Attendum to the paper
"Hardy spaces associated with semigroups generated by Bessel operators with potentials"
Houston Journal of Mathematics, 34 (2008), 205–234.

It was noticed by Anh Bui that the estimate (5.5) in the proof of Lemma 5.3 fails for k large. The aim of this attendum is explain that Proposition 5.1 and, consequently, Lemma 5.13 and 5.14 remain true. These are due to the facts that Lemma 5.3 and 5.8 hold for k from the interval $[1, 1+\delta_p)$ with $\delta_p > 0$, which suffices (see [11] pages 95–96).

To be more precise we reformulate Lemma 5.3 and 5.8 as follows.

Lemma 5.3. *For every $1 < p < \infty$ and every $k \geq 1$ such that $k(p-\alpha-1)+\alpha+1 > 0$ there exists a constant $C > 0$ such that for every $B = (a, b)$ and every $u \in C^1_c(a, b)$ we have*

$$\Big(\frac{1}{\mu(B)}\int_B |u|^{kp}\,d\mu\Big)^{1/(kp)} \leq C(b-a)\Big(\frac{1}{\mu(B)}\int_B |u'|^p\,d\mu\Big)^{1/p}. \tag{5.4}$$

Proof. The proof remains unchanged. □

Lemma 5.8. *Let $1 < k < \frac{\alpha+1}{\alpha-1}$, $\frac{1}{k}+\frac{1}{k'} = 1$. There exists a constant $C > 0$ such that for every v being a nonnegative local subsolution of $\mathcal{L}_0$ in (a,b) and every real valued function $\phi \in C^1_c(a, b)$ we have*

$$\int_a^b \phi^2 v^2\,d\mu \leq C(b-a)^2\Big(\frac{\mu(\{x \in (a,b) : \phi(x)v(x) \neq 0\})}{\mu((a,b))}\Big)^{1/k'}\int_a^b |\phi'|^2 v^2\,d\mu. \tag{5.9}$$

Proof. The proof is unchanged. □

Proof of Proposition 5.1. Since the proof is a compilation of ideas of [15] and [11], we present details.

For every constant $s \in \mathbb{R}$ the function

$$v(x) = \max(u(x), s) - s = u(x) - \min(u(x), s))$$

is a local subsolution of $\mathcal{L}_0$ in (a, b). Fix $x_0 \in (a + \frac{b-a}{6}, b - \frac{b-a}{6})$. Let $R = \frac{b-a}{24}$, $\rho < R$. Let $\phi \in C^\infty_c(a, b)$ be such that $\phi \geq 0$, $\phi \equiv 1$ on $B(x_0, \rho)$, $\phi \equiv 0$ outside $B(x_0, R)$, $|\phi'| \leq \frac{2}{R-\rho}$. Let

$$A(s, \rho) = \{y :\ u(y) \geq s\} \cap B(x_0, \rho).$$

By Lemma 5.8,

(A)

$$\begin{aligned}\int_{A(s,\rho)} (u-s)^2\, d\mu &\le \int_a^b v^2\phi^2 d\mu \\ &\le C(b-a)^2\Big(\frac{\mu(\{x\in(a,b):\phi(x)v(x)\neq 0\})}{\mu((a,b))}\Big)^{1/k'}\int_a^b |\phi'|v^2 d\mu \\ &\le C\frac{(b-a)^2}{(R-\rho)^2}\Big(\frac{\mu(\{x\in(a,b):\phi(x)v(x)\neq 0\})}{\mu((a,b))}\Big)^{1/k'}\int_{A(s,R)}(u-s)^2 d\mu.\end{aligned}$$

For constants $h>s$ we have

(B) $$(h-s)^2\mu(A(h,\rho)) \le \int_{A(h,\rho)}(u-s)^2 d\mu \le \int_{A(s,R)}(u-s)^2 d\mu.$$

Define

$$a(h,\rho)=\mu(A(h,\rho)),$$

$$u(h,\rho)=\int_{A(h,\rho)}(u-h)^2 d\mu.$$

From (B) and (A) we have

(C) $$a(h,\rho)\le \frac{u(s,R)}{(h-s)^2},$$

(D)

$$\begin{aligned}u(h,\rho) &= \int_{A(h,\rho)}(u-h)^2 d\mu \\ &\le \int_{A(h,\rho)}(u-s)^2 d\mu \\ &\le \int_{A(s,\rho)}(u-s)^2 d\mu \\ &\le \frac{C(b-a)^2}{\mu(a,b)^{1/k'}}\cdot\frac{1}{(R-\rho)^2}a(s,R)^{1/k'}u(s,R) \\ &= K\frac{1}{(R-\rho)^2}a(s,R)^{1/k'}u(s,R),\end{aligned}$$

where $K=\frac{C(b-a)^2}{\mu(a,b)^{1/k'}}$. For exponents ξ and η (which will be fixed latter on) we define

$$\boldsymbol{\varphi}(h,\rho)=u(h,\rho)^\xi a(h,\rho)^\eta.$$

We have for $s<h<R$

(E) $$\boldsymbol{\varphi}(h,\rho)\le \frac{K^\xi}{(h-s)^{2\eta}(R-\rho)^{2\xi}}u(s,R)^{\xi+\eta}a(s,R)^{\xi/k'}.$$

Let θ be the positive root of the equation $0 = \theta^2 - \theta - \frac{1}{k'}$. We have $\theta > 1$. Let $\eta = 1$, $\xi = \frac{\eta}{\theta - 1}$. We also have $\frac{\xi}{k'} = \theta\eta$. Hence,

$$\boldsymbol{\varphi}(h,\rho) \leq \frac{K^{\xi}}{(h-s)^{2\eta}(R-\rho)^{2\xi}} \boldsymbol{\varphi}(s,R)^{\theta}, \quad 0 < s < h < R. \tag{F}$$

Applying Lemma C.7 of [15] with $k_0 = 0$, $R_0 = 2\rho$, $\sigma = 1/2$, we have $\boldsymbol{\varphi}(d,\rho) = 0$, where

$$d^2 = \frac{2^{(2+2\xi)\theta/(\theta-1)} K^{\xi} \varphi(0, 2\rho)^{\theta-1}}{\frac{1}{2^{2\xi}}(2\rho)^{2\xi}}.$$

This means that $u(x_0) \leq d$. Finally, by simple calculations, taking $\rho = (b-a)/64$, we get

$$u(x_0) \leq d \leq M\Big(\frac{1}{\mu((a,b))} \int_{B(x_0,2\rho)} u^2 d\mu\Big)^{1/2}.$$

□

The remaining part of Section 5 stays unchanged except the definition of the fundamental solution, namely let

$$\Gamma_0(x,y) = \int_0^{\infty} p_t(x,y)\, dt$$

be the fundamental solution of $\mathcal{L}_0$. We have

$$\Gamma_0(x,y) \sim (x+y)^{-2\nu} \sim \begin{cases} x^{-2\nu} & \text{for } x \geq y; \\ y^{-2\nu} & \text{for } x < y, \end{cases}$$

where $\nu = (\alpha - 1)/2$.

Acknowledgment. I want to thank Anh Bui for pointing out the wrong statement of Lemma 5.3.

References

[11] E. B. Fabes, C. E. Kenig, R. P. Seraponi, *The local regularity of solutions of degenerate elliptic equations*. Comm. Part. Diff. Eq. 7(1) (1982), 77–116.
[15] D. Kinderlehrer, G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press 1980, N.Y.