ADDENDUM TO THE PAPER
SPECTRAL MULTIPLIER THEOREM FOR HARDY SPACES ASSOCIATED WITH SCHRÖDINGER OPERATORS WITH POLYNOMIOAL POTENTIALS, BULL. LONDON MATH. SOC. 32 (2000), 571–581.

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Abstract. The aim of this addendum is to explain actions of multiplier operators on Hardy spaces associated with Schrödinger operators with polynomial potentials. In particular we show that boundedness of multiplier operators $F(A)$ on atoms proved in [5] imply existence of their continuous extensions on the Hardy space H^p_A.

1. Introduction

Let $T_t(x,y)$ be the integral kernels a semigroup of linear operators $\{T_t\}_{t>0}$ generated by a Schrödinger operator $-A = \Delta - V(x)$, where $V(x) = \sum_{\beta\le\alpha} a_\beta x^\beta$ is a nonzero, nonnegative polynomial potential on $\mathbb{R}^d$.

Denote

$$\rho(x)^{-1} = m(x,V) = \sum_{\beta\le\alpha} |D^\beta V(x)|^{1/(\beta+2)}.$$

It is not difficult to check that there exists a constants $\kappa > 1$ and $C > 0$ such that

$$C^{-1}\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{-\kappa} \le \frac{\rho(y)}{\rho(x)} \le C\Big(1+\frac{|x-y|}{\rho(x)}\Big)^{\frac{\kappa}{\kappa+1}}. \tag{1.1}$$

Note that $\rho(x) \sim \rho(y)$ if $|x-y| \le C'\rho(x)$. Moreover, since V is a nonzero polynomial, there is C such that $\rho(y) \le C$.

Fix $0 < p \le 1$. Following [4] we say that a function a is a $(1,\infty)$-atom for H^p_A, if there exists a ball $B = B(y_0, r)$, $r \le \rho(y_0)$, such that

$$\operatorname{supp} a \subset B \quad\text{and}\quad \|a\|_{L^\infty(\mathbb{R}^d)} \le |B|^{-1/p}; \tag{1.2}$$

$$\text{if } \ r \le \rho(y_0)/4, \ \text{ then } \int a(x)x^\gamma\,dx = 0 \quad \text{for all} \quad \beta,\ \ |\beta| \le d(1/p-1). \tag{1.3}$$

It follows from (1.1)-(1.3) that the atoms form a bounded set in the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$, that is, there are constants C and N such that

$$\Big|\int a(y)\varphi(y)\,dx\Big| \le \|\varphi\|_{(N)}, \tag{1.4}$$

where $\|\varphi\|_{(N)} = \max_{|\gamma|\le N}\{\sup_{x\in\mathbb{R}^d} |D^\gamma\varphi(x)|(1+|x|)^N\}$ is a seminorm in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. Therefore, for any sequence a_n of atoms and any sequence of numbers $c_n \in \mathbb{C}$ such that $\sum |c_n|^p < \infty$, we define by means of the series

$$\sum_{n=1}^\infty c_n a_n(y) \tag{1.5}$$

a tempered distribution, by the formula:

$$\langle\sum_{n=1}^{\infty} c_n a_n(y),\varphi\rangle = \sum_{n=1}^{\infty} c_n \int a_n(y)\varphi(y)\,dx. \tag{1.6}$$

Furthermore,

$$\big|\langle\sum_{n=1}^{\infty} c_n a_n(y),\varphi\rangle\big| \le C\sum_{n=1}^{\infty}|c_n|\|\varphi\|_{(N)} \le C\Big(\sum_{n=1}^{\infty}|c_n|^p\Big)^{1/p}\|\varphi\|_{(N)}. \tag{1.7}$$

Let $f\in L^2(\mathbb{R}^d)$. We say that f belongs to that H^p_A space associated with A if the maximal function $\mathcal{M}_A f(x)=\sup_{t>0}|T_t f(x)|$ belongs to $L^p(\mathbb{R}^d)$. Then we set

$$\|f\|_{H^p_A} = \|\mathcal{M}_A f\|_{L^p}. \tag{1.8}$$

It was proved in [4] that there is a constant $C>0$ such that for any H^p_A-atom a one has

$$\|\mathcal{M}_A a\|_{L^p}\le C. \tag{1.9}$$

Since the bottom of the spectrum of A is bigger than 0 (see e.g., [2]), for every $t>0$ there exists a function $e_t(\lambda)\in\mathcal{S}_0([0,\infty))$ such that

$$T_t=\int_0^\infty e^{-t\lambda}dE_A(\lambda)=\int_0^\infty e_t(\lambda)dE_A(\lambda),$$

where E_A is the spectral decomposition of A. Recall that $\phi\in\mathcal{S}_0([0,\infty))$ if $\phi\in\mathcal{S}([0,\infty))$ and $\frac{d^k}{d\lambda^k}\phi(0^+)=0$ for $k=1,2,...$ (see [3, (3.3)]). Now Proposition 3.10 of [3] applied to $\phi(\lambda)=e_t(\lambda)$ and $\mu=0$ gives that for every $b>0$ and any muli-indexes γ, γ' there is $C_{b,\gamma,\gamma'}>0$ such that

$$|D^\gamma_x D^{\gamma'}_y T_t(x,y)|\le C_{b,\gamma,\gamma'}(1+|x-y|)^{-b}. \tag{1.10}$$

Hence, for every $t>0$ and $x\in\mathbb{R}^d$, the function $\mathbb{R}^d\ni y\mapsto T_t(x,y)$ belongs to the Schwartz class $\mathcal{S}(\mathbb{R}^d)$. By (1.9) the series (1.5) defines a tempered distribution f such that

$$\mathcal{M}_A f(x)=\sup_{t>0}|\langle f, T_t(x,\ \cdot\ )\rangle|\in L^p(dx)$$

and

$$\|\mathcal{M}_A f(x)\|^p_{L^p}\le C\sum_n |c_n|^p.$$

On the other hand the following theorem was actually proved in [4].

Theorem 1.11. *There exists a constant $C>0$ such that for any $f\in L^2\cap H^p_A$ there is a sequence of numbers c_n and a sequence of atoms $a_n(x)$ such that $f=\sum_n c_n a_n$ in the sense of distributions, that is,*

$$\int f(x)\varphi(x)\,dx = \langle\sum_{n=1}^{\infty}c_n a_n,\varphi\rangle \quad \textit{for} \quad \varphi\in\mathcal{S}(\mathbb{R}^d)$$

and

$$\sum_{n=1}^{\infty}|c_n|^p\le C\|\mathcal{M}_A f\|^p_{L^p}=C\|f\|^p_{H^p_A}.$$

We may now define the spaces H^p_A as a completion of $\{f\in L^2 : \mathcal{M}_A f(x)\in L^p(dx)\}$ in the quasinorm $\|\ \|_{H^p_A}$.

2. Action of multiplier operators on H^p_A

Assume that a multiplier $F(\lambda)$, defined for $\lambda > 0$ satisfies the assumptions of Theorem 1.2 of [5], that is, for certain $b > 0$,

$$\sup_{t>0} \|\psi(\,\cdot\,)F(t\,\cdot\,)\|_{C(d/2+b)} = C_0 < \infty, \tag{2.1}$$

where $\psi \in C_c^\infty(0,\infty)$ is a fixed auxiliary nonzero function. Since $F(A) = \int_0^\infty F(\lambda)\, dE_A(\lambda)$ is a bounded operator on $L^2(\mathbb{R}^d)$, $F(A)a \in L^2(\mathbb{R}^d)$ for every atom a. It was actually proved in [5] that for $d/(d+b) < p \le 1$ there exists a constant C such that

$$\|F(A)a\|_{H^p_A} \le CC_0 \quad \text{for every atom } \; a. \tag{2.2}$$

We are now in a position to clarify the action of multipliers on the space H^p_A.

Proposition 2.3. *Let $d/(d+b) < p \le 1$. Assume that F satisfies (2.1). For $f \in H^p_A \cap L^2(\mathbb{R}^d)$, let $f = \sum_j c_j a(x)$ be its atomic decomposition. Then for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$ one has*

$$\int (F(A)f)(x)\overline{\varphi(x)}\, dx = \sum_{j=1}^\infty c_j \int (F(A)a_j)(x)\overline{\varphi(x)}\, dx. \tag{2.4}$$

Proof. We first prove (2.4) for φ of the form $\varphi = T_s\phi$ with $s > 0$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$. Let $\eta_n(x) = \eta(x/n)$, where $\eta \in C_c^\infty(\mathbb{R}^d)$, $\eta(x) = 1$ for $|x| < 1$, $0 \le \eta \le 1$.

$$\begin{aligned}\int (F(A)f)(x)\overline{\varphi(x)}\, dx &= \int f(x)\overline{(\bar F(A)T_s\phi)(x)}\, dx \\ &= \int f(x)\overline{(T_s\bar F(A)\phi)(x)}\, dx \\ &= \lim_{n\to\infty} \int f(x)\eta_n(x)\overline{(T_s\bar F(A)\phi)(x)}\, dx\end{aligned} \tag{2.5}$$

It follows from (1.10) that $\eta_n(x)\overline{((T_s\bar F(A)\phi))(x)} \in \mathcal{S}(\mathbb{R}^d)$. Hence,

$$\begin{aligned}\int (F(A)f)(x)\overline{\varphi(x)}\, dx &= \lim_{n\to\infty} \sum_j c_j \int a_j(x)\eta_n(x)\overline{(T_s\bar F(A)\phi)(x)}\, dx \\ &= \lim_{n\to\infty} \sum_j c_j \int (T_sF(A))(a_j\eta_n)(x)\overline{\phi(x)}\, dx.\end{aligned} \tag{2.6}$$

It is not difficult to prove using the fact that $\rho(x) \le C$ that multiplication by the functions η_n are uniformly bounded operators on the Hardy space H^p_A and every function $a_j\eta_n$ can be written as a finite linear combination of atoms, that is,

$$a_j\eta_n = \sum_{k=1}^{m_{j,n}} c_{j,k,n}a_{j,k,n} \quad \text{and} \quad \sum_{k=1}^{m_{j,n}} |c_{j,k,n}|^p \le C.$$

Note that the functions $F^{(s)}(\lambda) = e^{-s\lambda}F(\lambda)$ satisfy (2.1) with a constant C_0 independent of $s > 0$. Hence, using (1.4), Theorem 1.11, and (2.2) with $F = F^{(s)}$, we get that

$$\Big|\int (T_sF(A))(a_j\eta_n)(x)\overline{\phi(x)}\, dx\Big| \le C\|\phi\|_{(N)} \tag{2.7}$$

with C independent of s, n, and j. Moreover, if $n \to \infty$, then the integral in (2.7) converges to

$$\int (T_sF(A)a_j)(x)\overline{\phi(x)}\,dx = \int (F(A)a_j)(x)\overline{\varphi(x)}\,dx$$

and, consequently, we may change the order of limit and summation in (2.6) and obtain (2.4) for $\varphi = T_s\phi$.

Now we remove the assumption $\varphi = T_s\phi$. Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Since $f \in L^2(\mathbb{R}^d)$, $F(A)f \in L^2(\mathbb{R}^d)$. Recall that T_t is a strongly continuous semigroup on $L^2(\mathbb{R}^d)$. Thus,

$$\begin{aligned}\int (F(A)f)(x)\overline{\varphi(x)}\,dx &= \lim_{s\to 0}\int (F(A)f)(x)\overline{T_s\varphi(x)}\,dx \\ &= \lim_{s\to 0}\sum_j c_j \int F(A)a_j(x)\overline{T_s\varphi(x)}\,dx \\ &= \lim_{s\to 0}\sum_j c_j \int (T_sF(A)a_j)(x)\overline{\varphi(x)}\,dx,\end{aligned} \tag{2.8}$$

where in the second equality we have used already proved (2.4) for $T_s\varphi$. Again, by the same arguments we used in the first part of the proof, $\|T_sF(A)a_j\|_{H^p_A} \leq C$ independently of s and j. Thus,

$$\Big|\int (T_sF(A)a_j)(x)\overline{\phi(x)}\,dx\Big| \leq C\|\varphi\|_{(N)}.$$

So we are allowed to change the order of limit and summation in (2.8) to get (2.4). $\square$

Corollary 2.9. *If $f \in L^2(\mathbb{R}^d) \cap H^p_A$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$, then*

$$\Big|\int F(A)f(x)\varphi(x)\,dx\Big| \leq CC_0\|f\|_{H^p_A}\|\varphi\|_{(N)},$$

$$\|F(A)f\|_{H^p_A} \leq CC_0\|f\|_{H^p_A}.$$

Finally, one can deduce from Proposition 2.3 and Corollary 2.9 that $F(A)$ has the unique continuous extension on H^p_A and $\|F(A)f\|_{H^p_A} \leq CC_0\|f\|_{H^p_A}$.

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