What isomorphisms between $C(K)$ spaces cannot forget

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Paseky nad Jizerou 2008
Stable properties

Let $\mathcal{P}$ be a class of compact spaces. Say that $\mathcal{P}$ is stable if for every $K \in \mathcal{P}$ and every compact $L$,

$$C(K) \simeq C(L) \Rightarrow L \in \mathcal{P}.$$ 

Here $C(K) \simeq C(L)$ denotes that Banach spaces $C(K)$, $C(L)$ are isomorphic as Banach spaces (of continuous functions with the supremum norm).
Examples of stable classes/properties

- **Metrizable spaces**
  \( K \) is metrizable iff \( C(K) \) is separable.

- **Eberlein compacta**
  \( K \) is Eberlein compact iff \( C(K) \) is WCG.

- **ccc spaces**
  \( K \) is ccc iff \( C(K) \) does not contain \( c_0(\omega_1) \), Rosenthal [1969].

- **Spaces with a strictly positive measure**
  \( K \) carries a strictly positive measure iff \( C(K)^* \) contains weak compact total subset, again Rosenthal, cf. Todorcevic [2000].

- **Rosenthal compacta**
  See below.
Unstable properties

- Separability is not a stable property:

\[ C(\beta\omega) = l_\infty \simeq L_\infty[0,1] = C(S), \]

\[ S = \text{the Stone space of the measure algebra of } \lambda \text{ on } [0,1]. \]

- By Miljutin’s theorem, \( C(2^\omega) \simeq C[0,1] \simeq C[0,1]^2 \): connectedness and dimension are not stable.
**Notation**

- \( C(K)^* = M(K); \)
- \( P(K) \subseteq M(K); \, M_1(K) \subseteq M(K); \)
- \( t \in K, \, \delta_t \in P(K) \) is the Dirac measure.

If \( T : C(K) \to C(L) \) then \( T^* : M(L) \to M(K), \) where for \( \nu \in M(L), \, T^* \nu \) is defined by \( T^* \nu(f) = \nu(Tf). \)
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If $T : C(K) \to C(L)$ then $T^* : M(L) \to M(K)$, where for $\nu \in M(L)$, $T^*\nu$ is defined by $T^*\nu(f) = \nu(Tf)$.

**Stability and spaces of measures**

If $T : C(K) \to C(L)$ is an isomorphism then $T^* : M(L) \to M(K)$, $T^*$ sends $\{\delta_t : t \in L\} = L$ to a bounded subset of $M(K)$.

**Conclusion.** A class $\mathcal{P}$ is stable provided

1. $K \in \mathcal{P}, L = \overline{L} \subseteq K \Rightarrow L \in \mathcal{P}$,
2. $K \in \mathcal{P} \Rightarrow M_1(K) \in \mathcal{P}$.

**Example.** Rosenthal compact spaces, see Godefroy [1980]
Corson compacta and first–countable spaces

$K$ is Corson compact if for some $\kappa$

$$K \hookrightarrow \{x \in \mathbb{R}^\kappa : |\{\alpha : x_\alpha \neq 0\}| \leq \omega\}.$$ 

REMARKS.

- Under CH, there are “pathological” first–countable Corson compacta $K$ (Haydon, Kunen, Talagrand . . .).
- There is such $K$ of size $c$ with $|P(K)| = 2^c$, Fremlin & GP [2003].
- Under MA + non CH, Corson compacta behave properly.
- Consistently, $M_1(K)$ is first–countable if (and only if) $K$ is first–countable, GP [2000].
- Under MA + non CH, first–countability is still unclear.
Sometimes stable

- Under MA + non CH, if $K$ is Corson compact then $M_1(K)$ is Corson compact, see AMN [1988] and then Corson compacta form a stable class.
- Consistently, first-countability is stable GP [2000].
- For $\kappa \geq \omega$, $\mathcal{P}_\kappa =$ the class of spaces admitting surjection onto $[0,1]^{\kappa}$. Then $\mathcal{P}_\omega$ is stable; for every $\kappa$, it is consistent that $\mathcal{P}_\kappa$ is stable, Fremlin [1997], GP [1997].
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Problems

Assume CH. Show that

1. Corson compactness not stable,
2. first–countability not stable,
3. the class $\mathcal{P}_{\omega_1}$ is not stable.
Upper bound

If $K$ is either Corson compact or first–countable then $C(K)$ has the Mazur property, i.e. every weak* sequentially continuous $\varphi$ on $M(K)$ is defined by some element of $C(K)$, GP [1993]. In particular, for any $L$ with $C(L) \simeq C(K)$, $C(L)$ cannot contain $l_\infty$ or $C[0, \omega_1]$. 
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Some results

Under CH,

1. there is first–countable Corson compact $K$, and a surjection $T : C(K) \to l_\infty$; in particular, $\beta\omega \hookrightarrow M_1(K)$;
2. there is first–countable Corson compact $K$, $L = \beta\omega \oplus L'$ and $T : C(K) \to C(L)$ which is 1–1 and has a dense image.
More than a conjecture

Under CH, the class of Corson compact spaces is **not** stable:

*There is a first–countable Corson compact $K$, and a compact $L$ containing the split interval, such that $C(L) \cong C(K)$.***