On isomorphisms and embeddings of $C(K)$ spaces

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$K$ and $L$ always stand for compact spaces. For a given $K$, $C(K)$ is the Banach space of all continuous real-valued functions $f : K \to \mathbb{R}$, with the usual norm: $\|g\| = \sup_{x \in K} |f(x)|$.

A linear operator $T : C(K) \to C(L)$ is an **isomorphic embedding** if there are $M, m > 0$ such that for every $g \in C(K)$

$$m \cdot \|g\| \leq \|Tg\| \leq M \cdot \|g\|.$$  

Here we can take $M = \|T\|$, $m = 1/\|T^{-1}\|$. Isomorphic embedding $T : C(K) \to C(L)$ which is onto is called an **isomorphism**; we then write $C(K) \sim C(L)$.  

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Isomorphisms of $C(K)$ spaces  
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Some ancient results

- **Banach-Stone**: If $C(K)$ is isometric to $C(L)$ then $K \cong L$.
- **Amir, Cambern**: If $T : C(K) \to C(L)$ is an isomorphism with $\|T\| \cdot \|T^{-1}\| < 2$ then $K \cong L$.
- **Jarosz (1984)**: If $T : C(K) \to C(L)$ is an embedding with $\|T\| \cdot \|T^{-1}\| < 2$ then $K$ is a continuous image of some compact subspace of $L$.
- **Miljutin**: If $K$ is an uncountable metric space then $C(K) \sim C([0, 1])$. In particular $C(2^\omega) \sim C[0, 1]; C[0, 1] \times \mathbb{R} = C([0, 1] \cup \{2\}) \sim C[0, 1]$. 

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Some ancient problems

Problem

For which spaces \( K \), \( C(K) \sim C(K + 1) \)?

Here \( C(K + 1) = C(K) \times \mathbb{R} \).
This is so if \( K \) contains a nontrivial converging sequence:
\[ C(K) = c_0 \oplus X \sim c_0 \oplus X \oplus \mathbb{R} \sim C(K + 1). \]
Note that \( C(\beta\omega) \sim C(\beta\omega + 1) \) (because \( C(\beta\omega) = l_\infty \)) though \( \beta\omega \) has no converging sequences.

Problem

For which spaces \( K \) there is a totally disconnected \( L \) such that \( C(K) \sim C(L) \) ?
Some more recent results

- **Koszmider (2004):** There is a compact connected space $K$ such that every bounded operator $T : C(K) \to C(K)$ is of the form $T = g \cdot I + S$, where $S : C(K) \to C(K)$ is weakly compact. cf. GP(2004).
  Consequently, $C(K) \not\sim C(K+1)$, and $C(K)$ is not isomorphic to $C(L)$ with $L$ totally disconnected.

- **Aviles-Koszmider (2011):** There is a space $K$ which is not Radon-Nikodym compact but is a continuous image of an RN compactum; it follows that $C(K)$ is not isomorphic to $C(L)$ with $L$ totally disconnected.
Some questions

- Suppose that $C(K)$ and $C(L)$ are isomorphic. How $K$ is topologically related to $L$?
- Suppose that $C(K)$ can be embedded into $C(L)$, where $L$ has some property $\mathcal{P}$. Does $K$ has property $\mathcal{P}$?
Results on positive embeddings

An embedding \( T : C(K) \to C(L) \) is **positive** if \( C(K) \ni g \geq 0 \) implies \( Tg \geq 0 \).

**Theorem**

Let \( T : C(K) \to C(L) \) be a positive isomorphic embedding. Then there is \( p \in \mathbb{N} \) and a finite valued mapping \( \varphi : L \to [K]^{\leq p} \) which is onto \( (\bigcup_{y \in L} \varphi(y) = K) \) and upper semicontinuous (i.e. \( \{y : \varphi(y) \subseteq U\} \subseteq L \) is open for every open \( U \subseteq K \)).

**Corollary**

If \( C(K) \) can be embedded into \( C(L) \) by a positive operator then 
\( \tau(K) \leq \tau(L) \) and if \( L \) is Frechet (or sequentially compact) then \( K \) is Frechet (sequentially compact).

Remark: \( p \) is the integer part of \( \|T\| \cdot \|T^{-1}\| \).
Theorem

If $C(K) \sim C(L)$ then there is nonempty open $U \subseteq K$ such that $\overline{U}$ is a continuous image of some compact subspace of $L$. In fact the family of such $U$ forms a $\pi$-base in $K$.

Corollary

If $C[0, 1]^\kappa \sim C(L)$ then $L$ maps continuously onto $[0, 1]^\kappa$. 
Corson compacta

*K* is **Corson compact** if *K* $\hookrightarrow \Sigma(\mathbb{R}^\kappa)$ for some $\kappa$, where

$$\Sigma(\mathbb{R}^\kappa) = \{ x \in \mathbb{R}^\kappa : |\{ \alpha : x_\alpha \neq 0 \}| \leq \omega \}.$$

This is equivalent to saying that $C(K)$ contains a point-countable family separating points of *K*.

**Problem**

*Suppose that* $C(K) \sim C(L)$, where *L* is Corson compact. *Must* *K* be Corson compact?*

The answer is ‘yes’ under $MA(\omega_1)$.

**Theorem**

*If* $C(K) \sim C(L)$ *where* *L* *is Corson compact then* *K* *has a* $\pi$ *— base of sets having Corson compact closures. In particular,* *K* *is itself Corson compact whenever* *K* *is homogeneous.*
Basic technique

If $\mu$ is a finite regular Borel measure on $K$ then $\mu$ is a continuous functional $C(K)$: $\mu(g) = \int g \, d\mu$ for $\mu \in C(K)$. In fact, $C(K)^*$ can be identified with the space of all signed regular measures of finite variation (i.e. is of the form $\mu_1 - \mu_2$, $\mu_1, \mu_2 \geq 0$).

Let $T : C(K) \to C(L)$ be a linear operator. Given $y \in L$, let $\delta_y \in C(L)^*$ be the Dirac measure.

We can define $\nu_y \in C(K)^*$ by $\nu_y(g) = Tg(y)$ for $g \in C(K)$ ($\nu_y = T^*\delta_y$).

**Lemma**

Let $T : C(K) \to C(L)$ be an embedding such that for $g \in C(K)$

$$m \cdot \|g\| \leq \|Tg\| \leq \|g\|.$$

Then for every $x \in K$ and $m' < m$ there is $y \in L$ such that $\nu_y(\{x\}) > m'$.
An application

Theorem (W. Marciszewski, GP (2000))

Suppose that $C(K)$ embeds into $C(L)$, where $L$ is Corson compact. Then $K$ is Corson compact provided

- $K$ is linearly ordered compactum, or
- $K$ is Rosenthal compact.

Problem

Can one embed $C(2^{\omega_1})$ into $C(L)$, $L$ Corson?

No, under MA+ non CH.
No, under CH (in fact whenever $2^{\omega_1} > \mathfrak{c}$).