A construction of a Banach space $C(K)$ with few operators

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Abstract

We present a construction, carried entirely in ZFC, of a compact connected space K such that every bounded operator $T : C(K) \to C(K)$ can be written as $T = g \cdot I + S$, where $g \in C(K)$ and S is a weakly compact operator. This extends a result due to Koszmider [17] who constructed such a space assuming the continuum hypothesis.

1. Introduction

There are several constructions of Banach spaces X which admit only operators $T : X \to X$ of some specific type. For example, Shelah [25] (assuming certain additional axioms) and Shelah & Steprāns [26] (in the usual set theory ZFC) proved the existence of a nonseparable Banach space X, such that every bounded operator T on X can be written as $cI + S$, where the operator S has a separable range. Wark [28] constructed a similar space X which is moreover reflexive. Argyros & Tolias [3] gave an example of a Banach space X on which every operator is the multiple of the identity plus a weakly compact operator; see also [4]. Spaces having few operators are connected with hereditarily indecomposable Banach spaces and the celebrated Gowers dichotomy, see Gowers [11], Gowers & Maurey [12]; Maurey [19] presents a detailed survey of this subject.

In a Banach space $C(K)$ of continuous functions on a compact space K there are of course operators of the form $g \cdot I$, where $g \in C(K)$. In a recent paper [17] Koszmider proved the following result.

Theorem 1.1 (Koszmider) *Assuming the continuum hypothesis (CH), there is an infinite compact connected Hausdorff space K such that every bounded operator $T : C(K) \to C(K)$ is of the form $T = g \cdot I + S$, where $g \in C(K)$ and the operator $S : C(K) \to C(K)$ is weakly compact.*

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We shall say that an infinite compact Hausdorff space K is a **Koszmider space** if every $T : C(K) \to C(K)$ can be written as in the theorem above, i.e. as $T = g \cdot I + S$ for some weakly compact S.

Spaces of the form $C(K)$ can hardly concur with other Banach spaces, when we look for examples admiting few operators. However, Theorem 1.1 contributes considerably to the isomorphic theory of Banach spaces of continuous functions, as it is outlined in the next theorem.

Theorem 1.2 (Koszmider) *(1) If K is a Koszmider space then the Banach space $C(K)$ is not isomorphic to any of its proper subspaces; consequently, the spaces $C(K)$ and $C(K+1) = C(K) \times \mathbb{R}$ are not isomorphic.*

(2) If K is a connected Koszmider space then $C(K)$ is isomorphic to no space $C(L)$, where L is compact and zerodimensional.

Thus Theorem 1.2 solves negatively two long standing problems on isomorphisms between $C(K)$ spaces, see Semandeni [24], page 381. The solutions based on Theorem 1.1 are given under CH; however in [17] the existence of a space K as in Theorem 1.2(1) is proved in ZFC by another argument*. Recall that by the classical Banach–Mazur theorem if $C(K)$ is **isometric** to $C(L)$ then K and L are homeomorphic, but isomorphisms between $C(K)$ and $C(L)$ can ignore topological structure of the underlying compact spaces: by Miljutin's theorem, $C(K)$ is isomorphic to $C[0, 1]$ for every uncountable compact metric space K, see [24], 21.5.10. This implies that $C(K)$ is isomorphic to $C(K+1)$ for every compact metrizable K (here $K + 1$ stands for a space K with an additional isolated point). In fact one can check that $C(K)$ is isomorphic to $C(K+1)$ if K contains a nontrivial converging sequence. This may be derived from the fact that in such a case $C(K)$ contains a complemented copy of c_0, see Pełczyński [21]. $C(K)$ is also isomorphic to $C(K + 1)$ whenever $C(K)$ contains an isomorphic copy of l_∞.

Part (1) of Theorem 1.2 shows that in general $C(K + 1)$ is different from $C(K)$. A related result was earlier proved by Marciszewski [20], who constructed a compact space K for which $C_p(K)$ and $C_p(K + 1)$ are not isomorphic (C_p indicates that the topology of pointwise convergence is considered).

Part (2) may be contrasted with the case when K is uncountable and metrizable, in which $C(K)$ is isomorphic to $C(\{0, 1\}^\omega)$ by Miljutin's theorem. Pełczyński [21] proved an analogous result for some classes of nonmetrizable spaces; cf. Bessaga & Pełczyński [5] and Argyros & Arvanitakis [2].

* * *

The aim of the present paper is to eliminate CH from Koszmider's result described in Theorem 1.1; in other words we shall prove that the following is a theorem of ZFC.

Theorem 1.3 *There is a connected Koszmider space.*

*In November 2003, P. Koszmider informed us that the final version of [17] (Mathematische Annalen, to appear) presents also a ZFC proof of Theorem 1.2(2).

Theorem 1.1 is obtained in [17] by quite an involved inductive construction using inverse limits, and it seems that there is no shortcut to such a result. We use several fine ideas and delicate arguments from [17] but our general strategy is different, and may be outlined as follows.

In section 2 we give a preliminary analysis of connections between various types of operators, based on [17]. In section 3 we single out a purely topological property of a given space K that enables us to handle all operators on $C(K)$. This property called (H) prevents K from containing converging sequence as well as too many copies of $\beta\omega$. Designing (H) we build on ideas due to Haydon [14], who constructed a Banach space $C(K)$ with the so called Grothendieck property, not containing l_∞. Our property can be expressed in terms of some lattice of basic closed sets in K. The next step is to show that there is a lattice $\mathfrak{L}$ which satisfies a lattice analogue of property (H). Such $\mathfrak{L}$ is constructed as a sublattice of a measure algebra $\mathfrak{A}$ of Maharam type $\mathfrak{c}$, where there are enough independent elements (let us recall here that such algebras provide a framework for various measure–theoretic constructions, see e.g. Plebanek [22]). In section 4 we develop techniques of extending lattices in $\mathfrak{A}$ to larger ones that have additional properties. Using those auxiliary results, in section 5 we construct the desired lattice $\mathfrak{L}$, and then define a Koszmider space K as the Wallman representation of $\mathfrak{L}$. Basic facts on Wallman duality are mentioned in the appendix.

Working in the algebra $\mathfrak{A}$ mentioned above has this advantage that we can carry out induction of length $\mathfrak{c}$ in the usual set theory. Having CH or some weaker axioms granted, we might work in the power set of ω, see section 6 for more details. This would produce a separable connected Koszmider space, as in [17]. It should be remarked that the space we mean in Theorem 1.3 has density character $\mathfrak{c}$. However, we lose very little: our space satisfies the countable chain condition (*ccc*), so in particular every weakly compact subset of $C(K)$ is separable, and therefore every weakly compact operator on $C(K)$ has a separable range (by a result due to Rosenthal [23], see also Todorčević [27]).

We have already explained why it is worth troubling with getting a connected example. We remark in section 6 that our way to a zerodimensional Koszmider space would be much shorter: we could consider algebras instead of lattices and we need only one result from the preparatory section 4, where much space is devoted to handling connectedness and normality (note that we still get a Koszmider space K in this way; the zerodimensional space mentioned in [17] has weaker properties).

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2. Weakly compact and centripetal operators

The letter K will always denote a compact Hausdorff space and $C(K)$ the (real) Banach space of continuous functions equipped with supremum norm. Recall that the conjugate space $C(K)^*$ is usually identified with the space $M(K)$ of all signed Radon measures μ on K of bounded variation $|\mu|$, see e.g. [6] or [24]; thus we often write $\mu(f)$ rather than $\int f \, \mathrm{d}\mu$.

We say that a sequence $(f_n)_n$ in $C(K)$ is *disjoint* if $f_n \cdot f_k = 0$ for $n \neq k$. By definition an operator $T : X \to Y$ between Banach spaces X, Y is *weakly compact* if it sends the unit ball of X into a relatively weakly compact subset of Y. Weak compactness of operators from $C(K)$ to any Banach space Y may be characterized in the following convenient way, see [6], Corollary 17 on page 160.

Theorem 2.1 *A bounded operator $T : C(K) \to Y$ is weakly compact if and only if $\lim_{n\to\infty} ||Tf_n|| = 0$ for every bounded disjoint sequence $(f_n)_n$ in $C(K)$.*

Following Koszmider [17] we shall compare the condition appearing in the last theorem with a weaker notion of centripetality.

Definition 2.2 *We say that a bounded operator $T : C(K) \to C(K)$ is* centripetal *if for every bounded disjoint sequence $(f_n)_n \subseteq C(K)$ we have*

$$\lim_{n\to\infty} \sup\{|Tf_n(x)| : \ x \in K \setminus S_n\} = 0,$$

where $S_n = \mathrm{supp}(f_n) = \overline{\{x \in K : \ f_n(x) \neq 0\}}$.

We shall also use a certain topological property of a space K considered in [17]; it will become crucial at some stage so one should give it a proper name.

Definition 2.3 *We say that a space K contains an* open butterfly *if there are open sets $U, V \subseteq K$ such that $\overline{U} \cap \overline{V}$ is a singleton.*

Note that an isolated point of K is also an open butterfly by this definition (a degenerated one!). It is easy to construct an open butterfly in a space having a countable base at some point. On the other hand, if either K is an extremally disconnected space without isolated points or an uncountable product of nontrivial metric spaces then K contains no such butterflies. The absence of open butterflies in a compact space K implies that for every $x \in K$ the space $K \setminus \{x\}$ is C^*–embedded in K, see Lemma 2.4 below. The latter property has been investigated in the space $\omega^* = \beta\omega \setminus \omega$, see Hart & van Mill [13], Answer 17 and Question 15; consistently, for every $p \in \omega^*$ the space $\omega^* \setminus \{p\}$ is not C^*–emdedded in ω^* and thus ω^* contains open butterflies.

We shall consider here compact spaces containing no **open** butterflies. There exists a notion of a butterfly point (or b–point) which has a different meaning: x is a *butterfly point* in a space K if x is a cluster point of two closed sets F, H in K such that $\{x\} = F \cap H$; see [13] for futher references.

The following observation is taken form [17].

Lemma 2.4 *Let K be a compact space containing no open butterfly, and let $\varphi : K \to \mathbb{R}$ be a bounded function such that the set D of points of discontinuoity of φ is of cardinalty less than $\mathfrak{c}$. Then $\varphi_{|K\setminus D}$ admits an extension to a continuous function g on K.*

Proof. Note first that since K contains no open butterfly, for every two open sets $V, V' \subseteq K$ the set $\overline{V} \cap \overline{V'}$ cannot have isolated points. Therefore, using the usual dyadic construction it is easy to verify the following.

If $V, V' \subseteq K$ are open and $\overline{V} \cap \overline{V'} \neq \emptyset$ then $\operatorname{card}(\overline{V} \cap \overline{V'}) \geq \mathfrak{c}$.

To prove that $\varphi_{|K\setminus D}$ admits a continuous extension we shall check that if $x \in D$ then for every $\varepsilon > 0$ there is a neighbourhood V of x and $r \in \mathbb{R}$ such that $|\varphi(y) - r| < \varepsilon$ whenever $y \in V \setminus D$.

Suppose otherwise; choose open interval J, J' in $\mathbb{R}$ with disjoint closures, such that for every open $V \ni x$ there are $y, y' \in V \setminus D$ such that $\varphi(y) \in J, \varphi(y') \in J'$. For some open $V, V' \subseteq K$ we have

$$\varphi^{-1}[J] \setminus D = V \setminus D, \quad \varphi^{-1}[J'] \setminus D = V' \setminus D.$$

It follows that $\overline{V} \cap \overline{V'} \neq \emptyset$ (since x is in this intersection) so $\operatorname{card}(\overline{V} \cap \overline{V'}) \geq \mathfrak{c}$. On the other hand, φ is discontinuous at each $y \in \overline{V} \cap \overline{V'}$; indeed, if $U \ni y$ then $U \cap V \neq \emptyset$ so $U \cap V \setminus D \neq \emptyset$ (as K has no isolated points, every nonempty open set is of cardinality $\geq \mathfrak{c}$). Hence there is $z \in U$ such that $\varphi(z) \in J$; accordingly there is $z' \in U$ with $\varphi(z') \in J'$. We get a contradiction, and the proof is complete. ★

The theorem given below shows how one can represent centripetal operators; this is Lemma 6.3 from [17]. We enclose the proof, mainly for completeness, but also to point out some simplifications.

Theorem 2.5 *Suppose that a compact space K contains no open butterfly. If $T : C(K) \to C(K)$ is a centripetal operator then there is $g \in C(K)$ and a weakly compact operator $S : C(K) \to C(K)$ such that $T = g \cdot I + S$.*

Proof. We shall first find a candidate for such a function g. For $x \in K$ we denote by $\delta_x \in M(K)$ the Dirac measure at x. Consider the conjugate operator $T^* : M(K) \to M(K)$ defined by the formula $T^*\nu(f) = Tf(\nu)$, where $\nu \in M(K)$, $f \in C(K)$. Given $x \in K$, there is a unique real number $\varphi(x)$ such that

$$T^*\delta_x = \varphi(x)\delta_x + \mu_x, \text{ i.e. } Tf(x) = \varphi(x)f(x) + \mu_x(f) \text{ for every } f \in C(K),$$

where the measure μ_x satisfies $\mu_x(\{x\}) = 0$. We now examine properties of the so defined function $\varphi : K \to \mathbb{R}$.

Note first that since μ_x vanishes at x and $|\mu_x|$ is outer regular, we can find for every ε an open set $V \ni x$ such that $|\mu_x|(V) < \varepsilon$. Taking a continuous function $f : K \to [0, 1]$ such that $f(x) = 1$, $f = 0$ outside V, we infer that $|\mu_x(f)| \leq |\mu_x|(V) < \varepsilon$, and so

$|\varphi(x)| \leq ||T|| + \varepsilon$. Therefore φ is uniformly bounded by $||T||$; for the rest of the proof we simply assume that $||T|| \leq 1$.

We write $\mathrm{osc}_x(\varphi)$ for the oscillation of φ at x.

Claim. If $\varepsilon > 0$ then $\mathrm{osc}_x(\varphi) > \varepsilon$ for at most finite number of points $x \in K$. In particular the set D of points of discontinuity of φ is countable.

Suppose otherwise and choose a sequence $(x_n)_n$ of distinct points in K with $\mathrm{osc}_{x_n}(\varphi) > \varepsilon$. Passing to a subsequence if necessary, we can already assume that x_n's are isolated, that is there are disjoint open sets $U_n \ni x_n$. We fix some $\eta > 0$, and for every n do the following.

Choose open V_n such that $x_n \in V_n \subseteq U_n$ and $|\mu_{x_n}|(V_n) < \eta$. Next fix a continuous function $f_n : K \to [0,1]$ which satisfies $f_n(x_n) = 1$ and vanishes outside V_n. Using continuity of the functions f_n and Tf_n, and the fact that $\mathrm{osc}_{x_n}(\varphi) > \varepsilon$ we can now choose a point $y_n \in V_n$ so that the following are satisfied

$$|1 - f_n(y_n)| < \eta;$$

$$|Tf_n(x_n) - Tf_n(y_n)| < \eta;$$

$$|\varphi(x_n) - \varphi(y_n)| \geq \varepsilon.$$

Finally we choose an open set H_n such that $y_n \in H_n \subseteq V_n$, $\mu_{y_n}(H_n) < \eta$, and a continuous function $h_n : K \to [0,1]$ such that $h_n = f_n$ on some neighbourhood of y_n, while h_n is 0 outside H_n. Now

$$T(f_n)(x_n) = \varphi(x_n) + \mu_{x_n}(f_n),$$

$$T(h_n)(y_n) = \varphi(y_n)h_n(y_n) + \mu_{y_n}(h_n).$$

Since $h_n(y_n) = f_n(y_n)$ is close to 1, $|\varphi(y_n)| \leq 1$ and $|\mu_{x_n}(f_n)| < \eta$ we get

$$|T(f_n)(x_n) - Th_n(y_n)| \geq \varepsilon - 3\eta.$$

Consequently,

$$|T(f_n - h_n)(y_n)| = |Tf_n(y_n) - Th_n(y_n)| =$$

$$|Tf_n(y_n) - Tf_n(x_n) + Tf_n(x_n) - Th_n(y_n)| \geq \varepsilon - 4\eta.$$

Taking any $\eta < \varepsilon/4$ we get a contradiction, since T is to be centripetal but y_n is not in the support of $f_n - h_n$. The claim is verified.

We now apply Lemma 2.4; let $g \in C(K)$ be a function extending $\varphi_{|K\setminus D}$. It remains to check that the operator $T - g \cdot I$ is weakly compact.

Suppose that $(f_n)_n$ is a bounded disjoint sequence in $C(K)$ for which $||Tf_n - gf_n|| > \varepsilon > 0$. Write $V_n = \{x : f_n(x) \neq 0\}$; we can for every n find $x_n \in V_n \setminus D$ such that $|Tf_n(x_n) - g(x_n)f_n(x_n)| > \varepsilon$. Again we find open set H_n such that $x_n \in H_n \subseteq V_n$,

$|\mu_{x_n}|(H_n) < \varepsilon/2$, and continuous functions $h_n : K \to [0,1]$ such that $h_n = f_n$ on some neighbourhood of x_n, while $h_n = 0$ outside H_n. Then

$$|T(f_n - h_n)(x_n)| = |Tf_n(x_n) - Th_n(x_n)| \geq$$

$$\geq |Tf_n(x_n) - g(x_n)f_n(x_n)| - |g(x_n)f_n(x_n) - Th_n(x_n)| \geq \varepsilon - |\mu_{x_n}(h_n)) \geq \varepsilon/2,$$

where we made use of $f_n(x_n) = h_n(x_n)$ and $g(x_n) = \varphi(x_n)$. This contradicts centripetality; the proof is complete. ★

3. Koszmider spaces

In this section we single out a purely topological property of a compact space K which makes all the operators $T : C(K) \to C(K)$ centripetal. The proof of Theorem 3.2 below uses ideas from Lemma 6.2 of [17] and our Lemma 3.3 and Lemma 3.4. We denote by ω the set of natural numbers and $[\omega]^\omega$ stands for the family of all infinite subsets of ω.

Definition 3.1 *We shall say that a compact space K has property (H) if there is a dense set $D \subseteq K$ such that whenever*

(i) $(F_n)_n$ is a sequence of closed subsets of K;

(ii) $(V_n)_n$ is a pairwise disjoint sequence of open sets with $F_n \subseteq V_n$ for every n;

(iii) $(d_n)_n$ is a sequence in D such that $\overline{\{d_n : n < \omega\}} \cap \overline{\bigcup_{n<\omega} V_n} = \emptyset$;

then there are infinite sets $\tau \subseteq \sigma \subseteq \omega$ such that

$$\overline{\bigcup_{n\in\tau} F_n} \subseteq \mathrm{int}\overline{\bigcup_{n\in\sigma} V_n} \quad \textit{and} \quad \overline{\{d_n : n \in \tau\}} \cap \overline{\{d_n : n \in \omega \setminus \sigma\}} \neq \emptyset.$$

In the zerodimensional case we might consider somewhat more transparent property (H') mentioned in section 6. Note that (H) does not allow $C(K)$ to contain a copy of l_∞, see Haydon [14], Proposition 1C; we shall see below that (H) prevents K from containing converging sequences.

The rest of the present section is devoted to proving the following.

Theorem 3.2 *If K is an infinite compact space with property (H) then every bounded operator $T : C(K) \to C(K)$ is centripetal. If moreover K contains no open butterfly then K is a Koszmider space.*

The second statement in the theorem above follows directly from the first one and Theorem 2.1 so we shall concentrate on proving the first part. We first analyse some consequences of property (H). A sequence $(F_n)_n$ as in (ii) of Definition 3.1 will be called strongly disjoint.

Lemma 3.3 *Let K be a compact space with property (H).*

(1) For any strongly disjoint sequence $(F_n)_{n<\omega}$ of closed subsets of K there are two infinite sets $\tau_1, \tau_2 \subseteq \omega$ such that

$$\overline{\bigcup_{n\in\tau_1} F_n} \cap \overline{\bigcup_{n\in\tau_2} F_n} = \emptyset.$$

(2) If F_n, V_n satisfy (i) and (ii) of Definition 3.1 and $(d_n)_n$ is a sequence in $D \setminus \bigcup_n \overline{V_n}$ then there is infinite $\sigma \subseteq \omega$ and open sets U_n such that $F_n \subseteq U_n \subseteq \overline{U_n} \subseteq V_n$ and

$$\overline{\{d_n : n \in \sigma\}} \cap \overline{\bigcup_{n\in\sigma} U_n} = \emptyset.$$

Proof. Recall first that if U, W are disjoint open sets (in any space) then

$$(*) \qquad \mathrm{int}\overline{U} \cap \mathrm{int}\overline{W} \subseteq \overline{U \cap W},$$

for if $x \in \mathrm{int}\overline{U} \cap \mathrm{int}\overline{W}$ and V is a neighbourhood of x then there is open H, $x \in H \subseteq V$, such that $H \subseteq \overline{U} \cap \overline{W}$; then $H \cap V \neq \emptyset$, and $H \cap V \subseteq \overline{W}$ so $H \cap U \cap W \neq \emptyset$.

ad (1) If open sets V_n separate F_n then we apply (H) twice, first to $(F_{2n})_n$, $(V_{2n})_n$, and then to $(F_{2n+1})_n$, $(V_{2n+1})_n$. In this way we get two disjoint infinite σ_1, σ_2 and infinite $\tau_1 \subseteq \sigma_1$, $\tau_2 \subseteq \sigma_2$ such that

$$\overline{\bigcup_{n\in\tau_1} F_n} \subseteq \mathrm{int}\overline{\bigcup_{n\in\sigma_1} V_n}, \quad \overline{\bigcup_{n\in\tau_2} F_n} \subseteq \mathrm{int}\overline{\bigcup_{n\in\sigma_2} V_n}.$$

Hence we get the result by (*).

ad (2). Let $\Sigma \subseteq [\omega]^\omega$ be an uncountable almost disjoint family. For every n we choose an open set U_n with $F_n \subseteq U_n \subseteq \overline{U_n} \subseteq V_n$. Applying (H) to each pair of sequences $(\overline{U_n})_{n\in\sigma}$, $(V_n)_{n\in\sigma}$, we find an infinite set $\tau(\sigma) \subseteq \sigma$ such that

$$(**) \qquad \overline{\bigcup_{n\in\tau(\sigma)} U_n} \subseteq \mathrm{int}\overline{\bigcup_{n\in\sigma} V_n}.$$

Note that it follows from (*) that for every k we can have $d_k \in \mathrm{int}\overline{\bigcup_{n\in\sigma} V_n}$ for at most one $\sigma \in \Sigma$. As Σ is uncountable, we may hence find $\sigma \in \Sigma$ such that $\mathrm{int}\overline{\bigcup_{n\in\sigma} V_n}$ contains no d_k and we get a required subsequence for $n \in \tau(\sigma)$ by (**). ★

Lemma 3.4 *Let K be a space with property (H) and suppose that we are given a sequence $(\mu_k)_{k<\omega}$ of nonnegative Radon measures on K. If $(F_n)_{n<\omega}$ is a strongly disjoint sequence of closed sets in K then there is an infinite set $\sigma \subseteq \omega$ such that for every $\tau \subseteq \sigma$ and for every k we have*

$$\mu_k(\overline{\bigcup_{n\in\tau} F_n}) = \sum_{n\in\tau} \mu_k(F_n).$$

Proof. For any $\sigma \subseteq \omega$ write

$$F_\sigma = \bigcup_{n\in\sigma} F_n; \quad F^*_\sigma = \overline{\bigcup_{n\in\sigma} F_n} \setminus \bigcup_{n\in\sigma} F_n.$$

We construct an almost disjoint family $(\sigma_\alpha)_{\alpha<\omega_1}$ in $[\omega]^\omega$ as follows. Using Lemma 3.3(1) we find σ_0, π_0 such that $\overline{F_{\sigma_0}} \cap \overline{F_{\pi_0}} = \emptyset$. Given σ_β, π_β for $\beta < \alpha < \omega_1$ we find a infinite set N that is almost contained in every π_β and again find $\sigma_\alpha, \pi_\alpha \subseteq N$ such that $\overline{F_{\sigma_\alpha}} \cap \overline{F_{\pi_\alpha}} = \emptyset$.

It follows from the construction that for every $\beta < \alpha < \omega_1$ we have

$$F^*_{\sigma_\alpha} \cap F^*_{\sigma_\beta} = \emptyset.$$

Indeed, σ_α is almost contained in π_β so

$$\overline{F_{\sigma_\alpha}} \cap \overline{F_{\sigma_\beta}} \subseteq \bigcup_{n\in\sigma_\alpha\cap\sigma_\beta} F_n.$$

It is now clear that $\mu_k(F^*_{\sigma_\alpha}) = 0$ for every k and all but countable many $\alpha < \omega_1$. Therefore we can fix $\alpha < \omega_1$ such that writing $\sigma = \sigma_\alpha$ we have $\mu_k(F^*_\sigma) = 0$ for all k, hence

$$(**) \qquad \mu_k(\overline{F_\sigma}) = \sum_{n\in\sigma} \mu_k(F_n).$$

Now if $\tau \subseteq \sigma$ then $F^*_\tau \subseteq F^*_\sigma$ since F_n's are strongly disjoint. Hence $\mu_k(F^*_\tau) = 0$, and we are done. ★

We are almost ready to present the proof of (the first statement of) Theorem 3.2; it remains to recall the classical Rosenthal lemma, see e.g. [6], page 18.

Theorem 3.5 *(Rosenthal lemma) Let $(\nu_n)_{n<\omega}$ be a uniformly bounded sequence of measures defined on a σ–algebra $\mathcal{A}$ of sets. Then for every disjoint sequence $(A_n)_{n<\omega}$ in $\mathcal{A}$ and $\varepsilon > 0$ there is a set $\sigma \in [\omega]^\omega$ such that for every $n \in \sigma$*

$$\nu_n\Big(\bigcup_{k\in\sigma\setminus\{n\}} A_k\Big) \le \varepsilon.$$

Proof. (of Theorem 3.2) Suppose that a bounded operator $T : C(K) \to C(K)$ is not centripetal and let $(f_n)_{n<\omega}$ be a disjoint sequence in the unit ball of $C(K)$ such that for some $\varepsilon > 0$ there are $x_n \in K \setminus S_n$ satisfying $|T(f_n)(x_n)| > 2\varepsilon$. Here we write S_n for the support of f_n. Since D is dense in K, we can moreover assume that $x_n \in D$ for every n. Let $\eta > 0$ be some smaller constant (to be chosen later).

For every n we make following choices. Put $\mu_n = T^*(\delta_{x_n})$, i.e. $\mu_n \in M(K)$ is such that $\mu_n(f) = \delta_{x_n}(Tf) = (Tf)(x_n)$ for all $f \in C(K)$. In particular, we have

$$\Big|\int f_n \, \mathrm{d}\mu_n\Big| = |\mu_n(f_n)| = |(Tf)(x_n)| > 2\varepsilon,$$

so there is a closed set $F_n \subseteq U_n = \{x : f_n(x) \neq 0\}$ such that $|\mu_n(F_n)| > \varepsilon$. Find an open set V_n such that

$$F_n \subseteq V_n \subseteq \overline{V_n} \subseteq U_n, \quad |\mu_n|(\overline{V_n} \setminus F_n) < \eta.$$

Now we shall carry out several reductions (passing to subsequences several times).

(1) We can assume that $x_n \notin \overline{V_m}$ for every n and m (recall that $\overline{V_m}$ are pairwise disjoint).

(2) We can assume by (1) and Lemma 3.3(2) that $\overline{\{x_n : n \in \omega\}} \cap \overline{\bigcup_{n\in\omega} V_n} = \emptyset$.

(3) We can assume, by Lemma 3.4, that whenever $\sigma \in [\omega]^\omega$ then for every k

$$|\mu_k|(\overline{\bigcup_{n\in\sigma} V_n}) = \sum_{n\in\sigma} |\mu_k|(\overline{V_n}).$$

(4) We can assume by Theorem 3.5 applied to $\nu_k = |\mu_k|$ that for every n

$$|\mu_n|(\bigcup_{k\neq n} \overline{V_k}) < \eta.$$

Having (1)–(4) granted, we at last apply property (H) and fix infinite sets $\tau \subseteq \sigma$ with the properties

(a) $F = \overline{\bigcup_{n\in\tau} F_n} \subseteq V = \mathrm{int}\overline{\bigcup_{n\in\sigma} V_n}$;

(b) $\overline{\{x_n : n \in \tau\}} \cap \overline{\{x_n : n \in \omega \setminus \sigma\}} \neq \emptyset$.

Then in view of (3) and (4)

(c) $|\mu_n|(V \setminus \overline{V_n}) < \eta$ for every n.

Now we take a continuous function $h : K \to [0,1]$ such that $h = 1$ on F and $h = 0$ outside V, and analyse the properties of $g = Th \in C(K)$.

If $i \in \tau$ then

$$|g(x_i)| = |\mu_i(h)| = |\int_V h \,\mathrm{d}\mu_i| \geq |\int_{\overline{V_i}} h \,\mathrm{d}\mu_i| - \int_{V\setminus\overline{V_i}} h \,\mathrm{d}|\mu_i| \geq |\int_{\overline{V_i}} h \,\mathrm{d}\mu_i| - \eta,$$

by (c) and since h vanish outside V. Moreover,

$$|\int_{\overline{V_i}} h \,\mathrm{d}\mu_i| \geq |\mu_i(F_i)| - \eta,$$

since $h = 1$ on F_i and $|\mu_i|(\overline{V_i} \setminus F) < \eta$. We conclude that $|g(x_i)| \geq \varepsilon - 2\eta$ for $i \in \tau$.

On the other hand, if $j \in \omega \setminus \sigma$ then using (c) again

$$|g(x_j)| = |\mu_j(h)| = |\int_V h \, \mathrm{d}\mu_j| \le |\mu_j|(\bigcup_{n\in\tau} \overline{V_n}) \le |\mu_j|(\bigcup_{n\neq j} \overline{V_n}) \le \eta.$$

Letting $\eta = \varepsilon/4$ we get

$$\{d_n:\ n \in \tau\} \subseteq \{g \ge \varepsilon/2\}, \quad \{d_n:\ n \in \omega \setminus \sigma\} \subseteq \{g \le \varepsilon/4\},$$

a contradiction with (b), and the proof is complete. ★

4. Lattices

In this section we develop techniques of constructing lattices in Boolean algebras with desired properties. We start by fixing the terminology that is used in the sequel, and then in a few subsections show how to enlarge a given lattice and preserve some properties at the same time. We shall work here in a fixed (σ–)complete Boolean algebra $\mathfrak{A}$ and denote by $\mathbf{S}$ the Stone space of (all ultrafilters on) $\mathfrak{A}$. Later we choose $\mathfrak{A}$ to be the measure algebra of Maharam type $\mathfrak{c}$ (see Fremlin [8] for the basic facts concerning measure algebras). Given $a \in \mathfrak{A}$, we write $\widehat{a} \subseteq \mathbf{S}$ for the corresponding clopen set.

We denote Boolean operations by $\vee$, $\wedge$, ^{c}; however we also write $a \cdot b$ for $a \wedge b$, with the convention that $a \cdot b \vee c = (a \wedge b) \vee c$ etc (this is to avoid too many brackets).

By a lattice in $\mathfrak{A}$ we mean a family $\mathfrak{L} \subseteq \mathfrak{A}$ containing 0 and 1, and closed under the operation $\vee$ and $\wedge$. If $\mathfrak{L}$ is a lattice then

$$\mathfrak{L}^c = \{a \in \mathfrak{A}:\ a^c \in \mathfrak{L}\},$$

is also a lattice (called the dual lattice).

Definition 4.1 *Let $\mathfrak{L}$ be a lattice in $\mathfrak{A}$.*

(i) $\mathfrak{L}$ *is* connected *if* $\mathfrak{L} \cap \mathfrak{L}^c = \{0, 1\}$.

(ii) $\mathfrak{L}$ *is said to be* normal *if for every disjoint $a, b \in \mathfrak{L}$ there are disjoint $u, v \in \mathfrak{L}^c$ such that $a \le u$, $b \le v$.*

(iii) $\mathfrak{L}$ *is* disjunctive *if for every $a, b \in \mathfrak{L}$ such that $a \cdot b^c \neq 0$ there is $x \in \mathfrak{L}$ with $x \cdot a \neq 0$, $x \cdot b = 0$.*

Definition 4.2 *Let P and Q be subsets of the Stone space $\mathbf{S}$. We say that $a \in \mathfrak{A}$* isolates *P from Q if $P \subseteq \widehat{a} \subseteq S \setminus Q$, in other words $a \in \mathcal{F}$ for every $\mathcal{F} \in P$ and $a^c \in \mathcal{F}$ for every $\mathcal{F} \in Q$.*

We also say that a lattice $\mathfrak{L}$ isolates *(P, Q) if there is an element x lying in the algebra generated by $\mathfrak{L}$ such that x isolates P from Q.*

Definition 4.3 *Given $x, y \in \mathfrak{A}$, we say that a lattice $\mathfrak{L}$* separates (x, y) *if there are disjoint $a, b \in \mathfrak{L}$ such that $x \leq a$ and $y \leq b$.*

If $\mathfrak{L}$ is a lattice and $x, y \in \mathfrak{A}$ then we write $\mathfrak{L}(x, y)$ for the lattice generated by $\mathfrak{L} \cup \{x, y\}$. The proof of the following fact is straighforward.

Lemma 4.4 *For any lattice $\mathfrak{L}$ and disjoint $a, b \in \mathfrak{A}$ we have*

$$\mathfrak{L}(a, b) = \{a \cdot x \vee b \cdot y \vee z : \ x, y, z \in \mathfrak{L}\}.$$

4.1 Adding new elements

In this subsection we consider a fixed lattice $\mathfrak{L} \subseteq \mathfrak{A}$ of cardinality $< \mathfrak{c}$, a fixed sequence $(a_n)_n$ in $\mathfrak{L}$, and a pairwise disjoint $(v_n)_n$ in $\mathfrak{L}^c$, such that $a_n \leq v_n$ for every n. For $\sigma \subseteq \omega$ we write

$$a_\sigma = \bigvee_{n \in \sigma} a_n, \qquad v_\sigma = \bigvee_{n \in \sigma} v_n.$$

Note that infinite operation are taken in $\mathfrak{A}$, and in general $a_\sigma \notin \mathfrak{L}$. Our aim here is to show that by careful choice of σ, τ the lattice $\mathfrak{L}(a_\tau, v_\sigma^c)$ will preserve some properties of $\mathfrak{L}$. In the sequel, we consider an almost disjoint family $\Sigma \subseteq [\omega]^\omega$ which is of cardinality $\mathfrak{c}$. We say that some property holds *for almost all* $\sigma \in \Sigma$ if the exceptional set is of cardinality $< \mathfrak{c}$. Note that

$$a_{\sigma \cap \tau} = \bigvee_{n \in \sigma \cap \tau} a_n \in \mathfrak{L},$$

whenever $\sigma, \tau \in \Sigma$ are distinct. This basic trick is taken from Haydon [14].

Lemma 4.5 *Suppose that the pair (p, q) of elements from $\mathfrak{A}$ is not separated by $\mathfrak{L}$ of cardinality $< \mathfrak{c}$. Then for almost all $\sigma \in \Sigma$ the lattice $\mathfrak{L}(a_\sigma, v_\sigma^c)$ does not separate (p, q).*

Proof. Otherwise, since $|\mathfrak{L}| < \mathfrak{c}$, the extended lattices are not connected 'for the same reasons', i.e. there are $x, x', y, y', z, z' \in \mathfrak{L}$ such that for two different $\sigma, \tau \in \Sigma$ the elements

$$l = x \cdot a_\sigma \vee y \cdot v_\sigma^c \vee z \qquad l' = x' \cdot a_\sigma \vee y' \cdot v_\sigma^c \vee z',$$

$$k = x \cdot a_\tau \vee y \cdot v_\tau^c \vee z \qquad k' = x' \cdot a_\tau \vee y' \cdot v_\tau^c \vee z',$$

satisfy $l \cdot l' = 0$, $k \cdot k' = 0$, $p \leq l, k$ and $q \leq l', k'$. Consider m, m', where

$$m = x \cdot a_\sigma \cdot a_\tau \vee y \cdot (v_\sigma^c \vee v_\tau^c) \vee z, \qquad m' = x' \cdot a_\sigma \cdot a_\tau \vee y' \cdot (v_\sigma^c \vee v_\tau^c) \vee z'.$$

Then $a_\sigma \cdot a_\tau = a_{\sigma \cap \tau} \in \mathfrak{L}$, $v_\sigma^c \vee v_\tau^c = (v_\sigma \cdot v_\tau)^c = v_{\sigma \cap \tau}^c \in \mathfrak{L}$, and hence $m, m' \in \mathfrak{L}$.

Now we have $p \leq m$, $q \leq m'$ and $m \cdot m' = 0$ so $\mathfrak{L}$ separates (p, q), a contradiction. ★

Lemma 4.6 *Suppose the pair (P, Q) in the Stone space is not isolated by $\mathfrak{L}$ of cardinality $< \mathfrak{c}$. Then for almost all $\sigma \in \Sigma$ the lattice $\mathfrak{L}(a_\sigma, v_\sigma^c)$ does not isolate (P, Q).*

Proof. Here we can repeat the argument from Lemma 4.5, applied to the algebra $\mathcal{A}$ of subset of $\mathbf{S}$ generated by $\{\widehat{x} : x \in \mathfrak{L}\}$ in place of $\mathfrak{L}$. Note that separation and isolation with respect to $\mathcal{A}$ means the same. ★

Lemma 4.7 *Suppose that $\mathfrak{L}$ is connected and* $\operatorname{card}(\mathfrak{L}) < \mathfrak{c}$.

(1) For almost all $\sigma \in \Sigma$ the lattice $\mathfrak{L}(a_\sigma)$ is connected.

(2) For almost all $\sigma \in \Sigma$ the lattice $\mathfrak{L}(v_\sigma^c)$ is connected.

Proof. ad (1) Suppose otherwise; as $\operatorname{card}(\mathfrak{L}) < \mathfrak{c}$, $\operatorname{card}(\Sigma) = \mathfrak{c}$ it follows that for some $x, z, x', z' \in \mathfrak{L}$, distinct $\sigma, \tau \in \Sigma$, writing

$$l = x \cdot a_\sigma \vee z \qquad l' = x' \cdot a_\sigma \vee z',$$

$$k = x \cdot a_\tau \vee z \qquad k' = x' \cdot a_\tau \vee z',$$

we have

$$l \cdot l' = 0, \quad l \vee l' = 1, \quad l, l' \neq 0;$$

$$k \cdot k' = 0, \quad k \vee k' = 1, \quad k, k' \neq 0.$$

But then $m, m' \in \mathfrak{L}$, where

$$m = x \cdot a_\sigma \cdot a_\tau \vee z, \quad m' = x' \cdot a_\sigma \cdot a_\tau \vee z',$$

since $\sigma \cap \tau$ is finite. Moreover $m \vee m' = 1$ and $m \cdot m' = 0$, so for instance $m = 1$, as $\mathfrak{L}$ is connected. This gives $l = 1$, a contradiction.

ad (2) We again argue by contradiction and infer that for some $y, z, y', z' \in \mathfrak{L}$, distinct $\sigma, \tau \in \Sigma$, writing

$$l = y \cdot v_\sigma^c \vee z \qquad l' = y' \cdot v_\sigma^c \vee z',$$

$$k = y \cdot v_\tau^c \vee z \qquad k' = y' \cdot v_\tau^c \vee z',$$

k, k', l, l' are as above. This time we define m, m' by

$$m = y \cdot (v_\sigma^c \vee v_\tau^c) \vee z, \quad m' = y' \cdot (v_\sigma^c \vee v_\tau^c) \vee z'.$$

Again $m, m' \in \mathfrak{L}$ since $v_\sigma^c \vee v_\tau^c = (v_\sigma^c \cdot v_\tau^c)^c = v_{\sigma\cap\tau}^c \in \mathfrak{L}$. As $m \vee m' = 1$ and $m \cdot m' = 0$, we have for instance $m' = 0$. This gives $l' = 0$, a contradiction. ★

Proposition 4.8 *Let $\mathfrak{L} \subseteq \mathfrak{A}$ be a connected lattice,* $\operatorname{card}(\mathfrak{L}) = \kappa < \mathfrak{c}$*, and suppose that we are given*

(i) a sequence $((P_\xi, Q_\xi))_{\xi<\kappa}$ of disjoint pairs in $\mathbf{S}$, which are not isolated by $\mathfrak{L}$.

(ii) *a sequence* $((p_\xi, q_\xi))_{\xi<\kappa}$ *of disjoint pairs in* $\mathfrak{A}$*, which are not separated by* $\mathfrak{L}$*.*

(iii) $(a_n)_n$ *in* $\mathfrak{L}$*, a disjoint sequence* $(v_n)_n$ *in* $\mathfrak{L}^c$ *such that* $a_n \le v_n$ *for every* n*;*

(iv) *a sequence* $(\mathcal{F}_n)_n$ *in* $\mathbf{S}$ *for which there is* $t \in \mathfrak{L}$ *which is disjoint from* $\bigvee_n v_n$ *and such that* $t \in \mathcal{F}_n$ *for every* n*.*

Then there are $\sigma, \tau \in [\omega]^\omega$*,* $\tau \subseteq \sigma$ *such that the lattice* $\mathfrak{K} = \mathfrak{L}(a_\tau, v_\sigma^c)$ *is connected and*

(a) $\mathfrak{K}$ *does not separate* (p_ξ, q_ξ) *and does not isolate* (P_ξ, Q_ξ) *for* $\xi < \kappa$*;*

(b) $\mathfrak{K}$ *does not isolate* $\{\mathcal{F}_n :\ n \in \tau\}$ *and* $\{\mathcal{F}_n :\ n \in \omega \setminus \sigma\}$*.*

Proof. We first note that if Σ is an almost disjoint family of size $\mathfrak{c}$ then for almost all $\sigma \in \Sigma$, whenever $\tau \subseteq \sigma$ is infinite then $\{\mathcal{F}_n :\ n \in \tau\}$ and $\{\mathcal{F}_n :\ n \in \omega \setminus \sigma\}$ are not isolated by $\mathfrak{L}$, because if $\tau_1 \subseteq \sigma_1$, $\tau_2 \subseteq \sigma_2$, where $\sigma_1, \sigma_2 \in \Sigma$ are distinct then the pairs

$$(\{\mathcal{F}_n :\ n \in \tau_1\}, \{\mathcal{F}_n :\ n \in \omega \setminus \sigma_1\}) \quad (\{\mathcal{F}_n :\ n \in \tau_2\}, \{\mathcal{F}_n :\ n \in \omega \setminus \sigma_2\})$$

can be isolated only by distinct elements of the algebra generated by $\mathfrak{L}$ (which is of size $\kappa < \mathfrak{c}$). For the rest of the proof we can hence assume that every $\sigma \in \Sigma$ has the above property.

Now by Lemmas 4.5, 4.6, 4.7 there is $\sigma \in \Sigma$ such that the lattice $\mathfrak{L}(v_\sigma^c)$ is connected and has the properties (a),(b) (note that in each case the exceptional set of 'bad' $\sigma \in \Sigma$ is of size $\le \kappa < \mathfrak{c}$). We now apply the same trick to an almost disjoint family $\Pi \subseteq [\sigma]^\omega$ of size $\mathfrak{c}$ to get $\tau \subseteq \sigma$ such that the lattice $\mathfrak{K} = \mathfrak{L}(a_\tau, v_\sigma^c)$ is connected and satisfies (a). Then $\mathfrak{K}$ satisfies also (b) by the above remark. Here we use the fact that there is $t \in \mathfrak{L}$ such that $t \in \mathcal{F}_n$ for every n, while $(\bigvee_n v_n) \cdot t = 0$, which implies that two parts of $\mathcal{F}_n$'s are not isolated by $\mathfrak{K}$ provided they are not isolated by $\mathfrak{L}$. ★

4.2 Working in a product space

Now it will become crucial that we are working in the measure algebra $\mathfrak{A}$ of Maharam type $\mathfrak{c}$. We let $\mathfrak{A}$ be the measure algebra of the usual product measure λ on $[0,1]^\mathfrak{c}$ (the measure on $\mathfrak{A}$ will be denoted again by λ). We shall recall some basic properties of λ, see Fremlin [8] for basic facts concerning measure algebras and Fremlin [8] 1.15–1.16, Fremlin [9] for properties of product measures.

The measure λ is defined on the product σ–algebra $\mathcal{B}$ in $[0,1]^\mathfrak{c}$. If $B \in \mathcal{B}$ then $B^\bullet$ denotes the corresponding element of $\mathfrak{A}$. Every $B \in \mathcal{B}$ is determined by coordinates in some countable set $I \subseteq \mathfrak{c}$, i.e. $B = B_0 \times [0,1]^{\mathfrak{c}\setminus I}$ for some Borel set $B_0 \subseteq [0,1]^I$. Given $I \subseteq \mathfrak{c}$, we denote by $\mathfrak{A}[I]$ the subalgebra of all $B^\bullet$, where B is determined by coordinates in I. Note that if $I, J \subseteq \mathfrak{c}$ are disjoint, $a \in \mathfrak{A}[I]$, $b \in \mathfrak{A}[J]$ are nonzero then $a \cdot b \neq 0$.

For every measurable set $T \subseteq [0,1]$ and $\zeta < \mathfrak{c}$ we write

$$C_\zeta^T = \{x \in [0,1]^\mathfrak{c} : \ x(\zeta) \in T\}, \quad c_\zeta^T = (C_\zeta^T)^\bullet,$$

for one dimensional cylinders. A special role is reserved for cylinders with a base $[0,1/2]$ or $[3/4,1]$, denoted by

$$c_\zeta = (\{x \in [0,1]^\mathfrak{c} : \ x(\zeta) \le 1/2\})^\bullet, \quad d_\zeta = (\{x \in [0,1]^\mathfrak{c} : \ x(\zeta) \ge 3/4\})^\bullet.$$

We shall again denote by $\mathbf{S}$ the space of all ultrafilters on $\mathfrak{A}$. For any $I \subseteq \mathfrak{c}$ we put

$$\mathbf{S}[I] = \{\mathcal{F} \in \mathbf{S} : \ c_\zeta \in \mathcal{F} \text{ for all } \zeta \in \mathfrak{c} \setminus I\}.$$

Lemma 4.9 *Let $\mathfrak{L} \subseteq \mathfrak{A}[I]$ be a lattice, $\zeta \in \mathfrak{c} \setminus I$ and $d \in \mathfrak{A}$ is such that $d \cdot c_\zeta = 0$.*

(1) If $\mathfrak{L}$ is connected then so is $\mathfrak{L}(d)$.

(2) Every pair (p,q), where $p,q \in \mathfrak{A}[I]$, separated by $\mathfrak{L}(d)$ is also separated by $\mathfrak{L}$.

(3) Every pair (P,Q), where $P,Q \subseteq \mathbf{S}[I]$, isolated by $\mathfrak{L}(d)$ is also isolated by $\mathfrak{L}$.

Exactly the same statements hold true when we consider $\mathfrak{L}(c_\zeta, d_\zeta)$ instead of $\mathfrak{L}(d)$.

Proof. (1) and (2) follow easily from the following remark: if $p \in \mathfrak{A}[I]$, $x,y \in \mathfrak{L}$ and $p \le x \cdot d \vee y$ then $p \cdot c_\zeta \le y \cdot c_\zeta \le y$, and hence $p \le y$ (otherwise, $p \cdot y^c \ne 0$ so $p \cdot y^c \cdot c_\zeta \ne 0$ by independence).

(3) is also easy since $c_\zeta \in \mathcal{F}$ for every $\mathcal{F} \in P \cup Q$. ★

Lemma 4.10 *Every connected lattice $\mathfrak{L} \subseteq \mathfrak{A}[I]$, where* $\text{card}(I) < \mathfrak{c}$ *can be extended to a connected disjunctive lattice $\mathfrak{K} \supseteq \mathfrak{L}$ such that* $\text{card}(\mathfrak{K}) = \text{card}(\mathfrak{L})$. *Moreover, $\mathfrak{K}$ can be taken so that*

(s) $\mathfrak{K}$ does not separate (p,q) whenever $p,q \in \mathfrak{A}[I]$ are not separated by $\mathfrak{L}$;

(i) $\mathfrak{K}$ does not isolate (P,Q) whenever $P,Q \subseteq \mathbf{S}[I]$ are not isolated by $\mathfrak{L}$.

Proof. If $a \in \mathfrak{L}$, $a \ne 1$ and we need to add nonzero d disjoint from a then we can apply the previous lemma for $d = a^c - c_\zeta$, where $\zeta \notin I$. Note that $\lambda(d) = (1/2)\lambda(a^c)$ so if we repeat this construction we can guarantee that a^c will be the joint of some sequence of elements from the extended lattice.

In such a way we define a connected lattice $\mathfrak{L}' \supseteq \mathfrak{L}$ with $\text{card}(\mathfrak{L}') = \text{card}(\mathfrak{L})$, which satisfies (s) and (i) and the following.

(*) For every $a \in \mathfrak{L}$, a^c can be written as $\bigvee_n b_n$ for some $b_n \in \mathfrak{L}'$.

Now we iterate the construction of $\mathfrak{L}'$ to get lattices $\mathfrak{L} \subseteq \mathfrak{L}' \subseteq \mathfrak{L}'' \subseteq \ldots$, and finally let $\mathfrak{K}$ be

$$\mathfrak{K} = \mathfrak{L} \cup \mathfrak{L}' \cup \mathfrak{L}'' \cup \ldots$$

It is clear that $\mathfrak{K}$ is disjunctive, for if $b \cdot a^c \ne 0$ then $a^c = \bigvee_n b_n$ for some $b_n \in \mathfrak{K}$; therefore there must be n such that $b \cdot b_n \ne 0$. ★

For the sake of the next lemma let us say that a lattice $\mathfrak{K}$ is normal at the pair (a, b) of disjoint elements of $\mathfrak{K}$ if there are disjoint $u, v \in \mathfrak{K}^c$ such that $a \leq u$ and $b \leq v$. In other words, for some $\tilde{a}, \tilde{b} \in \mathfrak{K}$ we have $a \leq \tilde{a}, b \leq \tilde{b}$, $\tilde{a} \cdot b = 0$, $\tilde{b} \cdot a = 0$, $\tilde{a} \vee \tilde{b} = 1$.

Lemma 4.11 *Let $\mathfrak{L} \subseteq \mathfrak{A}[I]$ be a connected lattice, where $I \subseteq \mathfrak{c}$ is of size $< \mathfrak{c}$. Given disjoint $a, b \in \mathfrak{L}$, there is a connected lattice $\mathfrak{K} \supseteq \mathfrak{L}$ which is normal at (a, b), and moreover $\mathfrak{K}$ satisfies conditions (s) and (i) from Lemma 4.10*

Proof. (I) In the first part of the proof we show how to make the lattice normal at (a, b) preserving connectedness.

We fix a sequence $(s_n)_n$ of nonzero elements of $\mathfrak{A}[I]$ such that

$$(a \vee b)^c = \bigvee_n s_n.$$

Next fix $\zeta \in \mathfrak{c} \setminus I$, and for $n = 1, 2, \ldots$ put

$$t_n = c_\zeta^{T_n}, \quad \text{where } \bigcup_n T_n = [0,1],\ T_n \cap T_k = \emptyset \text{ for } n \neq k.$$

Writing $u_{nk} = s_n \cdot t_k$ for every n, k, we define two new elements $\tilde{a}, \tilde{b}$ by

$$\tilde{a} = a \vee \bigvee_{n \neq k} u_{nk}, \quad \tilde{b} = b \vee \bigvee_{n \neq k+1} u_{nk}.$$

Then $\tilde{a} \geq a$, $\tilde{b} \geq b$, $\tilde{a} \cdot b = 0$, $\tilde{b} \cdot a = 0$. To check that $\mathfrak{L}(\tilde{a}, \tilde{b})$ is connected consider two elements $\tilde{l}, \tilde{l}' \in \mathfrak{L}(\tilde{a}, \tilde{b})$ such that $\tilde{l} \cdot \tilde{l}' = 0$, $\tilde{l} \vee \tilde{l}' = 1$. Then for some $x, x', y, y', \ldots \in \mathfrak{L}$

$$\tilde{l} = \tilde{a} \cdot x \vee \tilde{b} \cdot y \vee \tilde{a} \cdot \tilde{b} \cdot w \vee z, \quad \tilde{l}' = \tilde{a} \cdot x' \vee \tilde{b} \cdot y' \vee \tilde{a} \cdot \tilde{b} \cdot w' \vee z',$$

and we examine two auxiliary elements $l, l' \in \mathfrak{L}$, where

$$l = a \cdot x \vee b \cdot y \vee x \cdot y \vee z, \quad l' = a \cdot x' \vee b \cdot y' \vee x' \cdot y' \vee z'.$$

Claim 1. $l \leq \tilde{l}$ and $l' \leq \tilde{l}'$.

It is enough to check that $x \cdot y \cdot \tilde{l}' = 0$; because this gives $x \cdot y \leq \tilde{l}$ and $l \leq \tilde{l}$.

It is obvious that $x \cdot y$ is disjoint from the first three parts of $\tilde{l}'$, for instance, $(x \cdot y) \cdot \tilde{a} \cdot x' \leq (\tilde{a} \cdot x) \cdot (\tilde{a} \cdot x') = 0$. Suppose that $x \cdot y \cdot z' \neq 0$. As

$$a \cdot x \cdot y \cdot z' \leq \tilde{a} \cdot x \cdot z' \leq \tilde{l} \cdot \tilde{l}' = 0,$$

and similarly $b \cdot x \cdot y \cdot z' = 0$, there must be n such that $s_n \cdot x \cdot y \cdot z' \neq 0$; in particular, $s_n \cdot x \cdot z' \neq 0$. Since t_{n+2} is independent from $s_n \cdot x \cdot z'$ we get $u_{n(n+2)} \cdot x \cdot z' \neq 0$. This implies $\tilde{a} \cdot x \cdot z' \neq 0$, a contradiction.

CLAIM 2. $l \vee l' = 1$.

Suppose that $r = (l \vee l')^c \neq 0$. Then $r \cdot (a \vee b) = 0$ so there is n such that $r \cdot s_n > 0$. By independence, $r \cdot s_n \cdot t_n = r \cdot u_{nn} > 0$. But

$$r \cdot u_{nn} = r \cdot u_{nn} \cdot (\tilde{l} \vee \tilde{l}') \leq y \vee y'.$$

It follows that $r \cdot s_n \leq y \vee y'$ (as $r, s_n, y, y' \in \mathfrak{A}[I]$). By a similar argument, using $u_{n(n+1)}$, we check that $r \cdot s_n \leq x \vee x'$. Hence

$$r \cdot s_n \leq (x \vee x') \cdot (y \vee y'), \quad r \cdot x \cdot y = 0, \quad r \cdot x' \cdot y' = 0,$$

so for instance $x \cdot y' \cdot s_n \neq 0$. We get

$$(\tilde{a} \cdot x) \cdot (\tilde{b} \cdot y') \geq x \cdot y' \cdot u_{n(n+2)} \neq 0,$$

a contradiction.

Now it is clear from Claim 1 and Claim 2 that connectedness is preserved.

(II) We shall now attempt at preserving separation and isolation. Taking $\zeta \in \mathfrak{c} \setminus I$, $T_0 = [0, 1/2]$, $T_1 = [1/2, 1]$ and denoting

$$t_0 = c_\zeta^{T_0}, \quad t_1 = c_\zeta^{T_1}, \quad u = (a \vee b)^c,$$

we define new elements by,

$$\tilde{a} = a \vee u \cdot t_0, \quad \tilde{b} = b \vee u \cdot t_1.$$

Suppose that $p, q \in \mathfrak{A}[I]$ are separated by $\mathfrak{L}(\tilde{a}, \tilde{b})$; let $\tilde{l}, \tilde{l}' \in \mathfrak{L}(\tilde{a}, \tilde{b})$ be such that $\tilde{l} \cdot \tilde{l}' = 0$, $p \leq \tilde{l}$, $q \leq \tilde{l}'$. Write

$$\tilde{l} = \tilde{a} \cdot x \vee \tilde{b} \cdot y \vee z, \quad \tilde{l}' = \tilde{a} \cdot x' \vee \tilde{b} \cdot y' \vee z',$$

where $x, y, z, z', y', z' \in \mathfrak{L}$. We shall now examine two elements $l, l' \in \mathfrak{L}$, where

$$l = a \cdot x \vee b \cdot y \vee z \vee x \cdot y, \quad l' = a \cdot x' \vee b \cdot y' \vee z' \vee x' \cdot y'.$$

CLAIM 3. $p \leq l$ and $p' \leq l'$.

We shall prove the first part; note that $p \cdot a \leq \tilde{l} \cdot a \leq a \cdot x \vee z \leq l$; accordingly, $p \cdot b \leq l$, and it remains to check that $p \cdot u \leq l$.

Since $p \cdot u \cdot t_0 \leq x \vee z$, we have $p \cdot u \leq x \vee z$ (by indepenence: $p, u, x, z \in \mathfrak{A}[I]$ while $t_0 \in \mathfrak{A}[I^c]$). We check that $p \cdot u \leq y \vee z$ in a similar way, so

$$p \cdot u \leq (x \vee z) \cdot (y \vee z) = x \cdot y \vee z \leq l,$$

and this verifies the claim.

Claim 4. $l \cdot l' = 0$.

It is clear that $l \cdot l' \cdot a = 0$ and $l \cdot l' \cdot b = 0$, so it remains to check that $l \cdot l' \cdot u = 0$. This follows from

$$l \cdot u \leq z \vee x \cdot y \cdot u, \quad l' \cdot u \leq z' \vee x' \cdot y' \cdot u,$$

since (for instance) $x \cdot y \cdot u \cdot z' \leq \tilde{l} \cdot z' = 0$.

Claim 3 and Claim 4 show that every pair (p, q), where $p, q \in \mathfrak{A}[I]$, separated by $\mathfrak{L}(\tilde{a}, \tilde{b})$ is already separated by $\mathfrak{L}$.

If $P, Q \subseteq \mathbf{S}[I]$ then for every $\mathcal{F} \in P \cup Q$ we have $t_0 \in \mathcal{F}$. Suppose that the element x from the algebra generated by $\mathfrak{L}(\tilde{a}, \tilde{b})$ isolates the pair (P, Q). Then $x \cdot t_0 = y \cdot t_0$ for some y in the algebra generated by $\mathfrak{L}$; it is clear that y isolates (P, Q).

(III) Finally observe that if we define $\tilde{a}, \tilde{b}$ in the manner described in (II), this time with $T_0 = [0, 1/2]$ and $T_1 = [1/2, 3/4]$, and in the manner of (I), this time with $\bigcup_n G_n = [3/4, 1]$ then the lattice $\mathfrak{L}(\tilde{a}, \tilde{b})$ will have all the required properties. ★

Proposition 4.12 *Let $\mathfrak{L} \subseteq \mathfrak{A}[I]$ be a connected lattice, where $I \subseteq \mathfrak{c}$ and* card(I) = card$(\mathfrak{L}) = \kappa < \mathfrak{c}$. *Then there is a normal connected disjunctive lattice $\mathfrak{K} \supseteq \mathfrak{L}$ of cardinality κ, having the following properties:*

(s) $\mathfrak{K}$ does not separate (p, q) whenever $p, q \in \mathfrak{A}[I]$ are not separated by $\mathfrak{L}$;

(i) $\mathfrak{K}$ does not isolate (P, Q) whenever $P, Q \subseteq \mathbf{S}[I]$ are not isolated by $\mathfrak{L}$.

Proof. Applying Lemma 4.11 κ many times we find $\mathfrak{L}' \supseteq \mathfrak{L}$ which is connected and normal at each disjoint pair (a, b) in $\mathfrak{L}$, and satisfies (s) and (i). Use Lemma 4.10 to extend $\mathfrak{L}'$ to a disjunctive lattice $\mathfrak{L}''$. Repeating this procedure we define a sequence $\mathfrak{L} \subseteq \mathfrak{L}' \subseteq \mathfrak{L}'' \subseteq \ldots$ of appropriate lattices. Then $\mathfrak{K} = \mathfrak{L} \cup \mathfrak{L}' \cup \mathfrak{L}'' \cup \ldots$ is as required. ★

5. The construction

Once we have the auxiliary results from the previous section we may construct a desired lattice $\mathfrak{L}$ by a standard diagonal argument of length $\mathfrak{c}$. As in the previous section, $\mathfrak{A}$ is the measure algebra of type $\mathfrak{c}$ and $\mathbf{S}$ is its Stone space. For $\mathfrak{L} \subseteq \mathfrak{A}$ we write $\sigma(\mathfrak{L})$ for the family of all countable unions of elements of $\mathfrak{L}$ (we keep the notation of subsection 4.2; in particular c_ζ, d_ζ are defined there).

Theorem 5.1 *There is a normal connected disjunctive lattice $\mathfrak{L}$ of cardinality $\mathfrak{c}$ in $\mathfrak{A}$ and a dense set $\mathbf{S}_0 \subseteq \mathbf{S}$ with the following properties.*

(a) Whenever

$(a_n)_n \subseteq \mathfrak{L}$, $(v_n)_n \subseteq \mathfrak{L}^c$, $a_n \le v_n$ for every n, and v_n are pairwise disjoint;

$(\mathcal{F}_n)_n$ is a sequence in $\mathbf{S}_0$ for which there is $t \in \mathfrak{L}$ which is disjoint from $\bigvee_n v_n$ and such that $t \in \mathcal{F}_n$ for every n;

then there are infinite set $\tau \subseteq \sigma \subseteq \omega$ such that

$\bigvee_{n\in\tau} a_n \in \mathfrak{L}$, $\quad \bigvee_{n\in\sigma} v_n^c \in \mathfrak{L}^c$,

$\{\mathcal{F}_n:\ n \in \tau\}$, $\{\mathcal{F}_n:\ n \in \omega \setminus \sigma\}$ *are not isolated by* $\mathfrak{L}$.

(b) Whenever $p, q \in \sigma(\mathfrak{L})$ and $\mathfrak{L}$ does not separate (p, q) then there are disjoint $c, d \in \mathfrak{L}$ such that the pairs $(c \cdot p, c \cdot q)$ and $(d \cdot p, d \cdot q)$ are not separated by $\mathfrak{L}$.

Proof. Recall first that for every $a \in \mathfrak{L}$, $a \in \mathfrak{A}[I]$ for some countable $I \subseteq \mathfrak{c}$, therefore $\mathrm{card}(\mathfrak{A}) = \mathfrak{c}$. Consequently we can find a dense set $\mathbf{S}_0 \subseteq \mathbf{S}$ of cardinality $\mathfrak{c}$ with the property that for every $\mathcal{F} \in \mathbf{S}_0$, $c_\zeta \in \mathcal{F}$ for all but countable many $\zeta < \mathfrak{c}$ (see 4.2).

We want to define an increasing sequence of lattices $(\mathfrak{L}_\xi)_{\xi<\mathfrak{c}}$ in $\mathfrak{A}$, so that $\mathrm{card}(\mathfrak{L}_\xi) \le \mathrm{card}(\xi) + \omega$ for every $\xi < \mathfrak{c}$, in such a way that $\mathfrak{L} = \bigcup_{\xi<\mathfrak{c}} \mathfrak{L}_\xi$ will have the required properties. To begin we let $\mathfrak{L}_0$ be any countably infinite connected lattice in $\mathfrak{A}$.

Every lattice $\mathfrak{L}_\xi$ will be connected and this implies connectedness of $\mathfrak{L}$. Note that if we guarantee that $\mathfrak{L}_\xi$ is normal and disjunctive for cofinally many $\xi < \mathfrak{c}$ then $\mathfrak{L}$ will have the same properties; this may be done by Proposition 4.12.

It should be clear that Proposition 4.8 enables us to make $\mathfrak{L}$ satisfy condition (a). Indeed, with every lattice $\mathfrak{L}_\xi$ we can enumerate all the triples $(a_n)_n$, $(v_n)_n$, $(\mathcal{F}_n)_n$ as in (a), where $a_n \in \mathfrak{L}_\xi$, $v_n \in \mathfrak{L}_\xi^c$, and t mentioned in (a) is also in $\mathfrak{L}_\xi$ (there are $\mathfrak{c}$ many such objects), and fulfill (a) by a diagonal argument. Note that once we find τ, σ such that the pair (P, Q), where $P = \{\mathcal{F}_n:\ n \in \tau\}$, $Q = \{\mathcal{F}_n:\ n \in \omega \setminus \sigma\}$ is not isolated by $\mathfrak{L}_\xi$, we can by Proposition 4.8 and Proposition 4.12 prevent every $\mathfrak{L}_\eta$, $\eta > \xi$ from isolating (P, Q).

It remains to explain how to get condition (b). Here separation conditions come to play: With every lattice $\mathfrak{L}_\xi$ we fix a notation $(s^\xi_\alpha, t^\xi_\alpha)$, $\alpha < \mathfrak{c}$, of all disjoint pairs of elements of $\sigma(\mathfrak{L}_\xi)$, and a certain list SEP_ξ of cardinality $\mathrm{card}(\xi)$, consisting of pairs (p, q) in $\sigma(\mathfrak{L}_\xi)$ which are not separated by $\mathfrak{L}_\xi$. This is done according to the rule given below.

Suppose that $\mathfrak{L}_\xi$ and SEP_ξ are defined. First find a set $I \subseteq \mathfrak{c}$, $\mathrm{card}(I) < \mathfrak{c}$ which 'supports everything below ξ', i.e. $\mathfrak{L}_\xi \subseteq \mathfrak{A}[I]$, every $\mathcal{F}$ appearing in some triple as in (a), which was already treated, is in $\mathbf{S}[I]$, and every pair $(p, q) \in \mathrm{SEP}_\xi$ lies in $\mathfrak{A}[I]$. Fix some $\zeta \in \mathfrak{c} \setminus I$.

We shall define $\mathrm{SEP}_{\xi+1}$ to be SEP_ξ and new conditions chosen as follows. Consider any pair $(s^\eta_\alpha, t^\eta_\alpha)$, where $\eta, \alpha < \xi$. If this pair is already separated by $\mathfrak{L}_\xi$ then ignore it; otherwise add the pairs

$$(s^\eta_\alpha \cdot c_\zeta, t^\eta_\alpha \cdot c_\zeta), \quad (s^\eta_\alpha \cdot d_\zeta, t^\eta_\alpha \cdot d_\zeta)$$

to $\mathrm{SEP}_{\xi+1}$. At the same time be sure to add c_ζ, d_ζ to $\mathfrak{L}_{\xi+1}$, see Lemma 4.9.

Again, Proposition 4.8 and Proposition 4.12 say that other elements of the construction may be done in such a way that the pairs from SEP_ξ are separated never in the future. This makes $\mathfrak{L}$ satisfy condition (b). Indeed, if $p, q \in \sigma(\mathfrak{L})$ and $\mathfrak{L}$ does not separate (p,q) then the pair is split at some stage as explained above into two nonseparable parts. This finishes the proof of the theorem. ★

Given a normal disjunctive lattice $\mathfrak{L}$ we can construct a Hausdorff compact space K which has a lattice of basic closed sets isomorphic to $\mathfrak{L}$. We let $K = \mathrm{ULT}(\mathfrak{L})$ to be the space of all $\mathfrak{L}$–ultrafilters. For any $a \in \mathfrak{L}$ put

$$V(a) = \{\mathcal{F} \in \mathrm{ULT}(\mathfrak{L}) : \ a \notin \mathcal{F}\}, \quad F(a) = \{\mathcal{F} \in \mathrm{ULT}(\mathfrak{L}) : \ a \in \mathcal{F}\}.$$

Then the sets $V(a)$ form a basis of a compact Hausdorff topology on K and $a \mapsto F(a)$ is a lattice isomorphism. This is a Wallman–type construction, see Appendix for details.

Theorem 5.2 *If $\mathfrak{L}$ is a lattice as in Theorem 5.1 then its Wallman space $K = \mathrm{ULT}(\mathfrak{L})$ is a ccc compact Hausdorff connected space with property (H), and K does not contain any open butterfly.*

Proof. As it is explained in the appendix, K is indeed compact and Hausdorff, and connected whenever $\mathfrak{L}$ is connected as a lattice. We shall now define a dense set D that appears in property (H).

With every ultrafilter $\mathcal{F} \in \mathbf{S}$ we can associate its restriction $\mathcal{F}|\mathfrak{L}$ to $\mathfrak{L}$, which is clearly a prime $\mathfrak{L}$–filter, though need not to be maximal. Let us fix any extension of $\mathcal{F}|\mathfrak{L}$ to an $\mathfrak{L}$–ultrafilter and denote it by $\mathcal{F}^e$.

Let D be the set of all $\mathcal{F}^e$ for $\mathcal{F} \in \mathbf{S}_0$, where $\mathbf{S}_0$ is as in Theorem 5.1. Then clearly D is a dense subset of K.

Claim. If $P, Q \subseteq \mathbf{S}_0$ are so that

$$\overline{P^e} \cap \overline{Q^e} = \emptyset, \qquad \text{where } P^e = \{\mathcal{F}^e : \ \mathcal{F} \in P\}, \quad Q^e = \{\mathcal{F}^e : \ \mathcal{F} \in P\},$$

then $\mathfrak{L}$ isolates (P, Q).

Indeed, since K is normal there are disjoint basic closed sets $F(a) \supseteq P^e$, $F(b) \supseteq Q^e$, $a, b \in \mathfrak{L}$. In turn, $\mathfrak{L}$ is a normal lattice so there are $\tilde{a}, \tilde{b} \in \mathfrak{L}$ such that $\tilde{a} \geq a$, $\tilde{b} \geq b$, $\tilde{a} \vee \tilde{b} = 1$, $\tilde{a} \cdot b = 0$, $\tilde{b} \cdot a = 0$. Then $\tilde{a} \cdot \tilde{b}^c$ isolates (P, Q): take $\mathcal{F} \in P$; then $\tilde{a} \in \mathcal{F}$ or $\tilde{b} \in \mathcal{F}$. But $\tilde{b} \in \mathcal{F}$ implies $\tilde{b} \in \mathcal{F}^e$ which is not possible since $a \in \mathcal{F}^e$. Therefore $\tilde{a} \cdot \tilde{b}^c \in \mathcal{F}$ for each $\mathcal{F} \in P$; by a similar argument $\tilde{a}^c \cdot \tilde{b} \in \mathcal{F}$ for each $\mathcal{F} \in Q$.

Let $\{F_n : n < \omega\}$ and $\{V_n : n < \omega\}$ be as in Definition 3.1. We can without loss of generality assume that F_n and V_n are basic sets, i.e. $F_n = F(a_n)$ for some $a_n \in \mathfrak{L}$ and $V_n = V(r_n)$ for $r_n \in \mathfrak{L}$. Now we can apply (a) of Theorem 5.1 to a_n' and v_n's, where $v_n = r_n^c$. Write $a = \bigvee_{n \in \tau} a_n$, $r = \bigwedge_{n \in \sigma} r_n$. Now (H) follows by Claim and the following facts:

(i) $\overline{\bigcup_{n \in \tau} F_n} \subseteq F(a) \subseteq V(r)$;

(ii) $V(r) \subseteq \overline{\bigcup_{n\in\sigma} V_n}$;

(i) is clear. To check (ii) take any $\mathcal{F} \in V(r)$ and let $V(s)$ be a neighbourhood of $\mathcal{F}$. Then $r, s \notin \mathcal{F}$ so $r \vee s \notin \mathcal{F}$; in particular $r \vee s \neq 1$. But $r = \bigwedge_{n\in\sigma} r_n$ so there is $n \in \sigma$ with $s \vee r_n \neq 1$. It follows $\emptyset \neq V(r_n \vee s) = V(r_n) \cap V(s)$. This shows that $\mathcal{F} \in \overline{\bigcup_{n\in\sigma} V_n}$.

It remains to check that K contains no open butterfly. Clearly, K is *ccc*, so for every open $U, V \subseteq K$ we can find sequence $(a_n)_n$, $(b_n)_n$ in $\mathfrak{L}$ such that

$$\overline{U} = \overline{\bigcup_n F(a_n)}, \quad \overline{V} = \overline{\bigcup_n F(b_n)}.$$

Now if $\overline{U} \cap \overline{V} \neq \emptyset$ then $p = \bigvee_n a_n$ and $q = \bigvee_n b_n$ are not separated by $\mathfrak{L}$ and by property (b) of Theorem 5.1 the pair (p, q) can be split into two nonseparable pairs; hence $\overline{U} \cap \overline{V}$ contains two different points. The proof is complete. ★

6. Remarks

The space constructed in Theorem 5.2 has density character $\mathfrak{c}$, and not only is *ccc* but even carries a strictly positive Radon measure; we can define such a measure on ULT($\mathfrak{L}$) from the measure λ on $\mathfrak{A}$ as follows. Let μ be the restriction of λ to $\mathfrak{L}$; then μ is a regular modular set function on $\mathfrak{L}$, and it can be transferred to a set function $\widehat{\mu}$ on the corresponding lattice of closed subsets of ULT($\mathfrak{L}$) via the Wallman isomorphism. In turn, $\widehat{\mu}$ can be extended to a (necessarily strictly positive) Radon measure, see e.g. Kindler [16] for the terminology and facts mentioned here.

Assuming CH, one might obtain a separable Koszmider space in the following way. Working in the power set of ω instead of $\mathfrak{A}$, construct an appropriate lattice $\mathfrak{L} \subseteq [\omega]^\omega$, with a sequence of principal ultrafilters playing a role of a dense set. It seems that all the arguments from section 4 can be suitably adapted, using the fact, that if $\mathcal{A} \subseteq [\omega]^\omega$ is a countable family then there is $N \in [\omega]^\omega$ that reaps $\mathcal{A}$, i.e. $A \cap N, A \setminus N$ are infinite for every $A \in \mathcal{A}$. In fact CH might be weakened to the assumption that the so called reaping number is $\mathfrak{c}$.

There was much ado about connectedness (and normality) in section 4, but as explained in Introduction, connectedness is a very desirable property of the space K constructed above. Let us note, however, that the way to a zerodimensional Koszmider space is much shorter. This time we can forget about lattices and aim at constructing a suitable subalgebra $\mathfrak{L}$ of $\mathfrak{A}$. Then all the facts in Section 4 on connectedness, disjunctivity and normality become irrelevant. Moreover, there is no need to distinguish between separation and isolation, and the very property (H) may be replaced by somewhat more transparent property (H') given below, in this sense that (H') is sufficient to prove Theorem 3.2 for K zerodimensional.

Definition 6.1 *Say that a compact zerodimensional space K has property (H') if there is a dense set $D \subseteq K$ such that whenever* *(i)* $(A_n)_n$ *is a sequence of disjoint clopen subsets of* K*;*

(ii) $(d_n)_n$ *is a sequence in* D *such that* $\overline{\{d_n : n < \omega\}} \cap \overline{\bigcup_{n<\omega} A_n} = \emptyset$*;*

then there are infinite set $\sigma, \tau \subseteq \omega$ *such that*

$$\overline{\bigcup_{n\in\sigma} A_n} \cap \overline{\bigcup_{n\in\tau} A_n} = \emptyset, \quad \text{and} \quad \overline{\{d_n : n \in \sigma\}} \cap \overline{\{d_n : n \in \tau\}} \neq \emptyset.$$

It is interesting that a space K with property (H') was implicitly constructed by Haydon [14] more then twenty years ago!

We finally note that if K is a space with property (H) then it is rigid in quite a strong sense: whenever F_1, F_2 are two infinite disjoint closed subsets of K, where $F_1 = \overline{F_1 \cap D}$, then F_1 cannot be continuously mapped onto F_2 (in particular, there is no nontrivial autohomeomorpism of K).

7. Appendix: Wallman representation of lattices

We enclose the standard Wallman–type construction, associating a compact space to a given normal lattice, see Aarts [1] for a short survey and further references. Throughout this section assume that $\mathfrak{L}$ is a disjunctive normal distributive lattice.

Definition 7.1 *A family* $\mathcal{F} \subseteq \mathfrak{L}$ *is called an* $\mathfrak{L}$*–filter if* $1 \in \mathcal{F}, 0 \notin \mathcal{F}$*,* $\mathcal{F}$ *is closed under meets and for any* $a, b \in \mathfrak{L}$*, if* $a \le b, a \in \mathcal{F}$ *then* $b \in \mathcal{F}$*.*

An $\mathfrak{L}$*–ultrafilter is a maximal* $\mathfrak{L}$*–filter.*

Lemma 7.2 *(i) Every* $\mathcal{F}_0 \subseteq \mathfrak{L}$ *which is centered is contained in some* $\mathfrak{L}$*–ultrafilter.*

(ii) If $\mathcal{F}$ *is an* $\mathfrak{L}$ *ultrafilter,* $b \in \mathfrak{L}$ *has the property that* $b \cdot a \neq 0$ *for every* $a \in \mathcal{F}$ *then* $b \in \mathcal{F}$*.*

(iii) Every $\mathfrak{L}$*–ultrafilter* $\mathcal{F}$ *is prime, i.e.* $a \vee b \in \mathcal{F}$ *implies* $a \in \mathcal{F}$ *or* $b \in \mathcal{F}$*.*

Proof. (i) by Zorn's lemma; (ii) follows from (i).

(iii) If $a \notin \mathcal{F}$ and $b \notin \mathcal{F}$ then by (ii) there are $x, y \in \mathcal{F}$ such that $x \cdot a = 0$ and $y \cdot b = 0$. Then $x \cdot y \in \mathcal{F}$ and $x \cdot y \cdot (a \vee b) = 0$ so $a \vee b \notin \mathcal{F}$. ★

Let ULT($\mathfrak{L}$) be the set of all $\mathfrak{L}$–ultrafilters. For $a \in \mathfrak{L}$ we put

$$V(a) = \{\mathcal{F} \in \mathrm{ULT}(\mathfrak{L}) : a \notin \mathcal{F}\}.$$

Lemma 7.3 *(i)* $V(a) \cap V(b) = V(a \vee b)$ *and* $V(a) \cup V(b) = V(a \cdot b)$ *for any* $a, b \in \mathfrak{L}$*.*

(ii) $V(a) = \emptyset$ *if and only if* $a = 1$*.*

(iii) $V(a) = K(\mathfrak{L})$ *if and only if* $a = 0$*.*

Proof. (1) If $\mathcal{F} \in V(a) \cap V(b)$ then $a, b \notin \mathcal{F}$ and $a \vee b \notin \mathcal{F}$ since $\mathcal{F}$ is prime; therefore $\mathcal{F} \in V(a \vee b)$. The reverse inclusion is obvious. The second equality: if $\mathcal{F} \in V(a) \cup V(b)$ then say $\mathcal{F} \in V(a)$ so $a \notin \mathcal{F}$, and $a \cdot b \notin \mathcal{F}$ since $a \cdot b \leq a$. If $\mathcal{F} \in V(a \cdot b)$ then $a \cdot b \notin \mathcal{F}$ so one of them is not in $\mathcal{F}$ etc.

(2) If $a = 1$ then a is in every $\mathcal{F}$ so $V(a) = \emptyset$. If $a \neq 1$ then there is $b \neq 0$ such that $b \cdot a = 0$ (here we use the fact that $\mathfrak{L}$ is disjunctive). There is an ultrafilter $\mathcal{F}$ such that $b \in \mathcal{F}$; then $\mathcal{F} \in V(a)$.

(3) If $a = 0$ then $V(a) = K(\mathfrak{L})$ since no ultrafilter contains 0. If $a \neq 0$ then there is $\mathcal{F}$ containing a, and then $\mathcal{F} \notin V(a)$. ★

We declare all the sets $V(a)$ to be open and consider a topological space $\mathrm{ULT}(\mathfrak{L})$.

Theorem 7.4 *For a normal disjunctive lattice $\mathfrak{L}$ the space $K = \mathrm{ULT}(\mathfrak{L})$ is compact and Hausdorff. If $\mathfrak{L} \cap \mathfrak{L}^c = \{0, 1\}$ then K is connected.*

Proof. Let $\mathcal{F}, \mathcal{G}$ be two distinct $\mathfrak{L}$–ultrafilters. Then $\mathcal{F}$ is not contained in $\mathcal{G}$ so take $a \in \mathcal{F} \setminus \mathcal{G}$. By Lemma 7.2 (ii) there is $b \in G$ such that $a \cdot b = 0$. By normality of $\mathfrak{L}$ there are disjoint $u, v \in \mathfrak{L}^c$ such that $a \leq u$ and $b \leq v$. Then $\mathcal{F} \in V(v^c)$ and $\mathcal{G} \in V(u^c)$. Moreover, $V(u^c) \cap V(v^c) = V(u^c \vee v^c) = V(1) = \emptyset$. This shows Hausdorffness.

To check compactness consider a cover of K of the form $V(a_t)$, $t \in T$ without finite subcover. Then using Lemma 7.2 and 7.3

$$V(\bigwedge_{t \in I} a_t) = \bigcup_{t \in I} V(a_t) \neq K, \text{ so } \bigwedge_{t \in I} a_t \neq 0,$$

for any finite $I \subseteq T$. Hence a_t are centered and there is an ultrafilter $\mathcal{F}$ containing them all. It follow that $\mathcal{F} \notin V(a_t)$ for every $t \in T$, a contradiction.

Suppose that $M \subseteq K$ is a clopen set. Then by compactness and Lemma 7.3 $M = V(a)$ and $K \setminus M = V(b)$ for some $a, b \in \mathfrak{L}$. We have $K = V(a) \cup V(b) = V(a \cdot b)$ so $a \vee b = 0$; similarly $\emptyset = V(a) \cap V(b) = V(a \vee b)$ so $a \vee b = 1$. It follows that $a = b^c$ so $a = 0$ or $a = 1$. ★

Finally remark that the mapping $a \to F(a)$ is indeed a lattice isomorphism between $\mathfrak{L}$ and the corresponding lattice of closed subsets of $\mathrm{ULT}(\mathfrak{L})$, since such a mapping preserves lattice operations by Lemma 7.3 and is 1–1 in view of disjunctivity of the lattice.

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