsilon(t) = (t,u), \qquad \text{where } u(\xi,n) = \begin{cases} \varepsilon(\xi) & \text{for} \quad t \in F_{\xi n} \\ 0 & \text{for} \quad t \notin F_{\xi n}. \end{cases}$$

Write $C_{\xi n} = \{x \in \{0,1\}^{\lambda\times\mathbb{N}} : \ x(\xi,n) = 1\}$; note that for any $A \subseteq \{0,1\}^\kappa$ the set $f_\varepsilon^{-1}[(A \times C_{\xi n}) \cap X]$ is equal to $A \cap F_{\xi n}$ if $\varepsilon(\xi) = 1$, and is empty otherwise. It follows easily that $f_\varepsilon^{-1}[E]$ is a Borel subset of $\{0,1\}^\kappa$ whenever $E \subseteq X$ is closed–and–open.
With every $\varepsilon \in \{0,1\}^\lambda$ we can now associate a Radon measure ν_ε on X in the following way. We set $\nu_\varepsilon E = \mu_\kappa f_\varepsilon^{-1}[E]$ for every open–and–closed subset E of X; this uniquely defines ν_ε, see [8], 416Qa.

2B *$\{\nu_\varepsilon : \ \varepsilon \in \{0,1\}^\lambda\}$ is a family of mutually singular Radon probability measures on X.*

Indeed, take distinct $\varepsilon, \varepsilon' \in \{0,1\}^\lambda$, and assume for instance that for some $\xi < \lambda$ we have $\varepsilon(\xi) = 0$, $\varepsilon'(\xi) = 1$. Consider the set $E = \bigcup_{n\in\mathbb{N}}\{(t,u) \in$

$X : u(\xi, n) = 1\}$. Then $f_\varepsilon^{-1}[E] = \emptyset$ and $f_{\varepsilon'}^{-1}[E] = \bigcup_n F_{\xi n}$, so $\mu_\kappa f_{\varepsilon'}^{-1}[E] = 1$. Hence the measures ν_ε and $\nu_{\varepsilon'}$ are mutually singular. ■

Let us remark for future purpose that the way in which measures ν_ε are defined implies that the Maharam type of every ν_ε is less than or equal to κ.

3. Main result. Suppose that κ is such a cardinal number that $2^\theta \leq 2^\kappa$ for all $\theta < 2^\kappa$. Moreover, assume that the uniformity of the measure μ_κ equals 2^κ, i.e. for every $A \subseteq \{0,1\}^\kappa$ with $\#(A) < 2^\kappa$ we have $\mu_\kappa(A) = 0$.
Writing $\lambda = 2^\kappa$ and $\{0,1\}^\kappa = (t_\xi)_{\xi<\lambda}$, we can define sets $(N_\xi)_{\xi<\lambda}$ satisfying *(i)–(iii)* from section 2 by $N_\xi = \{t_\eta : \eta < \xi\}$. Then the above construction gives a compact space X with $\#(X) \leq \lambda$ carrying 2^λ mutually singular Radon probability measures, and we have the following result.

Theorem 3A. *Under Martin's axiom there is a compact space of cardinality $\mathfrak{c}$ carrying $2^\mathfrak{c}$ mutually singular Radon probability measures.*

Proof. We take $\kappa = \omega$; if Martin's axiom is assumed then the uniformity of the measure μ_ω is 2^ω and $2^\theta \leq \mathfrak{c}$ for every $\theta < \mathfrak{c}$, see [4], 22Hb and 21C . ■

We remark that if the continuum hypothesis is assumed then the space obtained in the proof of Theorem 3A is Corson compact. Indeed, in this case we can use our basic construction with $\kappa = \omega$ and $\lambda = \omega_1 = 2^\omega$, and by 2A the resulting space X is then contained in the subset of $\{0,1\}^\kappa \times \{0,1\}^{\lambda\times\mathbb{N}}$, consisting of elements with countable support.
The construction from section 2 gives a positive answer to problem FR under various set–theoretic assumptions.

Theorem 3B. *Assume that either*

(a) $\{0,1\}^\omega$ *is the union of* ω_1 *null sets and* $2^{\omega_1} > \mathfrak{c}$, *or*

(b) κ *is a strong limit cardinal of countable cofinality and* $2^\kappa = \kappa^+$.

Then there is a compact Hausdorff space X with a family $(\mu_s)_{s\in S}$ of mutually singular Radon probability measures on X such that $|S| > |X|$.

Proof. In case *(a)* we can apply the construction from section 2 to $\kappa = \omega$ and $\lambda = \omega_1$. Take a cardinal number as in *(b)* and denote $\lambda = \kappa^+$. Then the uniformity of the measure μ_κ is λ, see [5], 6.17e and the argument for 6.18d. Hence we can apply the construction from section 2, where $\{0,1\}^\kappa = (t_\xi)_{\xi<\lambda}$ and $N_\xi = \{t_\eta : \eta < \xi\}$. ■

4. Problems and remarks. Given a compact space, denote by $P(X)$ the set of all probability Radon measures on X. Since the topological weight $w(X)$ of X satisfies $w(X) \leq \#(X)$ and $\#(P(X)) \leq 2^{w(X)}$, we have the following inequalities

$$\#(X)^\omega \leq \#(P(X)) \leq 2^{\#(X)}.$$

In connection with his paper [12], Z. Lipecki asked if $\#(X)^\omega = \#(P(X))$ for every compact space X. We have seen that for instance Martin's axiom settles this question negatively, but the following problems seem to be open.

4.1 *Is it relatively consistent with ZFC that $\#(P(X)) = \mathfrak{c}$ for every compact space X with $\#(X) = \mathfrak{c}$?*

4.2 *Is there a ZFC example of a compact space X such that $\#(X)^\omega < \#(P(X))$?*

We remark that such questions can be easily answered if the assumption on compactness is omitted.

Example 4.3. *There is a completely regular space X of cardinality $\mathfrak{c}$ carrying $2^\mathfrak{c}$ mutually singular τ–additive probability measures.*

We let X to be

$$X = \{x \in \{0,1\}^\mathfrak{c} : \ \#(\{\xi < \mathfrak{c} : x(\xi) \neq 0\}) \leq \omega\},$$

so that X is a completely regular space of cardinality $\mathfrak{c}$. Moreover for the product measure $\mu_\mathfrak{c}$ on $\{0,1\}^\mathfrak{c}$ we have $\mu_\mathfrak{c}^*(X) = 1$. This follows immediately from Kakutani's theorem and the fact that X meets every nonempty G_δ set, see [7], 254 and [8], 416T. We can define a τ–additive probability measure μ on X by $\mu(B \cap X) = \mu_\mathfrak{c}(B)$ for every Borel set $B \subseteq \{0,1\}^\mathfrak{c}$, see 416A and 415B in [8].
For any $I \subseteq \mathfrak{c}$ we define the sets X_I and Z_I as

$$X_I = \{x \in X : \ x(\xi) = 0, \forall \xi \in \mathfrak{c} \setminus I\}, \quad Z_I = \{x \in X : \ x(\xi) = 0, \forall \xi \in I\}.$$

Note that $\mu(Z_I) = 0$ whenever $I \subseteq \mathfrak{c}$ is infinite.
Given $H \subseteq \mathfrak{c}$, we denote by π_H the continuous mapping $\pi_H : X \to X$, where $\pi_H(x)(\xi) = x(\xi)$ for $\xi \in H$ and 0 otherwise. Let μ_H be the corresponding image measure, defined by the formula $\mu_H(E) = \mu\pi_H^{-1}[E]$.
Now we take a family $\mathcal{H} \subseteq \mathcal{P}(\mathfrak{c})$ such that $\#(\mathcal{H}) = 2^\mathfrak{c}$ and for every $H, G \in \mathcal{H}$, if $H \neq G$ then $\#(H \triangle G) \geq \omega$, and claim that the family of measures $\{\mu_H : \ H \in \mathcal{H}\}$ is as required.
Take $H, G \in \mathcal{H}$, $H \neq G$, and suppose for instance that $\#(G \setminus H) \geq \omega$. we have $\mu_H(X_H) = 1$, while

$$\mu_G(X_H) = \mu\{x \in X : \ \pi_G(x) \in X_H\} \leq \mu(Z_{G\setminus H}) = 0,$$

so the measures μ_H and μ_G are mutually singular.

5. On the isometric structure of conjugate L^1 spaces. As it was mentioned in section 1, the question on the possible cardinality of a family of mutually singular Radon measures on a given compact space was implicitly asked in Haydon [10] in connection with the isometric structure of dual Banach spaces.

Building on the results of Haydon [9] and Argyros [1], Haydon noted, see [10], Theorem 3E, that subject to Martin's axiom if a dual Banach space E^* is an abstract L–space then it is isomorphic to an l^1–direct sum of Banach spaces that are of the form

$$(*) \quad C(\{0,1\}^\kappa)^* \text{ or } L^1(\mu_{\omega_1}) \oplus C[0,1]^*.$$

(Here and below all the direct sums of Banach spaces mean l^1–sums.) So he posed the problem (see 3F in [10]) if every abstract L–space E^* is actually *isometric* to such an l^1–direct sum of Banach spaces.
Basing on the remarks from [10] we can check that if X is a space from Theorem 3A then the Banach space $C(X)^*$ disproves this conjecture.
The point is that for every compact space X we can take a maximal family $\{\nu_s : s \in S\}$ of mutually singular homogenous Radon probability measures on X, and write

$$C(X)^* \equiv \sum_{s\in S}{}^{\oplus} L^1(\nu_s),$$

where $\equiv$ stands for the isometry. It follows from the Maharam theorem that for every $s \in S$, $L^1(\nu_s) \equiv L^1(\mu_{\kappa(s)})$ for some cardinal number $\kappa(s)$, so if we group together the terms with the same Maharam type it follows that the decomposition of $C(X)^*$ can be written as

$$C(X)^* \equiv \sum_{\omega\le\kappa<\lambda}{}^{\oplus} \left(\sum_{\alpha(\kappa)}{}^{\oplus} L^1(\mu_\kappa)\right) \oplus l^1(\#(X)),$$

where $\alpha(\kappa)$ is the multiplicity with which the factor $L^1(\mu_\kappa)$ occurs.
If X is a space from Theorem 3A then $\alpha(\omega) = 2^{\mathfrak{c}}$, since our space X carries as many mutually singular Radon measures of type ω (apply the remark made at the end of section 2). On the other hand, if $C(X)^*$ was to be written as an l^1–direct sum of spaces of the form (*) it would follow that $\alpha(\omega) \le \#(X)$ (see [10]).

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References

[1] S. Argyros, *On non–separable Banach spaces*, Trans. Amer. Math. Soc., 270 (1982) 193–216.

[2] V.V. Fedorchuk, *On a preservation of completeness of uniform spaces by the functor P_τ*, Topology Appl., 91 (1999) 25–45.

[3] R. Frankiewicz, G. Plebanek, C. Ryll–Nardzewski, *Between Lindelöf property and countable tightness*, Proc. Amer. Math. Soc., 129 (2001) 97–103.

[4] D.H. Fremlin, *Consequences of Martin's Axiom*, Cambridge Univ. Press, Cambridge 1984.

[5] D.H. Fremlin, *Measure algebras*, in: *Handbook of Boolean algebras*, Vol. **III**, Chap. 22, ed.: J.D. Monk, North–Holand 1989.

[6] D.H. Fremlin, *Problems* (July, 2001).

[7] D.H. Fremlin, *Measure Theory, Vol. 2: Broad Foundations,* Torres Fremlin, Colchester 2001.

[8] D.H. Fremlin, *Measure Theory, Vol. 4: Topological measure spaces,* in preparation. *Partial drafts available via*

`http://www.essex.ac.uk/maths/staff/fremlin/mt.htm`.

[9] R. Haydon, *On dual L^1–spaces and injective bidual Banach spaces*, Israel J. Math., 31 (1978) 142–152.

[10] R. Haydon, *Non–separable Banach spaces*, in: *Functional analysis: Surveys and recent results II*, edt.: K.D. Biersstedt, B. Fuchssteiner, North–Holland, Amsterdam (1980) 19–30.

[11] P. Holický, O. Kalenda, *Desriptive properties of spaces of measures* Bull. Pol. Acad. Sci., 47 (1999) 37–51.

[12] Z. Lipecki, *Cardinality of the set of extreme extensions of quasi–measures*, Manuscripta Math., 104 (2001) 333–341.