The complemented subspace problem for $C(K)$-spaces

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Rosenthal [1972]

Suppose that X is a complemented subspace of $C[0,1]$ and X^* is not separable. Then $X \simeq C[0,1]$.

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Salguero-Alarcón & P. [2022]

There are two separable scattered compacta K and L and a continuous surjection $\theta : L \to K$ such that $C(L) \simeq \theta^\circ[C(K)] \oplus X$ and the Banach space X is not a $\mathscr{C}$-space.

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3. Write $\Psi_{\mathscr{A}}$ for $\omega \cup \mathscr{A}$ and define a topology on $\Psi_{\mathscr{A}}$ by declaring that points in ω are isolated while basic neighbourhoods of $A \in \Psi_{\mathscr{A}}$ are of the form $\{A\} \cup A \setminus I$, with $I \subseteq \omega$ finite.

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There is a lot of research done on the interplay between combinatorial properties of $\mathscr{A}$ and topology of $\Psi_{\mathscr{A}}$ (or $K_{\mathscr{A}}$), see **Hrušák** [2014].

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- **Marciszewski & Pol [2009]:** If $\mathscr{A}$ and $\mathscr{A}'$ are AD families of branches of $2^{<\omega}$ and $\omega^{<\omega}$, respectively, then $C(K_{\mathscr{A}}) \not\approx C(K_{\mathscr{A}'})$.

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  - Every operator $T : C(K_{\mathscr{A}}) \to C(K_{\mathscr{A}})$ is of the form $T = c \cdot I + S$, where the range of S is separable;
  - $C(K_{\mathscr{A}}) \simeq c_0 \oplus C(K_{\mathscr{A}})$ is essentially the unique decomposition into a direct sum of infinitely dimensional summands.

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- **Pełczyński:** Suppose that φ_x is a probability measure on $\theta^{-1}(x)$, $x \in K$ and $K \ni x \to \varphi_x \in C(L)^*$ is *weak** continuous. Then $C(L) = \theta^\circ[C(K)] \oplus X$ because $Tf(x) = \int_L f \, \mathrm{d}\varphi_x$ defines $T : C(L) \to C(K)$ and $Pf = (Tf) \circ \theta$ is a projection.

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- If X is a $\mathscr{C}$-space then the ball in X^* contains a closed set F such that $X \ni x \to x|F \in C(F)$ is an isomorphism.

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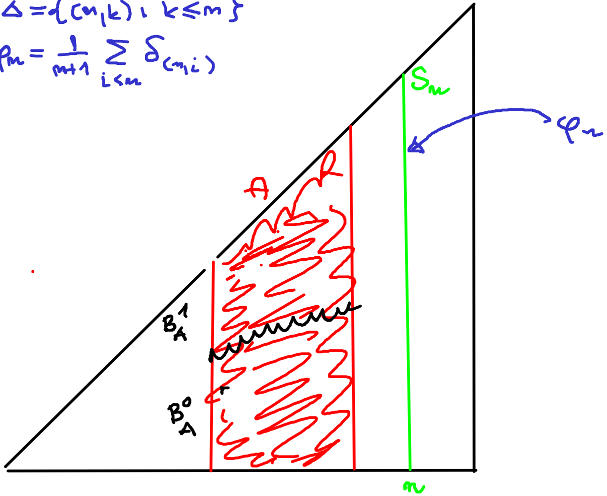

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4. Put $K = \mathrm{ult}(\mathfrak{B}_1)$, $L = \mathrm{ult}(\mathfrak{B}_2)$; $\theta : L \to K$ is the obvious surjection.
5. Property (3) enables us to define a projection from $C(L)$ onto $\theta^\circ[C(K)]$ so $C(L) = \theta^\circ[C(K)] \oplus X$.