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Independent families in measure algebras

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Definition. A family \( \{a_\xi : \xi < \kappa\} \) in a Boolean algebra \( \mathcal{A} \) is

(i) centred if \( \bigwedge_{\xi \in I} a_\xi \neq 0 \) for every finite \( I \subseteq \kappa \);

(ii) independent if \( \bigwedge_{\xi \in I} a_{\xi}^{\phi(\xi)} \neq 0 \) for every finite \( I \subseteq \kappa \) and every \( \phi : I \to \{0,1\} \).

Definition. A cardinal number \( \kappa \) is a precalibre of a Boolean algebra \( \mathcal{A} \) if for every family \( \{a_\xi : \xi < \kappa\} \subseteq \mathcal{A}^+ \) there is \( X \in [\kappa]^\kappa \) such that \( \{a_\xi : \xi \in X\} \) is centred.

Definition. A cardinal number \( \kappa \) is an independence precalibre of a Boolean algebra \( \mathcal{A} \) if for every family \( \{a_\xi : \xi < \kappa\} \) of distinct elements of \( \mathcal{A} \) there is \( X \in [\kappa]^\kappa \) such that \( \{a_\xi : \xi \in X\} \) is independent.
Definition. For a measure algebra \((\mathcal{A}, \mu)\) a cardinal number \(\kappa\) is a **measure precalibre** if in every family \(\{a_\xi : \xi < \kappa\} \subseteq \mathcal{A}\) with \(\inf_{\xi<\kappa} \mu(a_\xi) > 0\) there is a centred subfamily of size \(\kappa\).

Definition. For a measure algebra \((\mathcal{A}, \mu)\)

(i) a family \(\{a_\xi : \xi < \kappa\} \subseteq \mathcal{A}\) is **separated** if there is \(\varepsilon > 0\) such that \(\mu(a_\xi \Delta a_\eta) \geq \varepsilon\) for \(\xi \neq \eta\);

(ii) a cardinal number \(\kappa\) is a **measure independence precalibre** of \(\mathcal{A}\) if every separated family \(\{a_\xi : \xi < \kappa\} \subseteq \mathcal{A}\) contains an independent subfamily of size \(\kappa\).
Definition. $\kappa$ is $(\ast)$–precalibre of measure algebras if for every $(\mathcal{A}, \mu)$ and $\{a_\xi : \xi < \kappa\} \subseteq \mathcal{A}$ satisfying $(A)$ there is $X \in [\kappa]^\kappa$ such that $\{a_\xi : \xi \in X\}$ has property $(P)$, where

<table>
<thead>
<tr>
<th>$(\ast)$</th>
<th>$(A)$</th>
<th>$(P)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>precal.</td>
<td>$a_\xi \neq 0$</td>
<td>centred</td>
</tr>
<tr>
<td>ind. precal.</td>
<td>$a_\xi \neq a_\eta$</td>
<td>independent centred</td>
</tr>
<tr>
<td>m. precal.</td>
<td>$\mu(a_\xi) \geq \varepsilon$</td>
<td>independent centred</td>
</tr>
<tr>
<td>m. ind. precal.</td>
<td>$\mu(a_\xi \triangle a_\eta) \geq \varepsilon$</td>
<td>independent</td>
</tr>
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</table>

Remarks.

(1) If $\text{cof}(\kappa) > \omega$ then $\kappa$ is a measure precalibre iff $\kappa$ is a precalibre (of measure algebras).

(2) Every measure independence precalibre of measure algebras is a measure precalibre.

(3) No $\kappa \leq c$ is independence precalibre of measure algebras.
Measure precalibres.

(1) \( \omega \) is a measure precalibre.

(2) \( \kappa < c \) may be (under MA + non CH) and may not be (under CH) a measure precalibre; Fremlin vol. 5 of Measure Theory or Džamonja & G.P. [04].

(3) Problem. (Haydon) \ Let \( \kappa_n \) be regular and precalibre of measure algebras. Is \( \kappa = \sup_{n<\omega} \kappa_n \) a measure precalibre?

(4) Problem. (Fremlin) Is it rel. consistent that every regular \( \kappa \) is a precalibre of measure algebras?

Fact. (Shelah, Argyros & Tsarpalias, Fremlin) If \( \text{cof}(\kappa) = \omega \) and \( 2^\kappa = \kappa^+ \) then \( \kappa^+ \) is not a precalibre of measure algebras.
Measure independence precalibres.

(1) **Fact.** $\omega$ is a measure independence precalibre.

(2) **Theorem.** (Argyros & Tsarpalias [82]) Assume that $\kappa = \text{cof}(\kappa)$ and $\tau^\omega < \kappa$ for $\tau < \kappa$ (for instance: $\kappa = c^+$. Then $\kappa$ is an independence precalibre of all ccc Boolean algebras, so in particular of all measure algebras (Haydon [77]).

(3) **Theorem.** (Shelah [99]; Džamonja & G.P. [04]) Suppose that for some $\theta$, 

$$\theta = \theta^\omega < \text{cof}(\kappa) \leq \kappa \leq 2^\theta.$$ 

Then $\kappa$ is a measure independence precalibre. If, moreover, $\kappa > 2^c$ then $\kappa$ is an independence precalibre of measure algebras.
Theorem. (Fremlin & G.P.)
For $\kappa \geq \omega_2$ TFAE

(i) $\kappa$ is a measure precalibre;

(ii) $\kappa$ is a measure independence precalibre.

Theorem. (Fremlin [97]) Under MA + nonCH, $\omega_1$ is a measure independence precalibre.

Theorem. (G.P. [97]) It is rel. consistent that $\omega_1$ is a measure precalibre but not measure independence precalibre.
About the proof.

**Theorem.** (Hajnal’s free set theorem) If \( \kappa \geq \omega_2 \) and \( J : \kappa \rightarrow [\kappa]^{\leq \omega} \) is a set mapping such that \( \xi \notin J_\xi \) for every \( \xi < \kappa \) then there is \( X \in [\kappa]^\kappa \) such that \( \eta \notin J_\xi \) for all \( \eta, \xi \in X \).

(1) **Lemma.** Let \( \kappa \geq \omega_2 \) have uncountable cofinality. If \( \{s_\xi : \xi < \kappa\} \subseteq [\kappa]^{< \omega} \) is a pairwise disjoint family, \( \{J_\xi : \xi < \kappa\} \subseteq [\kappa]^{\leq \omega} \) are such that \( s_\xi \cap J_\xi = \emptyset \) for every \( \xi < \kappa \) then there is \( X \subseteq \kappa \) of cardinality \( \kappa \) such that \( s_\xi \cap J_\eta = \emptyset \) whenever \( \xi, \eta \in X \).

(2) **Lemma.** Let \( B \subseteq \{0, 1\}^\kappa \) be a measurable set and \( X \subseteq \kappa \) be such that \( B^* \notin \mathcal{A}[X] \). Then there are a finite set \( s \subseteq \kappa \setminus X \), a countable set \( J \subseteq \kappa \setminus s \), nonempty clopen sets \( C(0), C(1) \sim s \), a set \( Z \sim J \) with \( \lambda(Z) > 0 \) such that \( Z \cap C(i) \subseteq B^i \) for \( i = 0, 1 \).
Consider a separated family \( \{ B_\xi : \xi < \kappa \} \) of subsets of \( \{0, 1\}^\kappa \). Let \( B_\xi \sim I_\xi, I_\xi \in [\kappa]^\omega \). We can assume that

\[
B_\xi \not\in \mathcal{A}[X_\xi], \quad \text{where} \quad X_\xi = \bigcup_{\eta < \xi} J_\eta.
\]

Apply Lemma 2 to every \( B_\xi \not\in \mathcal{A}[X_\xi] \): there are pairwise disjoint finite sets \( s_\xi \) in \( \kappa \), nonempty clopen sets \( C_\xi(0), C_\xi(1) \sim s_\xi \), and sets \( Z_\xi \) of positive measure, where every \( Z_\xi \sim J_\xi \subseteq \kappa \setminus s_\xi \).

We can assume that
\[
J_\xi \cap s_\eta = \emptyset \text{ for all } \xi, \eta;
\]
\( \{ Z_\xi : \xi < \kappa \} \) is centred in \( \mathcal{A} \).

For any finite set \( a \subseteq \kappa \) and \( \varphi : a \to \{0, 1\} \)

\[
\lambda(\bigcap_{\xi \in a} B_\xi^{\varphi(\xi)}) \geq \lambda(\bigcap_{\xi \in a} Z_\xi \cap C_\xi(\varphi(\xi))) = \lambda(\bigcap_{\xi \in a} Z_\xi) \cdot \prod_{\xi \in a} \lambda(C_\xi(\varphi(\xi))) > 0.
\]
**Haydon’s property.** Say that $\kappa$ has Haydon’s property if every compact space $K$ that carries a Radon measures of Maharam type $\kappa$ can be continuously mapped onto $[0,1]^\kappa$.

**Theorem. (G.P. [97] + Haydon + Fremlin)** For $\kappa \geq \omega_2$, TFAE

(a) $\kappa$ has Haydon’s property;

(b) $\kappa$ is a measure precalibre.

(c) $\kappa$ is a measure independence precalibre.

**Theorem. (Fremlin [97])** Under MA $+$ non CH, $\omega_1$ has Haydon’s property and (therefore) is a measure independence precalibre.
Theorem. (G.P. [97]) Let $\mathcal{N}$ be the null ideal of the measure on $\{0, 1\}^{\omega_1}$. Suppose that $\text{cov}(\mathcal{N}) > \omega_1$ but there is a family $\{N_\xi : \xi < \omega_1\} \subseteq \mathcal{N}$ such that $\bigcup_{\xi < \omega_1} N_\xi$ meets every perfect set in $\{0, 1\}^{\omega_1}$. Then $\omega_1$ is a measure precalibre but does not have Haydon’s property and (therefore) is not a measure independence precalibre.

Theorem. (Kunen & van Mill [95] for (b) $\to$ (a); G.P. [95] for (a) $\to$ (b)) TFAE

(a) $\omega_1$ is a measure precalibre;

(b) every Radon measure on a first–countable (compact) space is of countable type.

Problem. Can we replace in (b) “first–countable” by “countably tight”? 
Problem. Suppose that $\omega_1$ is a measure pre-calibre; let $\{a_\xi : \xi < \omega_1\}$ be a separated family in a measure algebra. Can we find $X \in [\omega_1]^{\omega_1}$ such that $\{a_\xi : \xi \in X\}$ is free, i.e. $\bigwedge_{\xi \in I} a_\xi \wedge \bigwedge_{\xi \in J} a_\xi^C \neq 0$ for every finite $I, J \subseteq X$ with $\max I < \min J$?