Twisted sums with c_0 and the CCKY problem

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The CCKY Team

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Twisted sums with $C(K)$ spaces, Trans. Amer. Math. Soc. 355 (2003), no. 11, 4523–4541.

Twisted sums of Banach spaces

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From the introduction:

Let X and Y be real Banach spaces. Then we say $\mathrm{Ext}(X,Y)=\{0\}$ if every short exact sequence $0\to Y\to Z\to X\to 0$ splits; informally this means that if Z is a Banach space containing Y and so that $Z/Y\sim X$, then there is a bounded projection of Z onto Y. A space Z with a subspace isomorphic to Y so that Z/Y is isomorphic to X is often called a twisted sum of Y and X (order is important!). Thus $\mathrm{Ext}(X,Y)=\{0\}$ if and only if every twisted sum of Y and X is trivial (i.e. reduces to $Y\oplus X$).

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Homological Methods in Banach Spaces

F. Cabello Sánchez
J.M.F. Castillo

Cambridge University Press

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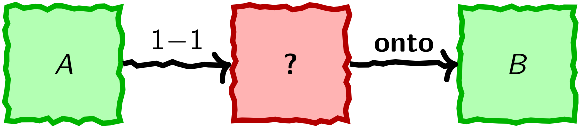

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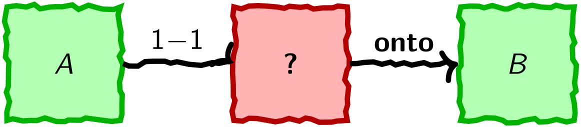

$$0 \longrightarrow Y \xrightarrow{\;j\;} Z \xrightarrow{\;\rho\;} X \longrightarrow 0$$

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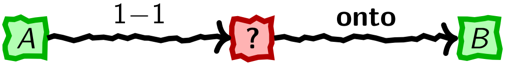

Theorem by CCKY

There is a nontrivial twisted sum $0 \to C(\omega^\omega) \to Z \to C(\omega^\omega) \to 0$

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Here $C(\omega^\omega) = C\big([0,\omega^\omega]\big)$; $[0,\omega^\omega]$ is the 'smallest' countable compactum K such that all its derivatives $K', K'', K''', \dots$ are nonempty. Then $C(K)$ is not isomorphic to c_0 (Bessaga-Pełczyński).

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From the end:

Finally, let us mention a non-separable problem related to the results of this paper. If X is a separable Banach space, then $\mathrm{Ext}(X, c_0) = \{0\}$ by Sobczyk's theorem: we do not know, however, if there is a non-metrizable compact Hausdorff space K such that $\mathrm{Ext}(C(K), c_0) = \{0\}$. It is known that...

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1. Marciszewski, GP (2018):
   NO.

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1. Marciszewski, GP (2018):
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2. Avilés, Marciszewski, GP (2020):
   YES.

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1. Marciszewski, GP (2018): **Assuming MA+¬CH,** NO.
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3. Under MA+$\neg$CH, K is sequentially compact, and if $D \subseteq K$ is a dense countable subset then every $x \in K$ is a limit of a converging sequence from D.

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4. For every κ there is a retraction $r : 2^{\mathfrak{c}} \cup \omega \to 2^{\mathfrak{c}}$ so $g \to g \circ r$ is a norm-one extension operator $C(2^{\kappa}) \to C(2^{\kappa} \cup \omega)$..

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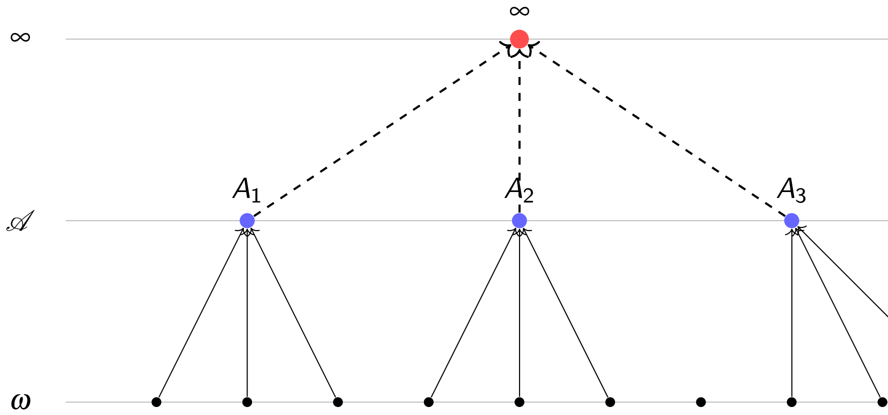
Here the ball B_{X^*} is equipped with its *weak** topology.
$B_{X^*} \cup \omega$ *that cannot be realized in* X^* means that the identity $B_{X^*} \to B_{X^*}$ does not extend to a continuous mapping $B_{X^*} \cup \omega \to X^*$.

Almost disjoint family $\mathscr{A}$ and the compact space $K_{\mathscr{A}}$



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Under Martin's axiom, every twisted sum of c_0 with $C(K_{\mathscr{A}})$ is trivial whenever $\omega_1 \leq |\mathscr{A}| < \mathfrak{c}$.

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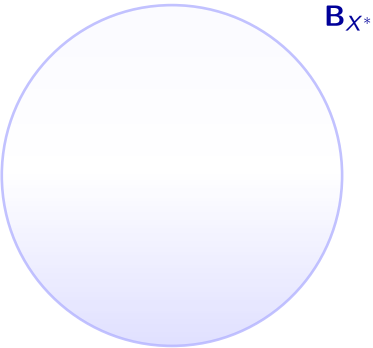
Building $B_{X^*} \cup \omega$ using $K_{\mathscr{A}} = \omega \cup K'_{\mathscr{A}}$



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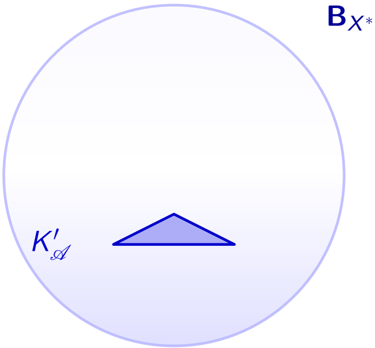

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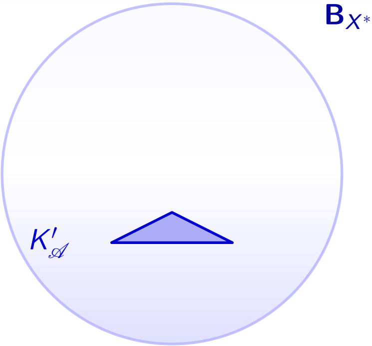

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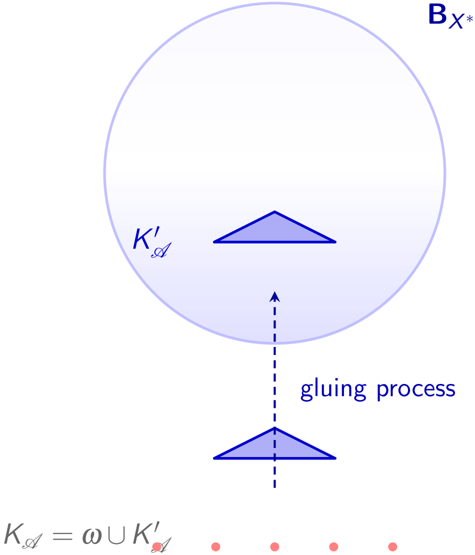

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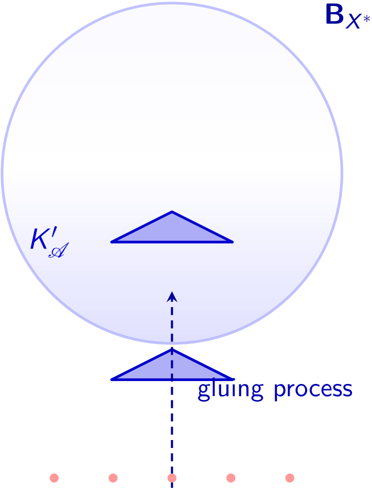

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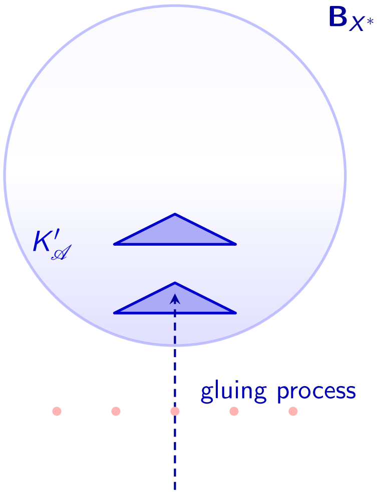

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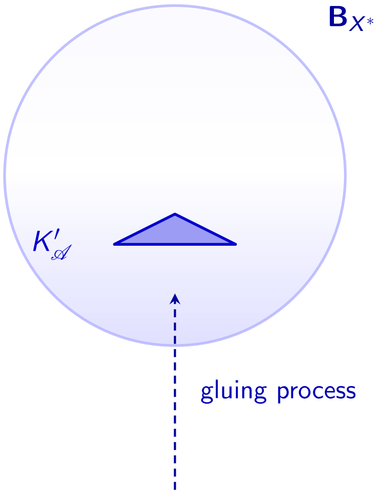

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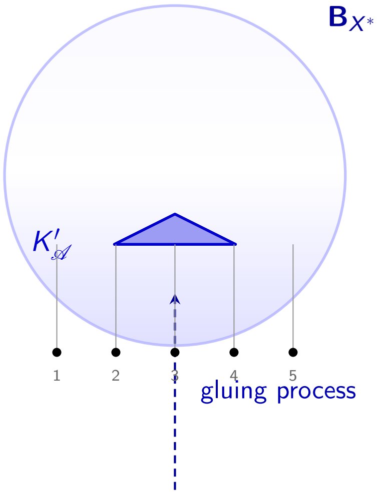

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Use $\mathscr{A}$ to form a 'difficult' space $B_{C(2^\kappa)^*} \cup 2^{<\omega}$.