Complemented subspaces of $C(K)$ spaces

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Ancient facts

1. Sobczyk: c_0 (isomorphic to $C[0, \omega]$) is 2-complemented in every separable superspace.
2. ℓ_∞ is injective (complemented in **every** superspace).
3. Phillips: c_0 is not complemented in ℓ_∞.

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Lindestrauss and Wulbert (1969), Benyamini (1973)

A 1-complemented **separable** subspace of $C(K)$ is isomorphic to a space of continuous functions.

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6. Every complemented subspace of $C]0,\omega^\omega]$ is isomorphic either to c_0 or to the whole space (Benyamini, 1978).
7. A subspace of $c_0(\Gamma)$ is complemented if and only if it is isomorphic to some $c_0(\Gamma')$ with $|\Gamma'| \leq |\Gamma|$ (Granero, 1998).

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The problem if such a result holds in ZFC was investigated, in particular, by **Koszmider** (2007), **Brech and Koszmider** (2014), **Dow** (2016), **Dow** and **Shelah** (2018).

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There are two (separable scattered) compacta K and L, such that $C(L) \simeq C(K) \oplus X$, X is 1-complemented, while X is not isomorphic to a Banach space of continuous functions.

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Antonio Acuaviva (2026)

The class of Banach lattices is not primary: There is L and a decomposition $C(L) = X \oplus \widetilde{X}$ such that neither summand is isomorphic to a Banach lattice.

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$$0 \longrightarrow c_0 \xrightarrow{j} B \xrightarrow{\rho} c_0(\mathfrak{c}) \longrightarrow 0$$

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Lemma.

Let $\theta : L \to K$ be a continuous surjection. Then $C(L)$ contains an isometric copy of $C(K)$, that is $\{g \circ \theta : g \in C(K)\}$.

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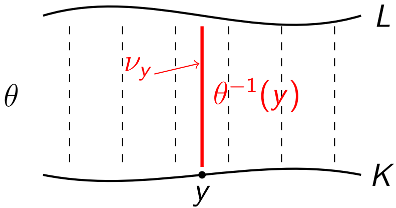

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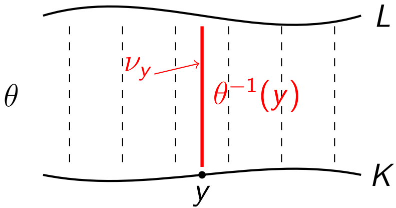

Proof: Define $T : C(L) \to C(K)$ by $Tf(y) = \int_L f \, \mathrm{d}\nu_y$ and $Pf = (Tf) \circ \theta$.

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A Banach space X is a space of continuous functions if and only if

- X^* is a L_1-space;
- the set of extreme points in B_{X^*} is *weak**-compact;
- there is an extreme point in B_X.

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Basic idea

1. Find X such that whenever $F \subseteq B_{X^*}$ is norming then
2. there is $h \in C(F)$ which is not an evaluation at some $x \in X$.

Almost disjoint families

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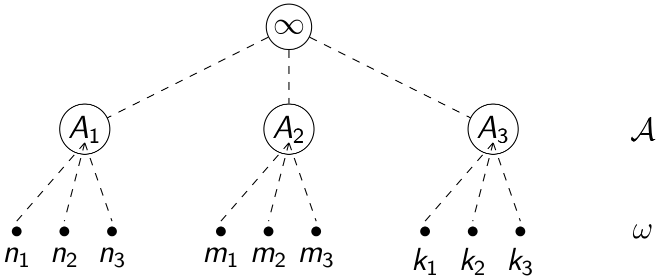
The Johnson-Lindestrauss space $\mathrm{JL}(\mathcal{A})$ is the closed subspace of ℓ_∞ spanned by $\chi_{\mathcal{A}}, A \in \mathcal{A}$ and c_{00}.
$\mathrm{JL}(\mathcal{A})$ is isometric to $C(K_{\mathcal{A}})$, where $K_{\mathcal{A}}$ is the Stone space of the algebra of sets generated by $\mathcal{A}$ and all finite subsets of ω.

Equivalently

$K_{\mathcal{A}} = \omega \cup \mathcal{A} \cup \{\infty\}$, where $\{A\} \cup (A \setminus I)$ with I finite is a neighbourhood of $A \in K_{\mathcal{A}}$.



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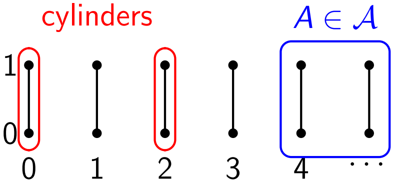
4. **Avilés, Martínez-Cervantes, Rodrígues** (2019, *weak**-topology), **Koszmider, Lautsen** (2021, few operators), **Guzmán, Hrušák, Koszmider** (2021), C^*-algebras),

The framework for our construction

- Consider $\omega \times \{0,1\}$.
- A cylinder is a set

$$c_n = \{(n,0),(n,1)\}.$$

- We construct an almost disjoint family $\mathcal{A}$ of size $\mathfrak{c}$, consisting of infinite unions of cylinders.



The framework (2)

1. Divide every $A \in \mathcal{A}$ into two parts $A = B_0^A \cup B_1^A$, and consider $\mathcal{B} = \{B_0^A, B_1^A : A \in \mathcal{A}\}$.
2. Show that $\mathrm{JL}(\mathcal{B}) = \mathrm{JL}(\mathcal{A}) \oplus X$, where X is not isomorphic to a space of continuous functions.