Twisted sums with c_0 and the CCKY problem

Grzegorz Plebanek

University of Wrocław

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The CCKY Team

- Félix Cabello Sánchez
- Jesús M.F. Castillo
- Nigel J. Kalton
- David T. Yost

. . . and their article

Twisted sums with $C(K)$ spaces, Trans. Amer. Math. Soc. 355 (2003), no. 11, 4523–4541.

Twisted sums of Banach spaces

From the introduction:

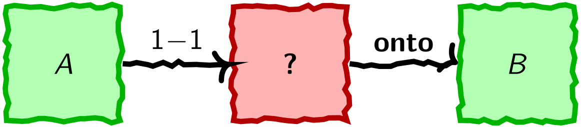
Let X and Y be real Banach spaces. Then we say $\mathrm{Ext}(X,Y)=\{0\}$ if every short exact sequence $0\to Y\to Z\to X\to 0$ splits; informally this means that if Z is a Banach space containing Y and so that $Z/Y\sim X$, then there is a bounded projection of Z onto Y. A space Z with a subspace isomorphic to Y so that Z/Y is isomorphic to X is often called a twisted sum of Y and X (order is important!). Thus $\mathrm{Ext}(X,Y)=\{0\}$ if and only if every twisted sum of Y and X is trivial (i.e. reduces to $Y\oplus X$).

Homological Methods in Banach Spaces

F. Cabello Sánchez

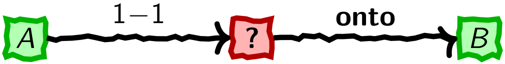
J.M.F. Castillo

Cambridge University Press



Sobczyk: c_0 is separably injective

Hence every $0 \to c_0 \to ? \to X \to 0$ is trivial for separable X.



Theorem by CCKY

There is a nontrivial twisted sum $0 \to C(\omega^\omega) \to Z \to C(\omega^\omega) \to 0$

Here $C(\omega^\omega) = C\big([0, \omega^\omega]\big)$; $[0, \omega^\omega]$ is the 'smallest' countable compactum K such that all its derivatives $K', K'', K''', \ldots$ are nonempty. Then $C(K)$ is not isomorphic to c_0 (Bessaga-Pełczyński).

From the end:

Finally, let us mention a non-separable problem related to the results of this paper. If X is a separable Banach space, then $\mathrm{Ext}(X, c_0) = \{0\}$ *by Sobczyk's theorem: we do not know, however, if there is a non-metrizable compact Hausdorff space K such that* $\mathrm{Ext}(C(K), c_0) = \{0\}$. *It is known that…*

The CCKY problem

Problem

Does there exist a nontrivial twisted sum

$$0 \to c_0 \to \boxed{?} \to C(K) \to 0$$

whenever K is a nonmetrizable compact space?

Two answers

1. Marciszewski, GP (2018): **Assuming MA+$\neg$CH,** NO.
2. Avilés, Marciszewski, GP (2020): **Assuming CH,** YES.

In particular,

Corollary (to MP 2018) and Correa and Tausk (2016)

The question if there exist a nontrivial twisted sum

$$0 \to c_0 \to \boxed{?} \to C(K) \to 0,$$

is undecidable in ZFC for the Cantor cube $K = \{0,1\}^{\omega_1}$.

1. $|C(K)| = \mathfrak{c}$.
2. The space K has $2^{\mathfrak{c}}$ points if $\omega_1 = \mathfrak{c}$ but only $\mathfrak{c}$ points in other models of set theory.
3. Under MA+$\neg$CH, K is sequentially compact, and if $D \subseteq K$ is a dense countable subset then every $x \in K$ is a limit of a converging sequence from D.

Basic idea: countable discrete extensions

1. If $K \cup \omega$ is a compact space then
$$0 \to c_0 \to \boxed{C(K \cup \omega)} \to C(K) \to 0.$$
2. This gives a nontrivial twisted sum iff there is no bounded extension operator $C(K) \to C(K \cup \omega)$.
3. **Correa and Tausk (2016):** there is nontrivial short exact sequence
$$0 \to c_0 \to \boxed{?} \to C(2^{\mathfrak{c}}) \to 0.$$
4. For every κ there is a retraction $r : 2^{\mathfrak{c}} \cup \omega \to 2^{\mathfrak{c}}$ so $g \to g \circ r$ is a norm-one extension operator $C(2^{\kappa}) \to C(2^{\kappa} \cup \omega)$..

Countable discrete extensions and twisted sums

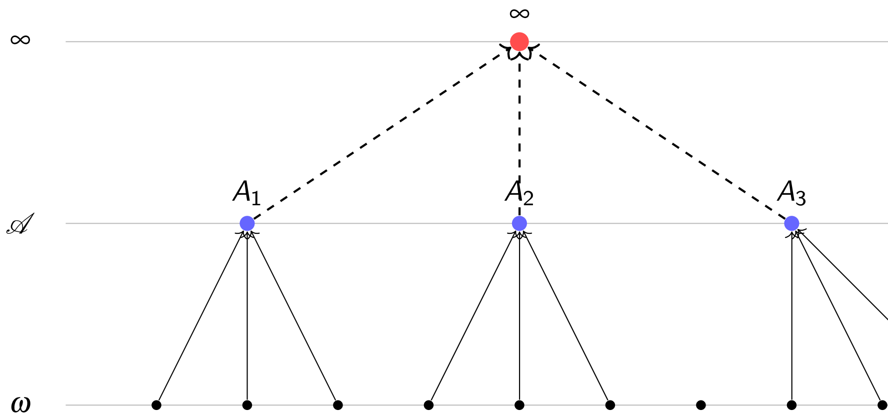
Theorem

For a Banach space X TFAE

1. there is nontrivial $0 \to c_0 \to \boxed{?} \to X \to 0$;
2. there is $B_{X^*} \cup \omega$ that cannot be realized in X^*.

Here the ball B_{X^*} is equipped with its *weak** topology.
$B_{X^*} \cup \omega$ *that cannot be realized in* X^* means that the identity $B_{X^*} \to B_{X^*}$ does not extend to a continuous mapping $B_{X^*} \cup \omega \to X^*$.

Almost disjoint family $\mathscr{A}$ and the compact space $K_{\mathscr{A}}$



Almost disjoint families and Banach spaces

Consider a family $\mathscr{A}$ of subsets if ω such that $A \cap B$ is finite for any distinct $A, B \in \mathscr{A}$.

The Johnson-Lindenstrauss space $\mathrm{JL}(\mathscr{A})$ is the closed linear span of $\chi_A, A \in \mathscr{A}$ and finitely supported elements of ℓ_∞.

$\mathrm{JL}(\mathscr{A})$ is isometric to $C(K_{\mathscr{A}})$, where $K_{\mathscr{A}}$ is a compact space associated with $\mathscr{A}$.

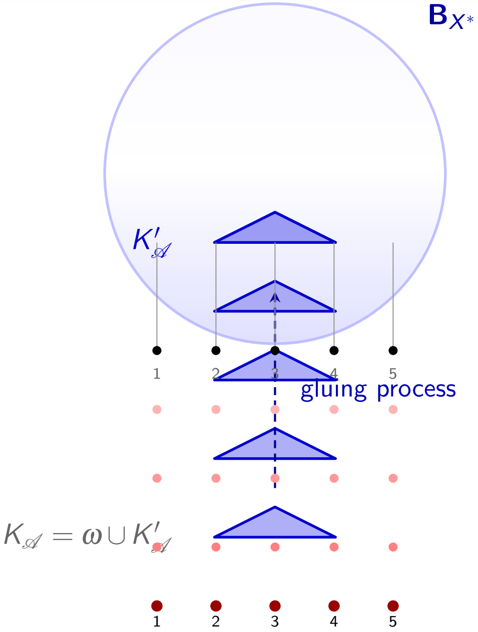
The second counterexample to CCKY

Under Martin's axiom, every twisted sum of c_0 with $C(K_{\mathscr{A}})$ is trivial whenever $\omega_1 \leq |\mathscr{A}| < \mathfrak{c}$.

Cabello Sánchez, Castillo, Marciszewski, Salguero-Alarcón:

Under Martin's axiom, there is only one (up to isomorphism) $\mathrm{JL}(\mathscr{A})$ of a given size $|\mathscr{A}| < \mathfrak{c}$ (while there are $2^{\mathfrak{c}}$ pairwise non-isomorphic $\mathrm{JL}(\mathscr{A})$ space for $|\mathscr{A}| = \mathfrak{c}$).

Building $B_{X^*} \cup \omega$ using $K_{\mathscr{A}} = \omega \cup K'_{\mathscr{A}}$



What else? First option

Escaping set theory:

Problem

Does there exist a nontrivial twisted sum

$$0 \to c_0 \to \boxed{?} \to C(K) \to 0$$

whenever K ~~is a nonmetrizable compact space~~ is of weight $\geq \mathfrak{c}$?

What else? Second option

Digging into cardinal numbers: Let κ_0 be the least κ admitting nontrivial

$$0 \to c_0 \to \boxed{?} \to C(2^\kappa) \to 0.$$

Theorem (Avilés, Marciszewski, GP)

$\kappa_0 \leq \mathrm{non}(\mathscr{E})$.

Here $\kappa = \mathrm{non}(\mathscr{E})$ is the least cardinality of $Z \subseteq [0,1]$ that cannot be covered by a sequence of closed sets of measure zero.
Assume that Z is a subset the Cantor set 2^ω.
Consider the almost disjoint family $\mathscr{A}$ of branches
$A(z) = \{z|n : n \in \omega\}$, $z \in Z$.
Use $\mathscr{A}$ to form a 'difficult' space $B_{C(2^\kappa)^*} \cup 2^{<\omega}$.