A NORMAL MEASURE ON A COMPACT CONNECTED SPACE

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Abstract. We present a construction of a compact connected space which supports a normal probability measure.

1. Introduction

If $K$ is a compact Hausdorff space then we denote by $P(K)$ the set of all probability regular Borel measures on $K$. We write $Z(K)$ for the family of all closed $G_δ$ subsets of $K$. Since every compact space is normal, $Z \in Z(K)$ if and only if $Z$ is a zero set, i.e. $Z = f^{-1}(0)$ for some continuous function $f : K \to \mathbb{R}$.

A measure $\mu \in P(K)$ is normal if $\mu$ is order-continuous on the Banach lattice $C(K)$. Equivalently, $\mu(F) = 0$ whenever $F \subseteq K$ is a closed set with empty interior ([1], Theorem 4.6.3). A typical example of a normal measure is the natural measure defined on the Stone space of the measure algebra $\mathfrak{A}$ of the Lebesgue measure $\lambda$ on $[0,1]$. Since the algebra $\mathfrak{A}$ is complete, its Stone space is extremely disconnected.

By a result from [2] if $K$ is a locally connected compactum then no measure $\mu \in P(K)$ can be normal, cf. [1], Proposition 4.6.20. Dales et al. posed a problem that can be stated as follows (Question 2 in [1]).

Problem 1.1. Suppose that $K$ is a compact and $\mu \in P(K)$ is a normal measure. Must $K$ be disconnected?

We show below that the answer is negative, namely we prove the following result.

Theorem 1.2. There is a compact connected space $L$ of weight $\mathfrak{c}$ which is the support of a normal measure.

2. Preliminaries

Recall that $\mu \in P(K)$ is said to be strictly positive or fully supported by $K$ if $\mu(U) > 0$ for every non-empty open set $U \subseteq K$.

Lemma 2.1. Let $K$ be a compact space, and suppose that $\mu$ is a strictly positive measure on $K$ such that $\mu(Z) = 0$ for every $Z \in Z(K)$ with empty interior. Then $\mu$ is a normal measure.

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Proof. Assume that there is a closed set \( F \subseteq K \) with empty interior but with \( \mu(F) > 0 \). Then we derive a contradiction by the following observation.

**Claim.** Every closed set \( F \subseteq K \) with empty interior is contained in some \( Z \in \mathcal{Z}(K) \) with empty interior.

Indeed, consider a maximal family \( \mathcal{F} \) of continuous functions \( K \to [0,1] \) such that \( f|F = 0 \) for \( f \in \mathcal{F} \) and \( f \cdot g = 0 \) whenever \( f, g \in \mathcal{F}, f \neq g \). Then \( \mathcal{F} \) is necessarily countable because \( K \), being the support of a measure, satisfies the countable chain condition. Write \( \mathcal{F} = \{ f_n : n \in \mathbb{N} \} \) and let \( f = \sum_n 2^{-n} f_n \) and \( Z = f^{-1}(0) \). Then the function \( f \) is continuous so that \( Z \subseteq \mathcal{Z}(K) \). We have \( Z \supseteq F \) and the interior of \( Z \) must be empty by the maximality of \( \mathcal{F} \). \( \square \)

If \( f : K \to L \) is a continuous map and \( \mu \in P(K) \) then the measure \( f[\mu] \in P(L) \) is defined by \( f[\mu](B) = \mu(f^{-1}(B)) \) for every Borel set \( B \subseteq L \).

We shall consider inverse systems of compact spaces with measures of the form

\[
\langle K_\alpha, \mu_\alpha, \pi_\alpha^\beta : \beta < \alpha < \kappa \rangle,
\]

where \( \kappa \) is an ordinal number and for all \( \gamma < \beta < \alpha < \kappa \) we have

2(i) \( K_\alpha \) is a compact space and \( \mu_\alpha \in P(K_\alpha) \);
2(ii) \( \pi_\alpha^\beta : K_\alpha \to K_\beta \) is a continuous surjection;
2(iii) \( \pi_\alpha^\beta \circ \pi_\beta^\gamma = \pi_\alpha^\gamma \);
2(iv) \( \pi_\alpha^\beta[\mu_\alpha] = \mu_\beta \).

The following summarises basic facts on inverse systems satisfying 2(i)-(iv).

**Theorem 2.2.** Let \( K \) be the limit of the system with uniquely defined continuous surjections \( \pi_\alpha : K \to K_\alpha \) for \( \alpha < \kappa \).

(a) \( K \) is a compact space and \( K \) is connected whenever all the space \( K_\alpha \) are connected.
(b) There is the unique \( \mu \in P(K) \) such that \( \pi_\alpha[\mu] = \mu_\alpha \) for \( \alpha < \kappa \).
(c) If every \( \mu_\alpha \) is strictly positive then \( \mu \) is strictly positive.

Engelking’s *General Topology* contains the topological part of 2.2 (measure-theoretic ingredients call for a proper reference). We also use the following fact on closed \( G_\delta \) sets and inverse systems of length \( \omega_1 \).

**Lemma 2.3.** Let \( K \) be the limit of an inverse system \( \langle K_\alpha, \pi_\alpha^\beta : \beta < \alpha < \omega_1 \rangle \). Then for every \( Z \in \mathcal{Z}(K) \), there are \( \alpha < \omega_1 \) and \( Z_\alpha \in \mathcal{Z}(K_\alpha) \) with \( Z = \pi_\alpha^{-1}(Z_\alpha) \).

**Proof.** Sets of the form \( \pi_\alpha^{-1}(V) \), where \( \alpha < \kappa \) and \( V \subseteq K_\alpha \) is open, give the canonical basis of \( K \) (closed under countable unions). Therefore if \( Z \in \mathcal{Z}(K) \) then \( Z = \bigcap_n \pi_\alpha^{-1}(V_n) \) for some \( \alpha_n < \omega_1 \) and some open \( V_n \subseteq K_{\alpha_n} \). Taking \( \alpha > \sup_n \alpha_n \) we can write \( Z = \bigcap_n \pi_\alpha^{-1}(W_n) \) for some open \( W_n \subseteq K_\alpha \). Let \( Z_\alpha = \bigcap_n W_n \). Then \( Z_\alpha \) is \( G_\delta \) in \( K_\alpha, \pi_\alpha^{-1}(Z_\alpha) = Z \) and \( Z_\alpha = \pi_\alpha(Z) \) is closed. \( \square \)
3. Proof of Theorem 1.2

We first describe a basic construction which will be used repeatedly.

**Lemma 3.1.** Let $K$ be a compact connected space, and let $\mu \in P(K)$ be a strictly positive measure. If $F \subseteq K$ is a closed set with $\mu(F) > 0$, then there are a compact connected space $\hat{K}$, a strictly positive measure $\hat{\mu} \in P(\hat{K})$ and a continuous surjection $f : \hat{K} \to K$ such that $f[\hat{\mu}] = \mu$ and $\text{int}(f^{-1}(F)) \neq \emptyset$.

*Proof.* Let $F_0$ be the support of $\mu$ restricted to $F$, that is

$$F_0 = F \setminus \bigcup \{U : U \text{ open and } \mu(F \cap U) = 0\}.$$

Let $\hat{K} = \{(x, t) \in K \times [0, 1] : x \in F_0 \text{ or } t = 0\}$. Then $\hat{K}$ is clearly a compact connected space and $f(x, t) = x$ defines a continuous surjection $f : \hat{K} \to K$. Moreover, the set $f^{-1}(F)$ contains $F_0 \times [0, 1]$, a set with non-empty interior. Hence $\text{int}(f^{-1}(F)) \neq \emptyset$.

We can define $\hat{\mu} \in P(\hat{K})$ with the required property by setting

$$\hat{\mu}(B) = \mu(f(B \cap (K \setminus F) \times \{0\})) + \mu \otimes \lambda(F \times [0, 1] \cap B),$$

for Borel sets $B \subseteq \hat{K}$, where $\lambda$ is the Lebesgue measure on $[0, 1]$. \hfill $\square$

**Lemma 3.2.** Let $K$ be a compact connected space, and let $\mu \in P(K)$ be a strictly positive measure. Then there are a compact connected space $K^\#$, a strictly positive measure $\mu^\# \in P(K^\#)$ and a continuous surjection $g : K^\# \to K$ such that $g[\mu^\#] = \mu$ and $\text{int}(g^{-1}(Z)) \neq \emptyset$ for every $Z \in \mathcal{Z}(K)$ with $\mu(Z) > 0$.

*Proof.* Let $\{Z_\alpha : \alpha < \kappa\}$ be an enumeration of all sets $Z \in \mathcal{Z}(K)$ of positive measure. Setting $K_0 = K$, $\mu_0 = \mu$, we define inductively an inverse system $\langle K_\alpha, \mu_\alpha, \pi^\beta_\alpha : \beta < \alpha < \kappa \rangle$ satisfying 2(i)-(iv). Assume the construction for all $\alpha < \xi$.

If $\xi$ is the limit ordinal we use Theorem 2.2 and let $K_\xi$ be the limit of $K_\alpha$, $\alpha < \kappa$, and $\mu_\xi$ be the unique measure as in 2.3.

If $\xi = \alpha + 1$ then we define $K_\xi$ and $\mu_\xi \in P(K_\xi)$ applying Lemma 3.1 to $K = K_\alpha$, $\mu = \mu_\alpha$, $F = (\pi^0_\alpha)^{-1}(Z_\alpha)$.

Then we can define $K^\#$ and $\mu^#$ as the limit of $\langle K_\alpha, \mu_\alpha, \pi^\beta_\alpha : \beta < \alpha < \kappa \rangle$ and set $g = \pi_0 : K^\# \to K$.

Indeed, if $Z \in \mathcal{Z}(K)$ and $\mu(Z) > 0$ then $Z = Z_\alpha$ for some $\alpha < \kappa$ so the interior of the set

$$(\pi^0_\alpha)^{-1}(Z_\alpha) = (\pi^{\alpha+1}_\alpha)^{-1}((\pi^0_0)^{-1}(Z_\alpha)),$$

is nonempty by the basic construction of Lemma 3.1. It follows that $\text{int}(g^{-1}(Z_\alpha)) \neq \emptyset$, and we are done. \hfill $\square$

We are now ready for the proof of Theorem 1.2. Let $L_0 = [0, 1]$ and $\mu_0 = \lambda$. Using Lemma 3.2 we define an inverse system $\langle L_\alpha, \mu_\alpha, \pi^\beta_\alpha : \beta < \alpha < \omega_1 \rangle$, where $L_{\alpha+1} = (L_\alpha)^\#$. 


and \( \mu_{\alpha+1} = (\mu_\alpha)^\# \). Consider the limit \( L \) of this inverse system with the limit measure \( \nu \in P(L) \).

We shall check that \( \nu \) is a normal measure using Lemma 2.1. Take \( Z \in \mathcal{Z}(L) \) with \( \nu(Z) > 0 \). It follows from Lemma 2.3 that \( Z = \pi_\alpha^{-1}(Z_\alpha) \) for some \( \alpha < \omega_1 \) and \( Z_\alpha \in \mathcal{Z}(L_\alpha) \). Then the set \( (\pi_\alpha^{\alpha+1})^{-1}(Z_\alpha) \) has non-empty interior in \( L_{\alpha+1} = (L_\alpha)^\# \) and, consequently, \( \text{int}(Z) \neq \emptyset \).

Note that in a compact space \( K \) of topological weight \( w(K) \leq c \) there are at most \( c \) many closed \( G_\delta \) sets. It follows from the proof of Lemma 3.2 that \( w(K^\#) \leq c \) whenever \( w(K) \leq c \). Therefore \( w(L_\alpha) \leq c \) for every \( \alpha < \omega_1 \) and \( w(L) = c \). This finishes the proof of our main result.

Let us remark that using Lemma 3.1 and the construction from Kunen [3] one can prove the following variant of Theorem 1.2.

**Theorem 3.3.** Assuming the continuum hypothesis, there is a perfectly normal compact connected space \( L \) supporting a normal probability measure.

Perfect normality of \( L \) means that every closed subset of \( L \) is \( G_\delta \) so in particular the space \( L \) from Theorem 3.3 is first-countable.

**References**


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