Baire measurability in $C(2^\kappa)$

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Terminology and notation

- For any family $\mathcal{A} \subseteq \mathcal{P}(X)$ we write $\sigma(\mathcal{A})$ for the $\sigma$-algebra generated by $\mathcal{A}$.
- If $\mathcal{F} \subseteq \mathbb{R}^X$ is a family of functions then $\sigma(\mathcal{F})$ denotes the $\sigma$-algebra generated by $\mathcal{F}$, i.e. the least $\sigma$-algebra making all $f \in \mathcal{F}$ measurable.

Baire and Borel sets

In every completely regular topological space $X$ there are two natural $\sigma$-algebras:

- $\text{Bor}(X)$ generated by all open sets, and
- $\text{Baire}(X)$ generated by all continuous functions $X \to \mathbb{R}$.

$\text{Baire}(X) = \text{Bor}(X)$ whenever $X$ is a metric space, in general $\text{Baire}(X) \subseteq \text{Bor}(X)$. 
Banach spaces $C(K)$

For a compact space $K$ we can equipp $C(K)$ with three natural topologies

- $(C(K), \|\cdot\|)$;
- $(C(K), \text{weak})$;
- $(C(K), \tau_p)$.

We can discuss five $\sigma$-algebras on $C(K)$. Recall that

- $Baire(C(K), \tau_p) = \sigma(\delta_x : x \in K)$, where $\delta_x(g) = g(x)$
- $Baire(C(K), \text{weak}) = \sigma(\mu : \mu \in C(K)^*)$, where $\mu(g) = \int g \, d\mu$. 

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Borel structures in $C(2^\kappa)$

$$\text{Bor}(C(2^\kappa), \tau_p) = \text{Bor}(C(2^\kappa), \text{weak}) = \text{Bor}(C(2^\kappa)),$$

for every $\kappa$ because $C(2^\kappa)$ has a $\tau_p$-Kadec renorming (Edgar).

Baire structures in $C(2^\kappa)$ for $\kappa \leq \mathfrak{c}$

$$\text{Baire}(C(2^\kappa), \tau_p) = \text{Baire}(C(2^\kappa), \text{weak}),$$

for $\kappa \leq \mathfrak{c}$ because every probability measure $\mu$ on $2^\mathfrak{c}$ is a $\text{weak}^*$-limit $\mu = \lim_n (1/n) \sum_{i \leq n} \delta_{x_i}$ for some sequence $x_i \in 2^\kappa$ (Fremlin).

For $\kappa \leq \mathfrak{c}$ we have thus the Baire $\sigma$-algebra on $C(2^\kappa)$ and its Borel $\sigma$-algebra.

Theorem

$$\text{Baire}(C(2^{\omega_1}), \tau_p) = \text{Bor}(C(2^{\omega_1}), \tau_p)$$ and, consequently, all the five algebras on $C(2^{\omega_1})$ coincide.
Why $Baire(C(2^{\omega_1}), \tau_p) = Bor(C(2^{\omega_1}), \tau_p)$?

**Lemma**

Suppose that $K$ is such a compact space that for every $n \in \mathbb{N}$ and every closed $F \subseteq K^n$, $F$ is a decreasing intersection of a sequence $(F_p)_{p \in \mathbb{N}}$ of closed separable subspaces $F_p \subseteq K^n$. Then $Baire(C(K), \tau_p) = Bor(C(K), \tau_p)$.

**Lemma**

Every closed $F \subseteq 2^{\omega_1}$ is a decreasing intersection of a sequence $(F_p)_{p \in \mathbb{N}}$ of closed separable subspaces $F_p \subseteq 2^{\omega_1}$. 

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Kunen cardinals

\( \kappa \) is a Kunen cardinal if \( \mathcal{P}(\kappa) \otimes \mathcal{P}(\kappa) = \mathcal{P}(\kappa \times \kappa) \), i.e. \( \sigma(\{A \times B : A, B \subseteq \kappa\}) \) contains all subsets of \( \kappa \times \kappa \).

- If \( \kappa \) is Kunen then \( \kappa \leq c \).
- \( \omega_1 \) is a Kunen cardinal.
- \( c \) is Kunen cardinal under MA + non CH, but, consistently, \( c = \omega_2 \) is not Kunen.
- If \( \kappa \) is a Kunen cardinal then there is no universal measure on \( \mathcal{P}(\kappa) \).
Fremlin’s result and a corollary

\[ \text{Baire}(l_1(\kappa), \text{weak}) = \text{Bor}(l_1(\kappa), \text{weak}) \iff \kappa \text{ is a Kunen cardinal.} \]

If \( \text{Baire}(C(2^\kappa), \tau_p) = \text{Bor}(C(2^\kappa), \tau_p) \) then \( \kappa \) is a Kunen cardinal.

Theorem - the main result

\[ \text{Baire}(C(2^\kappa), \tau_p) = \text{Bor}(C(2^\kappa), \tau_p) \iff \kappa \text{ is a Kunen cardinal.} \]
Corollary

$C(2^\kappa)$ is measure-compact whenever $\kappa$ is a Kunen cardinal.

A Banach space $E$ is measure compact if for every weakly measurable $f : \Omega \to E$ there is a Bochner measurable $g : \Omega \to E$ such that $x^*g = x^*f$ $\mu$-a.e. (for any probability space $(\Omega, \Sigma, \mu)$). Equivalently, for every finite measure $\nu$ on $\text{Baire}(E, \text{weak})$ there is a separable subspace $E_0$ such that $\mu^*(E_0) = \mu(E)$.

Remark

Assuming the absence of weakly inaccessible cardinals, $C(2^\kappa)$ is measure-compact for any $\kappa$. (Plebanek [1991])

Corollary

Under MA + non CH, $\text{Bor}(2^{\omega_1})$ is countable generated.

If $D \subseteq 2^{\omega_1}$ is a countable dense set then $\text{Bor}(2^{\omega_1}) = \sigma(\delta_x : x \in D)$. 

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