Weakly compact sets in separable Banach spaces

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29th Summer Conference on Topology and its Applications
New York, July 2014

joint work with A. Avilés and J. Rodríguez (University of Murcia)
$X$ denotes a (separable) Banach space and $B_X$ is its unit ball. $X^*$ is the dual space of all continuous functionals on $X$.

1. The space $l_p$ of all series summable in the $p > 1$ power;
   \[ \|x\| = \left( \sum_n |x(n)|^p \right)^{1/p} \]
   \( (l_p)^* = l_q, (l_p)^{**} = l_p \).

2. The space $c_0$ of sequences converging to 0.
   \[ \|x\| = \sup_n |x(n)|, \quad (c_0)^* = l_1. \]

3. The space $l_1$ of absolutely summable series.
   \[ \|x\| = \sum_n |x(n)|, \quad (l_1)^* = l_\infty. \]

4. The space $C[0,1]$ of continuous functions with the sup norm.
The topology \textit{weak} on \(X\) is the weakest topology making all \(x^* \in X^*\) continuous. Sets of the form
\[ V(x^*) = \{ x \in X : |x^*(x)| < \varepsilon \}, \quad x^* \in X^* \]
are the subbase of the weak topology at \(0 \in X\).

\textbf{Notation}

\(\mathcal{K}(B_X)\) denotes the family of weakly compact subsets of the ball.

\textbf{Main objective}

Classify \textit{separable} Banach spaces according to properties of \(\mathcal{K}(B_X)\), considered as a set partially ordered by inclusion and/or some other relations.

\textbf{Example}

\(B_X \in \mathcal{K}(B_X)\) iff \(X^{**} = X\).
Comparing posets \((P, \leq)\) and \((Q, \leq)\): Tukey reductions

**Definition**

We say that \(P\) is Tukey reducible to \(Q\) and write \(P \preccurlyeq Q\) if there is a function \(f : P \to Q\) such that \(f^{-1}(B)\) is bounded in \(P\) whenever \(B \subseteq Q\) is bounded.

**In other words...**

\(P \preccurlyeq Q\) means for every \(q \in Q\) there is \(h(q) \in P\) such that for every \(x \in P\), if \(f(x) \leq q\) then \(x \leq h(q)\).

\(h : Q \to P\) satisfies: \(h(C)\) is cofinal in \(P\) for every cofinal \(C \subseteq Q\). \(Q\) is reacher as a cofinal structure and \(\text{cf}(P) \leq \text{cf}(Q)\).

Here \(\text{cf}(Q)\) denotes the least cardinality of a set \(C \subseteq Q\) which is **cofinal**, i.e. for every \(q \in Q\) there is \(c \in C\) with \(q \leq c\).

**Notation**

\(P\) and \(Q\) are Tukey equivalent, \(P \sim Q\), whenever \(P \preccurlyeq Q\) and \(Q \preccurlyeq P\).

\(P \prec Q\) means \(P \preccurlyeq Q\) but **not** \(Q \preccurlyeq P\).
Tukey reductions, continued

Some simple posets

\[ \{0\} \prec \omega \prec \omega^\omega \prec \mathcal{H}(\mathbb{Q}) \prec [c]^{<\omega}. \]

ω\(^\omega\)

For \( g_1, g_2 \in \omega^\omega \), \( g_1 \leq g_2 \) if \( g_1(n) \leq g_2(n) \) for every \( n \in \omega \).

For the properties of \( \mathcal{H}(\mathbb{Q}) \) see Fremlin 91 and Gartside & Mamatelashvili 2014.

Remarks on cofinalities

- We have \( \text{cf}(\omega) = \omega \), \( \text{cf}([c]^{<\omega}) = c \)
- \( \text{cf}(\omega^\omega) \) is denoted by \( \mathfrak{d} \).
- \( \omega_1 \leq \mathfrak{d} \leq c \).
- Fremlin 1991: \( \text{cf}(\mathcal{H}(\mathbb{Q})) = \mathfrak{d} \).
## Classification of $\mathcal{K}(E)$ and its consequence

### Theorem (Fremlin 91)

If $E$ is coanalytic in some Polish space then either

1. $\mathcal{K}(E) \sim \mathcal{K}[0,1] \sim \{0\}$
   
   ($E$ compact), or

2. $\mathcal{K}(E) \sim \mathcal{K}(\mathbb{N}) \sim \omega$
   
   ($E$ loc. compact noncompact), or

3. $\mathcal{K}(E) \sim \mathcal{K}(\mathbb{R} \setminus \mathbb{Q}) \sim \omega^\omega$
   
   ($E$ Polish not loc. compact), or

4. $\mathcal{K}(E) \sim \mathcal{K}(\mathbb{Q})$
   
   ($E$ coanalytic but not Polish).

### Corollary

If $X$ is a Banach space with $X^*$ separable then

1. $\mathcal{K}(B_X) \sim \{0\}$
   
   ($X$ reflexive), or

2. does not occur: weakly loc. compact implies compact.

3. $\mathcal{K}(B_X) \sim \omega^\omega$
   
   ($X$ not reflexive, has PCP), or

4. $\mathcal{K}(E) \sim \mathcal{K}(\mathbb{Q})$
   
   ($X$ does not have PCP).

### Proof.

If $X^*$ is separable then $(B_X^{**}, weak^*)$ is compact metric and $(B_X, weak)$ is $F_{\sigma\delta}$.

$X$ has PCP if for every weakly closed bounded $A \subseteq X$, $(A, weak) \rightarrow (A, norm)$ has a point of continuity.

**Edgar & Wheeler:** $(B_X, weak)$ is Polish iff $X^*$ is separable and $X$ has PCP.
Example
If $X = C[0,1]$ then $\mathcal{K}(B_X) \sim [c]^{<\omega}$.

Conjecture
If $X$ is a separable Banach space then $\mathcal{K}(B_X)$ is Tukey equivalent to one of the following:

\[
\{0\}, \quad \omega^\omega, \quad \mathcal{K}(\mathbb{Q}), \quad [c]^{<\omega}.
\]
**Definition**

A Banach space $X$ is WCG if $X = \overline{\text{lin}}(K)$ for some $K \in \mathcal{K}(X)$.

Every separable $X$ is WCG.

**Definition**

A Banach space $X$ is SWCG if there is $L \in \mathcal{K}(X)$ such that for every $K \in \mathcal{K}(X)$ and $\varepsilon > 0$ there is $n$ such that $K \subseteq n \cdot L + \varepsilon \cdot B_X$.

**In other words**

$X$ is SWCG if and only if there are $L_n \in \mathcal{K}(B_X)$ such that for every $K \in \mathcal{K}(B_X)$ and $\varepsilon > 0$ we have $K \subseteq L_n + \varepsilon \cdot B_X$ for some $n$.

**Examples**

$L_1[0,1]$ is SWCG; try $L = \{f \in L_1[0,1] : |f| \leq 1\}$.

$l_1$ is SWCG; try $K_n = \{x \in B_{l_1} : x(k) = 0$ for $k \geq n\}$.

$c_0$ is not SWCG; how many weakly compact sets we need to generate $c_0$ strongly?
Asymptotic structures

**Definition**

Say that \((P, \leq_{\varepsilon}: \varepsilon > 0)\) is an asymptotic structure if every \(\leq_{\varepsilon}\) is a binary relation on \(P\) and for \(\eta < \varepsilon\), \(x \leq_{\eta} y\) implies \(x \leq_{\varepsilon} y\).

**Definition**

Given asymptotic structures \((P, \leq_{\varepsilon}: \varepsilon > 0)\) and \((Q, \leq_{\varepsilon}: \varepsilon > 0)\), we say that \(P \preceq Q\) if for every \(\varepsilon > 0\) there is \(\delta > 0\) such that

\[ (P, \leq_{\varepsilon}) \preceq (Q, \leq_{\delta}). \]

**Remarks**

Given an asymptotic structure \((P, \leq_{\varepsilon}: \varepsilon > 0)\) and an ordinary poset \((Q, \leq)\),

- \(P \preceq Q\) means \((P, \leq_{\varepsilon}) \preceq (Q, \leq)\) for every \(\varepsilon > 0\);
- \(Q \preceq P\) means \((Q, \leq) \preceq (P, \leq_{\delta})\) for some \(\delta > 0\).
Asymptotic structures of weakly compact sets

**Notation**

$\mathbb{AK}(B_X)$ is $\mathcal{K}(B_X)$ equipped with relations $\leq_{\varepsilon}$, where $K \leq_{\varepsilon}$ means $K \subseteq L + \varepsilon \cdot B_X$.

**Examples and remarks**

- $X$ is SWCG iff $\mathbb{AK}(B_X) \preceq \omega$.
- If $X = c_0$ then $\mathbb{AK}(B_X) \sim \mathcal{K}(\mathbb{Q})$. Hence $\text{cf}(\mathbb{AK}(B_X)) = 0$ so $c_0$ is strongly generated by 0 weakly compact sets.
- If $\mathbb{AK}(B_X) \sim P$ for some poset $P$ then $P \preceq \mathcal{K}(B_X) \preceq P^\omega$.
- To show that $P \preceq \mathbb{AK}(B_X)$ we need to define $f : P \to \mathcal{K}(B_X)$ such that for every $L \in \mathcal{K}(B_X)$ there is $p \in P$ such that whenever $f(x) \subseteq L + \varepsilon \cdot B_X$ then $x \leq p$. 
Open problem

**Problem**
Is it true that for every *separable* $X$, either $\mathbb{A}K(B_X) \preceq \omega$ or $\omega^\omega \preceq \mathbb{A}K(B_X)$?

**Remarks**
For every Banach space $X$, either $\mathcal{K}(B_X) \sim \{0\}$ or $\omega^\omega \preceq \mathcal{K}(B_X)$. Assuming $\vartheta > \omega_1$, for the *nonseparable* space $X = l_1(\omega_1)$,

- neither $\mathbb{A}K(B_X) \preceq \omega$ (because $X$ is not SWCG),
- nor $\omega^\omega \preceq \mathbb{A}K(B_X)$ (because $\text{cf}(\mathbb{A}K(B_X)) = \omega_1$).
Theorem

If a separable space $X$ does not contain an isomorphic copy of $l_1$ then $\mathcal{AK}(B_X) \sim \mathcal{H}(B_X)$ and, moreover, is Tukey equivalent to either

1. $\{0\}$ (if $X$ is reflexive), or
2. $\omega^\omega$ (if $X$ is not reflexive, $X^*$ is separable and $X$ has PCP), or
3. $\mathcal{H}(Q)$ (if $X$ is not reflexive, $X^*$ is separable and $X$ does not have PCP), or
4. $[c]^{<\omega}$ (if $X^*$ is not separable).

The proof uses a result of López Pérez & Soler Arias 2012 and some Ramsey type results due to Todorčević 2010 and others.
Theorem

Assuming the axiom of analytic determinacy, every separable space Banach space $X$ satisfies one of the following

1. $\text{AK}(B_X) \sim \mathcal{H}(B_X) \sim \{0\}$,
2. $\omega \not\leq \text{AK}(B_X) \leq \omega^\omega$ and $\mathcal{H}(B_X) \sim \omega^\omega$,
3. $\text{AK}(B_X) \sim \mathcal{H}(B_X) \sim \mathcal{H}(\mathbb{Q})$,
4. $\text{AK}(B_X) \sim \mathcal{H}(B_X) \sim [c]^{<\omega}$.

Theorem (under analytic determinacy)

If $\mathcal{I}$ is an analytic ideal on $\omega$, $\mathcal{I}^\perp = \{A \subseteq \omega : A \cap l \text{ finite for } l \in \mathcal{I}\}$ then $\mathcal{I}^\perp$ is Tukey equivalent to one of the following $\{0\}, \omega, \omega^\omega, \mathcal{H}(\mathbb{Q}), [c]^{<\omega}$.

The proof is based on results on analytic gaps due to Todorčević and analytic multigaps due to Avilés and Todorčević 2013-2014.
Subspaces

Two positive results

Let $Y$ be a subspace of $X$.

- $\mathcal{K}(B_Y) \lesssim \mathcal{K}(B_X)$.
  
  **Proof.** $\mathcal{K}(B_Y) \ni K \rightarrow K \in \mathcal{K}(B_X)$ is Tukey because if $K \subseteq L \in \mathcal{K}(B_X)$ then $K \subseteq L \cap Y \in \mathcal{K}(B_Y)$.

- If $Y$ is complemented in $X$ (i.e. $X = Y \oplus Z$ for some closed $Z$) then $\mathcal{AK}(B_Y) \lesssim \mathcal{AK}(B_X)$.
  
  **Proof.** Let $P : X \rightarrow Y$ be a projection. If $K \in \mathcal{K}(B_Y)$, $L \in \mathcal{K}(B_X)$ and $K \subseteq L + \varepsilon \cdot B_X$ then $K \subseteq P(L) + \varepsilon \cdot \|P\| \cdot B_Y$.

Following Mercourakis & Stamaki

There is a subspace $Y$ of $X = L_1[0,1]$ (which is SWCG so $\mathcal{AK}(B_X) \sim \omega$) such that $\mathcal{AK}(B_Y) \sim \omega^\omega$. 
Unconditional bases

- Let $E = \langle e_n : n \in \omega \rangle$ be an unconditional basic sequence in $X$, i.e. there is $C > 0$ such that $\| \sum_{n \in J} a_n \cdot e_n \| \leq C \cdot \| \sum_{n \in J} a_n \cdot e_n \|$ for any finite sets $I \subseteq J \subseteq \omega$ and any scalars $a_n \in \mathbb{R}$.

- **Lemma.** Let $\mathcal{N}(E) = \{ A \subseteq \omega : (e_n)_{n \in A} \text{ is weakly null} \}$. Then $\mathcal{N}(E) \preceq \mathbb{AK}(B_X)$.

- Let $\mathcal{A}$ be an adequate family on $\omega$, i.e. $\mathcal{A}$ is hereditary and $A \in \mathcal{A}$ whenever all finite subsets of $A$ are in $\mathcal{A}$.

- Following Argyros & Mercourakis 1993 define a norm $\| \cdot \|$ on $c_{00}$ by

\[
\| x \| = \sup_{T \in \mathcal{A}} \sum_{n \in T} |x(n)|.
\]

Let $X$ be the completion of $c_{00}$ with respect to such a norm.

- We have $\mathcal{N}(E) \sim \mathcal{A}^\perp \preceq \mathbb{AK}(B_X)$.

- Consistently, there is a Banach space $X$ and $E \subseteq X$ such that $\mathbb{AK}(E)$ is not Tukey equivalent to any of $\{0\}, \omega, \omega^\omega, \mathcal{K}(\mathbb{Q}), [c]^{<\omega}$. 