Complemented subspaces of $C(K)$ spaces

Grzegorz Plebanek

Instytut Matematyczny, Uniwersytet Wrocławski

FAME (June, 2026)

Complemented subspaces of Banach spaces

A subspace X of Y is complemented if there is closed subspace $Z \subseteq Y$ such that $Y = X \oplus Z$. Equivalently,
there is a bounded operator $P : Y \to Y$ which is a projection, that is $P \circ P = P$ and $X = P[Y]$. Then Z is the kernel of P.
If $\|P\| \leq c$ then X is said to be $c-$complemented.

Ancient facts

1. Sobczyk: c_0 (isomorphic to $C[0,\omega]$) is 2-complemented in every separable superspace.
2. ℓ_∞ is injective (complemented in **every** superspace).
3. Phillips: c_0 is not complemented in ℓ_∞.

CSP for spaces of continuous functions

Problem

Suppose that X is a complemented subspace of $C(K)$ (of continuous functions on a compact space K).
Must X be isomorphic to a space of the form $C(L)$?

Lindestrauss and Wulbert (1969), Benyamini (1973)

A 1-complemented **separable** subspace of $C(K)$ is isomorphic to a space of continuous functions.

Some results concerning CSP

1. A complemented subspace of $C(K)$ contains an isomorphic copy of c_0 (Pełczyński, 1962).
2. In particular, ℓ_p is not isomorphic to a complemented subspace of $C[0,1]$.
3. Every complemented subspace of ℓ_∞ is isomorphic to ℓ_∞ (Lindenstrauss, 1967).
4. Every complemented subspace of $C[0,1]$ with a non-separable dual is isomorphic to $C[0,1]$ (Rosenthal, 1972).
5. Consequently, $C[0,1]$ is primary: if $C[0,1] = X \oplus Y$ then either X or Y is isomorphic to $C[0,1]$.
6. Every complemented subspace of $C]0,\omega^\omega]$ is isomorphic either to c_0 or to the whole space (Benyamini, 1978).
7. A subspace of $c_0(\Gamma)$ is complemented if and only if it is isomorphic to some $c_0(\Gamma')$ with $|\Gamma'| \leq |\Gamma|$ (Granero, 1998).

Professor Lech Drewnowski

Recall that ℓ_∞/c_0 is isometric to $C(\beta\omega \setminus \omega)$, and is a natural candidate to be universal among Banach spaces of density $\leq \mathfrak{c}$.

L. Drewnowski, J.W. Roberts, *On the primariness of the Banach space ℓ_∞/c_0*, Proc. Amer. Math. Soc. 112 (1991).

Theorem

Under CH, the space ℓ_∞/c_0 is primary.

The problem if such a result holds in ZFC was investigated, in particular, by **Koszmider** (2007), **Brech and Koszmider** (2014), **Dow** (2016), **Dow** and **Shelah** (2018).

No to CSP

Alberto Salguero Alarcón and GP (2023)

There are two (separable scattered) compacta K and L, such that $C(L) \simeq C(K) \oplus X$, X is 1-complemented, while X is not isomorphic to a Banach space of continuous functions.

de Hevia, Martínez-Cervantes, ASA and Tradacete (2025)

The above space X is not isomorphic to a Banach lattice.

Antonio Acuaviva (2026)

The class of Banach lattices is not primary: There is L and a decomposition $C(L) = X \oplus \widetilde{X}$ such that neither summand is isomorphic to a Banach lattice.

Consequences for twisted sums

Alberto Salguero Alarcón and GP (2023)

There are short exact sequences

$$0 \longrightarrow c_0 \xrightarrow{j} C(L) \xrightarrow{\rho} A \longrightarrow 0$$

where A is a Banach space not isomorphic to a space of the form $C(K)$, and

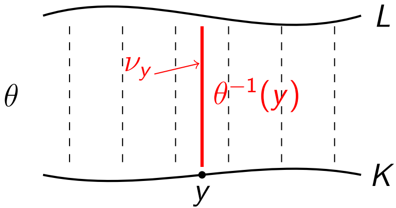
$$0 \longrightarrow c_0 \xrightarrow{j} B \xrightarrow{\rho} c_0(\mathfrak{c}) \longrightarrow 0$$

where B is a Banach space not isomorphic to a space of the form $C(K)$.

How to get $C(L) \simeq C(K) \oplus X$

Lemma.

Let $\theta : L \to K$ be a continuous surjection. Then $C(L)$ contains an isometric copy of $C(K)$, that is $\{g \circ \theta : g \in C(K)\}$.
Suppose that there is a continuous mapping $K \ni y \to \nu_y \in P(L)$ such that $\nu_y\big(\theta^{-1}(y)\big) = 1$. Then $C(K)$ is isometric to a complemented subspace of $C(L)$.



Proof: Define $T : C(L) \to C(K)$ by $Tf(y) = \int_L f \, \mathrm{d}\nu_y$ and $Pf = (Tf) \circ \theta$.

How to recognize $X \simeq C(F)$

Lindestrauss (1964)

A Banach space X is isomorphic to a space of continuous functions if and only if

- X^* is a L_1-space;
- the set of extreme points in B_{X^*} is *weak**-compact;
- there is an extreme point in B_X.

How to recognize $X \simeq C(F)$ (2)

Lemma.

Let $T : X \to C(F)$ is an isomorphism. Let

$$F' = \{T^*\delta_a : a \in F\} \subseteq X^*.$$

Then

$$X \ni x \to x|F' \in C(F')$$

is an isomorphism.

Basic idea

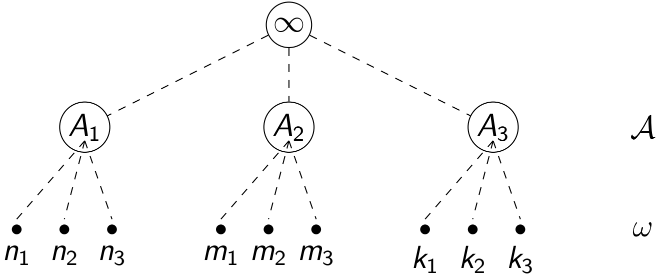
1. Find X such that whenever $F \subseteq B_{X^*}$ is norming then
2. there is $h \in C(F)$ which is not an evaluation at some $x \in X$.

Almost disjoint families

A family $\mathcal{A}$ of subsets of ω is almost disjoint if $A \cap B$ is finite for an distinct $A, B \in \mathcal{A}$.
The Johnson-Lindestrauss space $\mathrm{JL}(\mathcal{A})$ is the closed subspace of ℓ_∞ spanned by $\chi_\mathcal{A}, A \in \mathcal{A}$ and c_{00}.
$\mathrm{JL}(\mathcal{A})$ is isometric to $C(K_\mathcal{A})$, where $K_\mathcal{A}$ is the Stone space of the algebra of sets generated by $\mathcal{A}$ and all finite subsets of ω.

Equivalently

$K_{\mathcal{A}} = \omega \cup \mathcal{A} \cup \{\infty\}$, where $\{A\} \cup (A \setminus I)$ with I finite is a neighbourhood of $A \in K_{\mathcal{A}}$.



The world of $K_{\mathcal{A}}$

1. There are $2^{\mathfrak{c}}$ pairwise nonisomorphic spaces of the form $C(K_{\mathcal{A}})$ with $|\mathcal{A}| = \mathfrak{c}$.
2. **Avilés, Marciszewski, GP**: the question if for every nonmetrizable K there is a nontrivial short exact sequence

$$0 \longrightarrow c_0 \xrightarrow{j} ? \xrightarrow{\rho} C(K) \longrightarrow 0$$

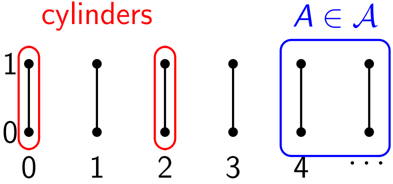
   is undecidable in ZFC.
3. **Cabello Sánchez, Castillo, Marciszewski, Salguero-Alarcón, GP**: Under MA, there is only one (up to isomorphism) $C(K_{\mathcal{A}})$ of a given size $|\mathcal{A}| < \mathfrak{c}$.
4. **Avilés, Martínez-Cervantes, Rodrígues** (2019, *weak**-topology), **Koszmider, Lautsen** (2021, few operators), **Guzmán, Hrušák, Koszmider** (2021), C^*-algebras),

The framework for our construction

- Consider $\omega \times \{0, 1\}$.
- A cylinder is a set

$$c_n = \{(n, 0), (n, 1)\}.$$

- We construct an almost disjoint family $\mathcal{A}$ of size $\mathfrak{c}$, consisting of infinite unions of cylinders.



The framework (2)

1. Divide every $A \in \mathcal{A}$ into two parts $A = B_0^A \cup B_1^A$, and consider $\mathcal{B} = \{B_0^A, B_1^A : A \in \mathcal{A}\}$.
2. Show that $\mathrm{JL}(\mathcal{B}) = \mathrm{JL}(\mathcal{A}) \oplus X$, where X is not isomorphic to a space of continuous functions.