A remark on infinite discrete isometry groups

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Abstract

We prove that arbitrary infinite discrete isometry groups of euclidean space are closely related to crystallographic groups.

Introduction. Crystallographic groups are discrete isometry groups of euclidean space containing large subgroup of translations. This subgroup of translations appear naturally in applications, but it is also natural to ask if different types of symmetry are possible. In particular, discrete isometry groups containing few (or no translations) are of some interest. The topic of crystallographic groups is classical and was studied quite extensively. Also discrete group actions (frequently on spaces more general than euclidean space) were (and are) subject of intensive research. So it is likely that the result contained in this note are known. However searching massive literature is much harder than proving theorems below. While I make no claim concerning novelty of the results the exposition still may be of some interest. In particular we get easy proof of main part of 0.5 using (hard) theorem of Gromov.

Our result essentially says that to get an infinite discrete isometry group of of euclidean space one takes a crystallographic group G_1 acting on a subspace. Then one takes representation of G_1 via isometries on the complementary subspace. Finally one add a finite extension. Since the image of representation in the step two is (in general) not closed, one has a lot of possible representations. Still, the overall group structure look simple.

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Notation and preliminaries By euclidean space we mean finite dimensional vector space over $\mathcal{R}$ equipped with the scalar product. Scalar product determines norm and metric. We denote by $O(H)$ group of orthogonal transformations of H. As usual $O(n)$ (where n is an positive integer) we denote group of orthogonal transformations of $\mathcal{R}^n$.

It is well known that each isometry of an euclidean space is a composition of translation and orthogonal transformation and the isometry group of an euclidean space is a cross product of translations with orthogonal group (translation being a normal subgroup).

If h_1 is an isometry of H_1 and h_2 is an isometry of H_2 then the product map: $(x, y) \mapsto (h_1(x), h_2(y))$ is an isometry of $H = H_1 \oplus H_2$ (with the scalar product on H being the sum of scalar products on H_1 and H_2. Product map gives an injective homomorphism from the product of isometry groups of H_1 and H_2 into isometry group of H. Not all isometries are in the image of the product map: all orthogonal transformation corresponding to product maps have H_1 and H_2 as invariant subspaces. In fact, if an orthogonal transformation corresponding to an isometry of H leave H_1 and H_2 invariant then the corresponding map may be written as a product map.

Given a subgroup G of the isometry group of H we say that the action of G splits if there are two orthogonal subspaces H_1 and H_2 of H such that $H = H_1 \oplus H_2$ and each isometry g from G can be written as a product of isometries g_1 of H_1 and g_2 of H_2. If the action of G splits we say that g_1 (respectively g_2) is the restriction of g to H_1 (respectively H_2). We denote by π_{H_1} the group homomorphism mapping g to g_1.

We say that a subgroup S of G is discrete if and only if subset topology on S is discrete. For subgroups of isometry groups of euclidean space an equivalent condition is: intersection of the S-orbit of any x has finite intersection with any compact set.

Results

Theorem 0.1 *Let G be an infinite discrete subgroup of the isometry group of $\mathcal{R}^3$ which contains no translations. G contains a finite index subgroup generated by a single element g. The component of g in $O(3)$ is a rotation by irrational angle, and the translational part along the axis of the rotation is nonzero.*

Proof: This follows immediately from 0.2 below. In detail, let $g \in G_1$. Rotational part of g must be a rotation by irrational angle (otherwise some power of g would give a nontrivial translation). So V is just axis of the rotation and is one dimensional. Consequently, G_1 is generated by a single element and g generate subgroup of finite index in G_1 (hence in G). $\diamond$

Theorem 0.2 *Let G be an infinite discrete subgroup of the isometry group of n-dimensional euclidean space H which contains no translations. Then there exists a subspace V such that the action of G splits into a direct sum of actions on V and orthogonal complement of V. Moreover there exists a normal subgroup $G_1 \subset G$ of finite index such that the restriction π_V of G_1 to V is one-to-one and the image of π_V is a discrete subgroup of the group of translation of V. In particular G_1 is finitely generated (of rank less then n), torsion free and commutative.*

Proof: Consider a quotient mapping ψ from $M(H)$ into $O(H)$. Let G_o be the closure of $\psi(G)$ in $O(H)$ and let T be the connected component of the identity in G_o.

Lemma 0.3 *If G_1 and G_2 are subgroups of a Lie group G_{Lie} and G_1 is a subgroup if finite index in G_2 then closure of G_1 in G_{Lie} and closure of G_2 in G_{Lie} have the same connected component of the identity.*

Proof: Let S_1 be the closure of G_1 in G_{Lie} and S_2 be the closure of G_2 in G_{Lie}. Since G_1 if of finite index in G_2, there are $x_1, \ldots, x_m$ such that $G_2 \subset \bigcup_i x_i G_1$. So also $G_2 \subset S_0 = \bigcup_i x_i S_1$. However S_0 is closed (as a finite sum of closed sets), so $S_2 \subset \bigcup_i x_i S_1$. Hence, S_2 has the same dimension as S_1. Now, connected component of the identity in S_1 is contained in the component of the identity in S_2. Since both contained in the components are of the same dimension they must be equal. $\diamond$

Lemma 0.4 *Let S be a compact Lie group. There exists a neighbourhood U of the identity id in S such that if G is a discrete subgroup of S and $g \neq id$ is an element of G closest to the id then g commutes with $G \cap U$*

Proof: commutator mapping $(x, y)- > x^{-1}yxy^{-1}$ is smooth and has value id at (id, id) (where id the identity matrix). So, in some neighbourhood of the identity in S we have $d(x^{-1}yxy^{-1}, id) \leq Cd(x, id)d(y, id)$ where d is the Riemannian metric in S. If $d(x, id)$ and $d(y, id)$ is small enough we have: $d(x^{-1}yxy^{-1}, id) < \min(d(x, id), d(y, id))$. We choose U a Riemannian ball around id such that the above inequality holds when $x \in U$ and $y \in U$. Now, if $g \notin U$ then $G \cap U = \{id\}$ and the claim is trivial. If $ginU$ and $yinG \cap U$ then $y_0 = g^{-1}ygy^{-1} \in G$ and $d(y_0, id) < (d(g, id)$ which (by definition of g) is possible only if $y_0 = id$. So also in this case g and y commute. $\diamond$

Lemma 0.5 *T is commutative*

Proof: First, consider the case of finitely generated G. Isometry group of euclidean space is of polynomial growth, so also G (being a closed subgroup) is of polynomial growth. By Gromow theorem [1] G has nilpotent subgroup G_1 of finite index. Consequently $\psi(G_1)$ and its closure is nilpotent. By 0.3 T is equal to the connected component of the identity in the closure of $\psi(G_1)$, so also is nilpotent. However, connected nilpotent subgroup of a compact group is commutative, which gives our claim for finitely generated G.

Let T_S be the the connected component of the identity in the closure of finitely generated subgroup S of G, let T_0 be T_S of maximal dimension and let S_0 be a fixed S corresponding to T_0. Note that $T_{G_1} \subset T_0$ for all finitely generated G_1. Namely, the sum of generators G_1 and S_0 generate a subgroup G_2 such that $T_{G_1} \subset T_{G_2}$ and $T_0 \subset T_{G_2}$. Since dimensions of T_0 and T_{G_2} are equal (and both are connected) we have $T_0 = T_{G_2}$. Now, maximality of T_0 implies that $\psi(G)$ normalizes T_0. Since T_0 is a (multidimensional) torus is group of authomorphisms is discrete. On the other hand inner automorphizm of $\psi(G)$ act trivially on T_0 if and only if it acts trivially on the Lie algebra of T_0. However, inner automorphizms of $O(H)$ corresponding to elements of $\psi(G)$ act on the Lie algebra of T_0 via orthogonal mappings, so they can give only finitely many authomorphisms of T_0. Hence, passing to a subgroup of finite index we may assume that $\psi(G)$ centralizes T_0. Consequently, T_0 is a central subgroup of T.

Let T_1 be the maximal central torus in T and let $S = T/T_1$. To finish the proof we need to show that $T_1 = T$ which is equivalent to S being trivial. We claim that nontrivial S contradict maximality of T_1. More precisely, we will show that nontrivial S contains a central one parameter subgroup (such subgroup would allow to enlarge T_1). Let η be the composition of ψ and the quotient map from T to S (passing if necessary to a subgroup of finite index we may assume that the image of ψ is contained in T). If there is a finitely generated subgroup of G_1 such that $\eta(G_1)$ is not discrete, then we can produce central one parameter subgroup in the same way as in the fist stage of the proof. So we need only consider the case when $\eta(G_1)$ is discrete for all finitely generated G_1. Fix enumeration $j \mapsto x_j$ of $G = \{x_1, x_2, \ldots\}$ and let S_j be the subgroup of S generated by $\eta(x_1), \ldots, \eta(x_j)$. Let $g_j \neq id$ be the element of S_j closest to the identity and let U be a neighbourhood of the identity in S given by 0.4. g_j converge to identity, so by compactness of the sphere in the Lie algebra of S we can choose natural numbers n_j such that the mappings $t \mapsto g_j^{[n_j t]}$ (where $[\cdot]$ denotes the integer part) converge to a one parameter subgroup $s(t)$ of S when j goes to infinity. By 0.4 for $j \geq i$ g_j commutes with $S_i \cap U$, so also $s(t)$ (as a limit) commutes with $S_i \cap U$. Since the sum of S_i is dense in S $s(t)$ commutes with all elements of U. Since S is connected U generates S, so $s(t)$ is central. $\diamond$

By the lemma G contains a subgroup G_1 such that $\psi(G_1) \subset T$. Since G contains no translations the kernel of ψ is trivial and ψ is injective. So G_1 is commutative. We put $V = \{x \in H : \forall_{g \in T} gx = x\}$. Since T as a connected component of the identity is a normal subgroup of the closure of $\psi(G)$, also $\psi(G)$ normalizes T and consequently V is invariant under the action of $\psi(G)$.

This gives the first part of the theorem: the action of G is a direct sum of actions on V and its orthogonal complement.

In the sequel (if necessary replacing G by current G_1) we may assume that $\psi(G) \subset T$ and G is commutative. Let G_1 be a finitely generated subgroup of G such that $V = \{x \in H : \forall_{g \in \psi(G_1)} gx = x\}$. Such subgroups exist: the set of fixed points of $\psi(G_1)$ is a linear space. If this set is bigger than V, then there is $x \notin V$ fixed by $\psi(G_1)$. By definition of V we can find an $g \in G$ such that $\psi(g)x \neq x$. Adding this g to G_1 we reduce the dimension of the set of fixed points of $\psi(G_1)$. After finitely many such steps we will get set of the same dimension as V, hence equal to V. Let $g_1, \ldots, g_n$ generate G_1 (we fix a set of generators) and let W be the orthogonal complement of V in H. We claim that there exists an $\epsilon > 0$ such that $\|\psi(g_k)x - x\| \geq \epsilon\|x\|$ for all $x \in W$. Namely, it is enough to prove the inequality for x belonging to the unit sphere of W. But then $\|\psi(g_k)x - x\|$ is a positive (since $\psi(g_k)x \neq x$) continuous function on a compact set, so it is bounded from below by a positive ϵ.

Now, consider restriction to W of an arbitrary $g \in G$. We can write $gx = R_g x + w_g$ where $R_g \in O(W)$ and $w_g \in W$.

Lemma 0.6 $\|w_g\| \leq 2\max_{i=1,\dots,n} \|w_{g_i}\|/\epsilon$

Proof: G is now commutative, so $gg_i = g_i g$. We write $gg_i x = R_g(R_{g_i}x + w_{g_i}) + w_g = R_g R_{g_i} x + R_g w_{g_i} + w_g$, $g_i g x = R_{g_i}(R_g x + w_g) + w_{g_i} = R_{g_i} R_g x + R_{g_i} w_g + w_{g_i}$, so $R_g w_{g_i} + w_g = R_{g_i} w_g + w_{g_i}$ and $R_{g_i} w_g - w_g = R_g w_{g_i} - w_{g_i}$ Now $\|w_g\| \leq \|R_{g_i} w_g - w_g\|/\epsilon = \|R_g w_{g_i} - w_{g_i}\|/\epsilon \leq 2\max_{i=1,\dots,n} \|w_{g_i}\|/\epsilon$. $\diamond$

Finally, consider kernel of π_V. If $g \in \ker \pi_V$ then g restricted to V is an identity. We can identify the isometry group of W with the subgroup of isometry group of H which restricted to V is an identity. So, we can identify $\ker \pi_V$ with a discrete subgroup of the isometry group of W. By the lemma above this subgroup is contained in a bounded (hence compact) subset, so $\ker \pi_V$ is finite. It follows that π_V is discrete. Since $\pi_V(G)$ consists of translation it is a free abelian group of rank not exceeding the dimension of V. Since $\pi_V(G)$ is free abelian we can build the inverse map giving us a subgroup G_1 of finite index in G such that $\pi_V(G_1)$ is a discrete subgroup of translation group of V. Passing if necessary to a subgroup of finite index we may assume that G_1 is normal in our original G. $\diamond$

Theorem 0.7 *The conclusion of 0.2 remains valid for arbitrary discrete subgroups of the isometry group of euclidean space H.*

Proof: Again, let ψ be the quotient map from isometry group to $O(H)$. Consider the subgroup T of translations contained in G. T is a normal subgroup, so corresponding vectors span a subspace W of H invariant under the action of $\psi(G)$. Consequently, the action of G splits into direct sum of actions on W and on its orthogonal complement U. Each element of T restricted to U gives identity, so the restriction to U gives mapping from G/T into isometry group of U. Since T is a co-compact subgroup of isometry group of W the restriction π_U considered as mapping from G/T is closed and has finite kernel. Also, the image of π_U contains no translations. Namely, if $\pi_U(g)$ is a translation then $gx = R_g x + u + w$ where $R_g \in O(W)$, $u \in U$, $w \in W$. Since R_g normalizes lattice spanned by T some power of R_g equals identity $R_g^k = id$. Consider now $g^k = u' + kw$. g^k is a translation with respect to a vector not contained in W. This is a contradiction with definition of W. Now we finish the proof applying 0.2 to $\pi_U(G)$.

References

[1] M. Gromov. *Groups of polynomial growth and expanding maps.* Publ. Ihes, 53, 1981, p.53-78.