On almost disjoint families and Johnson-Lindenstrauss Spaces

Grzegorz Plebanek

Instytut Matematyczny, Uniwersytet Wrocławski

Brazilian Workshop on Banach Spaces (August, 2026)

joint work with **Antonio Avilés** and **Luis Sáenz**

Almost disjoint families and compacta

Almost disjoint families and compacta

A family $\mathcal{A}$ of subsets of ω is almost disjoint if $A \cap B$ is finite for distinct $A, B \in \mathcal{A}$. To any $\mathcal{A}$ we can associate

Almost disjoint families and compacta

A family $\mathcal{A}$ of subsets of ω is almost disjoint if $A \cap B$ is finite for distinct $A, B \in \mathcal{A}$. To any $\mathcal{A}$ we can associate

- the locally compact topological space $\psi(\mathcal{A}) = \omega \cup \mathcal{A}$ where points in ω are isolated, and basic neighbourhoods of $A \in \mathcal{A}$ are of the form $\{A\} \cup A \setminus I$ with I finite.

Almost disjoint families and compacta

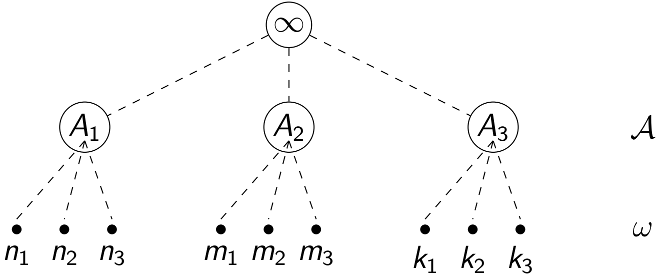
A family $\mathcal{A}$ of subsets of ω is almost disjoint if $A \cap B$ is finite for distinct $A, B \in \mathcal{A}$. To any $\mathcal{A}$ we can associate

- the locally compact topological space $\psi(\mathcal{A}) = \omega \cup \mathcal{A}$ where points in ω are isolated, and basic neighbourhoods of $A \in \mathcal{A}$ are of the form $\{A\} \cup A \setminus I$ with I finite.
- the compact space $K_{\mathcal{A}} = \psi(\mathcal{A}) \cup \{\infty\}$, the one-point compactification of $\psi(A)$.

The space $K_{\mathcal{A}}$

The space $K_{\mathcal{A}}$

$K_{\mathcal{A}} = \omega \cup \mathcal{A} \cup \{\infty\}$, where $\{A\} \cup (A \setminus I)$ with I finite is a neighbourhood of $A \in K_{\mathcal{A}}$.



Johnson-Lindenstrauss spaces

Johnson-Lindenstrauss spaces

The Johnson-Lindestrauss space $\mathrm{JL}_\infty(\mathcal{A})$ is the closed subspace of ℓ_∞ spanned by $\chi_{\mathcal{A}}, A \in \mathcal{A}$, constant functions and c_{00}.

Johnson-Lindenstrauss spaces

The Johnson-Lindestrauss space $\mathrm{JL}_\infty(\mathcal{A})$ is the closed subspace of ℓ_∞ spanned by $\chi_\mathcal{A}, A \in \mathcal{A}$, constant functions and c_{00}.

Fact

The space $\mathrm{JL}_\infty(\mathcal{A})$ is isometric to $C(K_\mathcal{A})$.

Johnson-Lindenstrauss spaces

The Johnson-Lindestrauss space $\mathrm{JL}_\infty(\mathcal{A})$ is the closed subspace of ℓ_∞ spanned by $\chi_{\mathcal{A}}, A \in \mathcal{A}$, constant functions and c_{00}.

Fact

The space $\mathrm{JL}_\infty(\mathcal{A})$ is isometric to $C(K_{\mathcal{A}})$.

The Johnson-Lindenstrauss space $\mathrm{JL}_p(\mathcal{A})$, for $p \in (1, \infty)$, is

$$\mathrm{JL}_p(\mathcal{A}) = \{f \in C(K_{\mathcal{A}}) : \|f\| < \infty\},$$

$$\|f\| = \max\{\|f\|_\infty, \|f\|_p\} \text{ and } \|f\|_p = \Big(\sum_{A \in \mathcal{A}} \big|f(A)\big|^p\Big)^{1/p}.$$

Johnson-Lindenstrauss spaces

The Johnson-Lindestrauss space $\mathrm{JL}_\infty(\mathcal{A})$ is the closed subspace of ℓ_∞ spanned by $\chi_\mathcal{A}, A \in \mathcal{A}$, constant functions and c_{00}.

Fact

The space $\mathrm{JL}_\infty(\mathcal{A})$ is isometric to $C(K_\mathcal{A})$.

The Johnson-Lindenstrauss space $\mathrm{JL}_p(\mathcal{A})$, for $p \in (1,\infty)$, is

$$\mathrm{JL}_p(\mathcal{A}) = \{f \in C(K_\mathcal{A}) : \|f\| < \infty\},$$

$$\|f\| = \max\{\|f\|_\infty, \|f\|_p\} \text{ and } \|f\|_p = \Big(\sum_{A\in\mathcal{A}} \big|f(A)\big|^p\Big)^{1/p}.$$

$$0 \longrightarrow c_0 \longrightarrow \mathrm{JL}_\infty(\mathcal{A}) \longrightarrow c_0(\kappa) \longrightarrow 0,$$

$$0 \longrightarrow c_0 \longrightarrow \mathrm{JL}_p(\mathcal{A}) \longrightarrow \ell_p(\kappa) \longrightarrow 0,$$

where $\kappa = |\mathcal{A}|$

Fifty years of JL-spaces

Fifty years of JL-spaces

Cabello Sánchez, Castillo, Marciszewski, Salguero-Alarcón, GP

There are $2^{\mathfrak{c}}$ pairwise nonisomorphic $\mathrm{JL}_\infty(\mathcal{A})$ spaces with $|\mathcal{A}| = \mathfrak{c}$.

Fifty years of JL-spaces

Cabello Sánchez, Castillo, Marciszewski, Salguero-Alarcón, GP

There are $2^{\mathfrak{c}}$ pairwise nonisomorphic $\mathrm{JL}_\infty(\mathcal{A})$ spaces with $|\mathcal{A}| = \mathfrak{c}$.
Consistently, there is only one $\mathrm{JL}_\infty(\mathcal{A})$ for $\mathcal{A}$ of size ω_1.

Fifty years of JL-spaces

Cabello Sánchez, Castillo, Marciszewski, Salguero-Alarcón, GP

There are $2^{\mathfrak{c}}$ pairwise nonisomorphic $\mathrm{JL}_\infty(\mathcal{A})$ spaces with $|\mathcal{A}| = \mathfrak{c}$.
Consistently, there is only one $\mathrm{JL}_\infty(\mathcal{A})$ for $\mathcal{A}$ of size ω_1.

1. **Avilés, Marciszewski, GP** (2018, 2020, twisted sums of c_0 and $C(K)$ spaces)

Fifty years of JL-spaces

Cabello Sánchez, Castillo, Marciszewski, Salguero-Alarcón, GP

There are $2^{\mathfrak{c}}$ pairwise nonisomorphic $\mathrm{JL}_\infty(\mathcal{A})$ spaces with $|\mathcal{A}| = \mathfrak{c}$. Consistently, there is only one $\mathrm{JL}_\infty(\mathcal{A})$ for $\mathcal{A}$ of size ω_1.

1. **Avilés, Marciszewski, GP** (2018, 2020, twisted sums of c_0 and $C(K)$ spaces)
2. **Avilés, Martínez-Cervantes, Rodrígues** (2019, *weak**-topology)

Fifty years of JL-spaces

Cabello Sánchez, Castillo, Marciszewski, Salguero-Alarcón, GP

There are $2^{\mathfrak{c}}$ pairwise nonisomorphic $\mathrm{JL}_\infty(\mathcal{A})$ spaces with $|\mathcal{A}| = \mathfrak{c}$. Consistently, there is only one $\mathrm{JL}_\infty(\mathcal{A})$ for $\mathcal{A}$ of size ω_1.

1. **Avilés, Marciszewski, GP** (2018, 2020, twisted sums of c_0 and $C(K)$ spaces)
2. **Avilés, Martínez-Cervantes, Rodrígues** (2019, *weak**-topology)
3. **Koszmider, Lautsen** (2021, few operators)

Fifty years of JL-spaces

Cabello Sánchez, Castillo, Marciszewski, Salguero-Alarcón, GP

There are $2^{\mathfrak{c}}$ pairwise nonisomorphic $\mathrm{JL}_\infty(\mathcal{A})$ spaces with $|\mathcal{A}| = \mathfrak{c}$. Consistently, there is only one $\mathrm{JL}_\infty(\mathcal{A})$ for $\mathcal{A}$ of size ω_1.

1. **Avilés, Marciszewski, GP** (2018, 2020, twisted sums of c_0 and $C(K)$ spaces)
2. **Avilés, Martínez-Cervantes, Rodrígues** (2019, *weak**-topology)
3. **Koszmider, Lautsen** (2021, few operators)
4. **Guzmán, Hrušák, Koszmider** (2021, C^*-algebras)

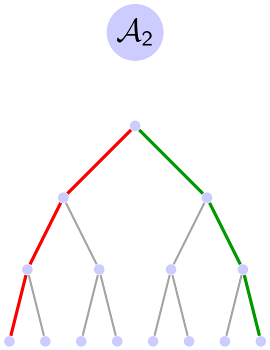
Fifty years of JL-spaces

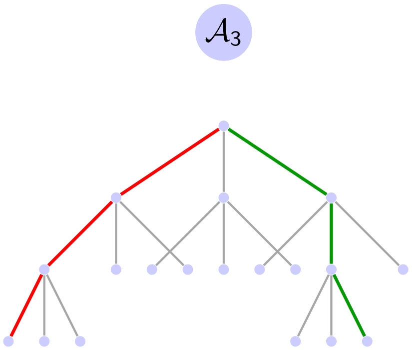
Cabello Sánchez, Castillo, Marciszewski, Salguero-Alarcón, GP

There are $2^{\mathfrak{c}}$ pairwise nonisomorphic $\mathrm{JL}_\infty(\mathcal{A})$ spaces with $|\mathcal{A}| = \mathfrak{c}$. Consistently, there is only one $\mathrm{JL}_\infty(\mathcal{A})$ for $\mathcal{A}$ of size ω_1.

1. **Avilés, Marciszewski, GP** (2018, 2020, twisted sums of c_0 and $C(K)$ spaces)
2. **Avilés, Martínez-Cervantes, Rodrígues** (2019, *weak**-topology)
3. **Koszmider, Lautsen** (2021, few operators)
4. **Guzmán, Hrušák, Koszmider** (2021, C^*-algebras)
5. **Salguero-Alarcón, GP** (2023, the complemented subspace problem)

Basic problem





Are $K_{\mathcal{A}_2}$ and $K_{\mathcal{A}_3}$ different? Are $\mathrm{JL}_p(\mathcal{A}_2)$ and $\mathrm{JL}_p(\mathcal{A}_3)$ isomorphic?

From subtrees of $\omega^{<\omega}$ to JL-spaces

From subtrees of $\omega^{<\omega}$ to JL-spaces

Problem

Let T and T' be subtrees of $\omega^{<\omega}$ and let $\mathcal{A}$ and $\mathcal{A}'$ be the almost disjoint of corresponding branches.
Are $K_{\mathcal{A}}$ and $K_{\mathcal{A}'}$ homeomorphic?
Are the Banach spaces $\mathrm{JL}_p(\mathcal{A})$ and $\mathrm{JL}_p(\mathcal{A}')$ isomorphic?

On $K_{\mathcal{A}}$ compacta

On $K_{\mathcal{A}}$ compacta

Barriga-Acosta and Hernández-Hernández (2019)

There are $\mathfrak{c}$ many pairwise nonhomeomorphic spaces of the form $K_{\mathcal{A}}$ with $|\mathcal{A}| = \omega_1$.

On $K_{\mathcal{A}}$ compacta

Barriga-Acosta and Hernández-Hernández (2019)

There are $\mathfrak{c}$ many pairwise nonhomeomorphic spaces of the form $K_{\mathcal{A}}$ with $|\mathcal{A}| = \omega_1$.

Theorem

Let $\mathcal{A}_m$ be the almost disjoint family of branches of $m^{<\omega}$ and let K_m denote the corresponding compact space.
If $m < n$ then there is no continuous surjection $K_m \to K_n$.

On $K_{\mathcal{A}}$ compacta

Barriga-Acosta and Hernández-Hernández (2019)

There are $\mathfrak{c}$ many pairwise nonhomeomorphic spaces of the form $K_{\mathcal{A}}$ with $|\mathcal{A}| = \omega_1$.

Theorem

Let $\mathcal{A}_m$ be the almost disjoint family of branches of $m^{<\omega}$ and let K_m denote the corresponding compact space.
If $m < n$ then there is no continuous surjection $K_m \to K_n$.

We partially build on Marciszewski and Pol (2009).

Distinguishing $C(K)$ spaces

Distinguishing $C(K)$ spaces

- Amir,Cambern: if there is an isomorphism $U : C(K) \to C(L)$ with $\|U\|\|U^{-1}\| < 2$ then K and L are homeomorphic.

Distinguishing $C(K)$ spaces

- Amir,Cambern: if there is an isomorphism $U : C(K) \to C(L)$ with $\|U\|\|U^{-1}\| < 2$ then K and L are homeomorphic.
- Miljutin: $C(2^\omega) \simeq C[0,1]$.

Distinguishing $C(K)$ spaces

- Amir,Cambern: if there is an isomorphism $U : C(K) \to C(L)$ with $\|U\|\|U^{-1}\| < 2$ then K and L are homeomorphic.
- Miljutin: $C(2^\omega) \simeq C[0,1]$.
- Pełczyński: $\ell_\infty \simeq L_\infty[0,1]$ so $C(\beta\omega) \simeq C(S)$ where S is a nonseparable compact space without isolated points.

Distinguishing $C(K)$ spaces

- Amir,Cambern: if there is an isomorphism $U : C(K) \to C(L)$ with $\|U\|\|U^{-1}\| < 2$ then K and L are homeomorphic.
- Miljutin: $C(2^\omega) \simeq C[0,1]$.
- Pełczyński: $\ell_\infty \simeq L_\infty[0,1]$ so $C(\beta\omega) \simeq C(S)$ where S is a nonseparable compact space without isolated points.
- Korpalski and GP (2026): every isomorphism U of ℓ_∞ and $L_\infty[0,1]$ satisfies $\|U\|\|U\|^{-1}\| > 7.41$.

Distinguishing $C(K_{\mathcal{A}})$ spaces

Distinguishing $C(K_{\mathcal{A}})$ spaces

Take any $X \subseteq 2^\omega$ and consider

$$\mathcal{A}_X = \{\{x|n : n < \omega\} : x \in X\},$$

which is an almost dishoint family on $2^{<\omega}$.

Distinguishing $C(K_{\mathcal{A}})$ spaces

Take any $X \subseteq 2^\omega$ and consider

$$\mathcal{A}_X = \{\{x|n : n < \omega\} : x \in X\},$$

which is an almost dishoint family on $2^{<\omega}$.

Marciszewski (1989), CCMSP (2020)

The Borel complexity of a set $X \subseteq 2^\omega$ is reflected by the isomorphic structure of $C\left(K_{\mathcal{A}_X}\right)$:

Distinguishing $C(K_{\mathcal{A}})$ spaces

Take any $X \subseteq 2^\omega$ and consider

$$\mathcal{A}_X = \{\{x|n : n < \omega\} : x \in X\},$$

which is an almost dishoint family on $2^{<\omega}$.

Marciszewski (1989), CCMSP (2020)

The Borel complexity of a set $X \subseteq 2^\omega$ is reflected by the isomorphic structure of $C\left(K_{\mathcal{A}_X}\right)$:
If X and Y are in different Borel classes Σ_α, $\alpha > \omega$ then $C\left(K_{\mathcal{A}_X}\right)$ and $C\left(K_{\mathcal{A}_Y}\right)$ are not isomorphic.

Distinguishing $\mathrm{JL}(\mathcal{A})$ spaces

Distinguishing $\mathrm{JL}(\mathcal{A})$ spaces

Avilés, Sáenz, GP

Let $\mathcal{A}_m$ be the almost disjoint family of branches of $m^{<\omega}$ and let K_m denote the corresponding compact space.
For any $m, n > 1$ the spaces $\mathrm{JL}_p(\mathcal{A}_m)$ and $\mathrm{JL}_p(\mathcal{A}_n)$ are isomorphic, and

Distinguishing $\mathrm{JL}(\mathcal{A})$ spaces

Avilés, Sáenz, GP

Let $\mathcal{A}_m$ be the almost disjoint family of branches of $m^{<\omega}$ and let K_m denote the corresponding compact space.
For any $m, n > 1$ the spaces $\mathrm{JL}_p(\mathcal{A}_m)$ and $\mathrm{JL}_p(\mathcal{A}_n)$ are isomorphic, and so are $C(K_m)$ and $C(K_n)$.

Distinguishing $\mathrm{JL}(\mathcal{A})$ spaces

Avilés, Sáenz, GP

Let $\mathcal{A}_m$ be the almost disjoint family of branches of $m^{<\omega}$ and let K_m denote the corresponding compact space.
For any $m, n > 1$ the spaces $\mathrm{JL}_p(\mathcal{A}_m)$ and $\mathrm{JL}_p(\mathcal{A}_n)$ are isomorphic, and so are $C(K_m)$ and $C(K_n)$.

Marciszewski and Pol

If $\mathcal{A}$ is an almost disjoint family of branches of $\omega^{<\omega}$ then $\mathrm{JL}_p(\mathcal{A}) \not\approx \mathrm{JL}_p(\mathcal{A}_2)$ (here $1 < p \leq \infty$).

Distinguishing $\mathrm{JL}(\mathcal{A})$ spaces (2)

Distinguishing $\mathrm{JL}(\mathcal{A})$ spaces (2)

Avilés, Sáenz, GP

Let $n > m \geq 2$ and $1 < p \leq \infty$. Suppose that $U : \mathrm{JL}_p(\mathcal{A}_n) \to \mathrm{JL}_p(\mathcal{A}_m)$ is an isomorphic embedding. Then

$$\|U\|\|U^{-1}\| \geq \frac{n^{1-1/p}}{4(m+1)}.$$

Distinguishing $\mathrm{JL}(\mathcal{A})$ spaces (2)

Avilés, Sáenz, GP

Let $n > m \geq 2$ and $1 < p \leq \infty$. Suppose that $U : \mathrm{JL}_p(\mathcal{A}_n) \to \mathrm{JL}_p(\mathcal{A}_m)$ is an isomorphic embedding. Then

$$\|U\|\|U^{-1}\| \geq \frac{n^{1-1/p}}{4(m+1)}.$$

Avilés, Sáenz, GP

If Υ is the tree for which every node at level n has $n+1$ immediate successors, then $\mathrm{JL}_p(\mathcal{A}_\Upsilon)$ is not isomorphic to a subspace of $\mathrm{JL}_p(\mathcal{A}_2)$ for every $p \in (1, \infty]$.

Small $\mathrm{JL}(\mathcal{A})$ spaces

Small $\mathrm{JL}(\mathcal{A})$ spaces

Avilés, Sáenz, GP

Under MA_κ(σ-linked), if $\mathcal{A}, \mathcal{B}$ are two almost disjoint families of size $\kappa < \mathfrak{c}$ then the spaces $\mathrm{JL}_p(\mathcal{A})$ and $\mathrm{JL}_p(\mathcal{B})$ are isomorphic for $p \in (1, \infty]$.

Small $\mathrm{JL}(\mathcal{A})$ spaces

Avilés, Sáenz, GP

Under MA_κ(σ-linked), if $\mathcal{A}, \mathcal{B}$ are two almost disjoint families of size $\kappa < \mathfrak{c}$ then the spaces $\mathrm{JL}_p(\mathcal{A})$ and $\mathrm{JL}_p(\mathcal{B})$ are isomorphic for $p \in (1, \infty]$.

Recall that there are $2^\mathfrak{c}$ pairwise nonisomorphic $\mathrm{JL}_p(\mathcal{A})$ spaces with $|\mathcal{A}| = \mathfrak{c}$; see Cabello Sánchez, Castillo, Marciszewski, Salguero-Alarcón, GP (2020).

$|\mathcal{A}| = |\mathcal{B}| < \mathfrak{c}$

$|\mathcal{A}| = |\mathcal{B}| < \mathfrak{c}$

Let $\mathcal{A}$ and $\mathcal{B}$ live on disjoint countable sets N and N'.

$|\mathcal{A}| = |\mathcal{B}| < \mathfrak{c}$

Let $\mathcal{A}$ and $\mathcal{B}$ live on disjoint countable sets N and N'. Fix a bijection $\varphi : \mathcal{A} \to \mathcal{B}$ and consider

$$\mathcal{C} = \{A \cup \varphi(A) : A \in \mathcal{A}\}.$$

$|\mathcal{A}| = |\mathcal{B}| < \mathfrak{c}$

Let $\mathcal{A}$ and $\mathcal{B}$ live on disjoint countable sets N and N'. Fix a bijection $\varphi : \mathcal{A} \to \mathcal{B}$ and consider

$$\mathcal{C} = \{A \cup \varphi(A) : A \in \mathcal{A}\}.$$

Main step: Using MA, show that

$$\mathrm{JL}_p(\mathcal{C}) \simeq c_0 \oplus \mathrm{JL}_p(\mathcal{A}).$$

$|\mathcal{A}| = |\mathcal{B}| < \mathfrak{c}$

Let $\mathcal{A}$ and $\mathcal{B}$ live on disjoint countable sets N and N'. Fix a bijection $\varphi : \mathcal{A} \to \mathcal{B}$ and consider

$$\mathcal{C} = \{A \cup \varphi(A) : A \in \mathcal{A}\}.$$

Main step: Using MA, show that

$$\mathrm{JL}_p(\mathcal{C}) \simeq c_0 \oplus \mathrm{JL}_p(\mathcal{A}).$$

Then

$$\mathrm{JL}_p(\mathcal{C}) \simeq c_0 \oplus \mathrm{JL}_p(\mathcal{C}) \simeq \mathrm{JL}_p(\mathcal{A}).$$

$|\mathcal{A}| = |\mathcal{B}| < \mathfrak{c}$

Let $\mathcal{A}$ and $\mathcal{B}$ live on disjoint countable sets N and N'. Fix a bijection $\varphi : \mathcal{A} \to \mathcal{B}$ and consider

$$\mathcal{C} = \{A \cup \varphi(A) : A \in \mathcal{A}\}.$$

Main step: Using MA, show that

$$\mathrm{JL}_p(\mathcal{C}) \simeq c_0 \oplus \mathrm{JL}_p(\mathcal{A}).$$

Then

$$\mathrm{JL}_p(\mathcal{C}) \simeq c_0 \oplus \mathrm{JL}_p(\mathcal{C}) \simeq \mathrm{JL}_p(\mathcal{A}).$$

Repeat for $\mathrm{JL}_p(\mathcal{B})$.

Splitting AD families

Splitting AD families

Consider the m-adic tree $m^{<\omega}$ and for every $x \in m^{\omega}$ and every $i \in \{0, \ldots, m-1\}$ write

$$A_x^i = \{\sigma \in m^{<\omega} : \sigma \frown i \prec x\}.$$

Splitting AD families

Consider the m-adic tree $m^{<\omega}$ and for every $x \in m^\omega$ and every $i \in \{0, \dots, m-1\}$ write

$$A_x^i = \{\sigma \in m^{<\omega} : \sigma \frown i \prec x\}.$$

Avilés and Todorcevic (2018)

Let $\mathcal{A}^{[m]}$ be the almost disjoint family of all sets of the form A_x^i which are infinite, for $x \in m^\omega$, $i < m$. Then $K_{\mathcal{A}^{[n]}}$ is not a continuous image of $K_{\mathcal{A}^{[m]}}$ if $n > m$, and none of these compact spaces is a continuous image of K_T for any subtree T of $\omega^{<\omega}$.

Splitting AD families

Consider the m-adic tree $m^{<\omega}$ and for every $x \in m^\omega$ and every $i \in \{0, \ldots, m-1\}$ write

$$A^i_x = \{\sigma \in m^{<\omega} : \sigma \frown i \prec x\}.$$

Avilés and Todorcevic (2018)

Let $\mathcal{A}^{[m]}$ be the almost disjoint family of all sets of the form A^i_x which are infinite, for $x \in m^\omega$, $i < m$. Then $K_{\mathcal{A}^{[n]}}$ is not a continuous image of $K_{\mathcal{A}^{[m]}}$ if $n > m$, and none of these compact spaces is a continuous image of K_T for any subtree T of $\omega^{<\omega}$.

Problem

Is $C(K_{\mathcal{A}^{[m]}})$ isomorphic to $C(K_{\mathcal{A}^{[n]}})$ when $n \neq m$?

$\mathcal{A}^{[m]} = \{A_x^i : x \in m^\omega, i < m\}$

$\mathcal{A}^{[m]} = \{A_x^i : x \in m^\omega, i < m\}$

Definition

The open degree $\mathrm{odeg}(K)$ of a compact space K is defined by saying that $\mathrm{odeg}(K) \leq n$ if there is a countable family $\mathcal{V}$ of open subsets of K such that whenever $x_0, \ldots, x_n$ are pairwise distinct points in K, there are $V_0, \ldots, V_n \in \mathcal{V}$ with $x_i \in V_i$ for every i and $\bigcap_i V_i = \emptyset$.

$\mathcal{A}^{[m]} = \{A^i_x : x \in m^\omega, i < m\}$

Definition

The open degree $\mathrm{odeg}(K)$ of a compact space K is defined by saying that $\mathrm{odeg}(K) \leq n$ if there is a countable family $\mathcal{V}$ of open subsets of K such that whenever $x_0, \ldots, x_n$ are pairwise distinct points in K, there are $V_0, \ldots, V_n \in \mathcal{V}$ with $x_i \in V_i$ for every i and $\bigcap_i V_i = \emptyset$.

Avilés and Todorcevic

- $\mathrm{odeg}(K_{\mathcal{A}^{[m]}}) = m + 1$;
- $\mathrm{odeg}(K_{\mathcal{A}_T}) \leq 2$ for every subtree T of $\omega^{<\omega}$.