Pluriharmonic functions on symmetric tube domains with BMO boundary values

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Abstract

Let $\mathcal{D}$ be a symmetric Siegel domain of tube type and S be a solvable Lie group acting simply transitively on $\mathcal{D}$. Assume that L is a real S-invariant second order operator that satisfies Hörmander's condition and annihilates holomorphic functions. Let $\mathbf{H}$ be the Laplace-Beltrami operator for the product of upper half planes imbedded in $\mathcal{D}$. We prove that if F is an L-Poisson integral of a BMO function and $\mathbf{H}F = 0$ then F is pluriharmonic. Some other related results are also considered.

1. Introduction.

Let $\mathcal{D}$ be a symmetric Siegel domain of tube type, i.e. $\mathcal{D} = V + i\Omega$, where Ω is an irreducible symmetric cone in a Euclidean space V. Let S be a solvable Lie group acting simply transitively on $\mathcal{D}$ which, as in previous papers [DHMP], [DHP], etc, we identify with $\mathcal{D}$. In a recent series of articles [BBDHPT], [BDH],[DHMP] pluriharmonic functions have been studied by means of S-invariant operators . More precisely, the operators of interest are real S-invariant second order, degenerate elliptic operators L that annihilate holomorphic functions F and, consequently, their real and imaginary parts. Such operators will be called admissible. If L is hypoelliptic then there is a bounded, integrable, positive function P_L on V such that Poisson integrals

$$F(w) = \int_V f(w \bullet x) P_L(x)dx = P_L f(w), \tag{1.1}$$

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$f \in L^p(V)$, $1 < p \leq \infty$ are L harmonic [DH], [DHP]. A real valued F is pluriharmonic if and only if the (distributional) Fourier transform $\hat{f}$ satisfies

$$\mathrm{supp}\hat{f} \subset \overline{\Omega} \cup -\overline{\Omega}. \tag{1.2}$$

Granted (1.1), condition (1.2) is equivalent to

$$\mathbf{H}F = 0, \tag{1.3}$$

$\mathbf{H}$ being the Laplace-Beltrami operator for the product of upper half planes imbedded in $\mathcal{D}$ [BDH]. If F is real, satisfies (1.1), (1.2) and $f \in L^2(V)$, then the conjugate pluriharmonic function $\tilde{F}$ has the same properties (up to an additive constant) and $F + i\tilde{F}$ is in the Hardy H^2 space, [DHMP]. It follows from what is shown in the present paper that the same holds if $f \in L^p(V), 1 < p < \infty$. In general, when F is a bounded function, then $\tilde{f}$ can be only BMO, as in the case example of the upper half-plane. The aim of this paper is to study the P_L-Poisson integrals in the sense of (1.1), where $f \in \mathrm{BMO}(V)$. We show that for $f \in \mathrm{BMO}(V)$ the integral (1.1) is absolutely convergent, conditions (1.2) and (1.3) are equivalent and, in turn, they are equivalent to pluriharmonicity of F. Moreover, for the conjugate pluriharmonic function $\tilde{F}$, its boundary value $\tilde{f}$ is also a $\mathrm{BMO}(V)$ function.

The group S being identified with $\mathcal{D}$, admissible operators are of the form

$$L = \sum_{j=1}^{m} \mathrm{X}_j^2 + \mathrm{X}_0, \tag{1.4}$$

where the X_j's are appropriately chosen elements of the Lie algebra of S. If $X_1, ... X_m$ generate the Lie algebra, then we say that L is of Hörmander type. If L is admissible of Hörmander type, the bounded L-harmonic functions are integrals of their boundary values on a nilpotent subgroup $N(L)$ of S against the corresponding Poisson kernel [DH].[2] The fact that L annihilate holomorphic functions implies that the Shilov boundary V is contained in $N(L)$ and not necessarily equal to $N(L)$ ([DHP]). However, there is a positive Poisson kernel P_L on V with the following properties:

1. P_L-Poisson integrals $F(w) = \int_V f(w \bullet x) P_L(x) dx$ of P_L-integrable functions f are L harmonic (see (2.11)).
2. bounded holomorphic and antyholomorphic functions are P_L-Poisson integrals.
3. $\int_V |x|^\varepsilon P_L(x) dx < \infty$ for an $\varepsilon > 0$.

This way we obtain a family of kernels analogous to the Poisson-Szegö kernel except of the fact that they are not Laplace-Beltrami harmonic, but L harmonic. In [BDH] the following theorem was proved:

[2]See (2.9). The origin of this research goes back to H.Furstenberg [F], Y.Guivarc'h [G] and A.Raugi [Ra] who developed a probabilistic approach to bounded functions on groups harmonic with respect to a probability measure.

THEOREM 1.5. *Let $\mathcal{D}$ be a symmetric tube domain and let L be a Hörmander type admissible operator. There is an elliptic degenerate operator $\mathbf{H}_\alpha$ (see (4.1)) such that if a bounded function F is annihilated by L and $\mathbf{H}_\alpha$, then F is pluriharmonic.*[3]

$\mathbf{H}_\alpha$ is the Laplace-Beltrami operator for the product of upper half planes with the usual metric, possibly scaled on each factor by an appropriate constant. The scaling is done in the way that $N(L+\mathbf{H}_\alpha) = V$. This allows to write F as the Poisson integral $P_{L+\mathbf{H}_\alpha} f$ on V. Generally $\mathbf{H}_\alpha$ depends on L, but there is quite a freedom in choosing it. Granted $F = P_{L+\mathbf{H}_\alpha} f$, (1.2) is equivalent to $\mathbf{H}_\alpha F = 0$. As a straightforward consequence of Theorem 1.5 we obtain that if $F = P_L f$, $f \in L^p$, $1 < p < \infty$ and $\mathbf{H}_\alpha F = 0$, then F is pluriharmonic. However, no information about the size of the conjugate function can be deduced from [BDH]. Here we prove more general theorems that, in particular, solve this problem.

THEOREM 4.3. *Let F be the P_L-Poisson integral of a BMO function f. Let $\mathbf{H}_\alpha$ be any operator of the form (4.1) and assume that $\mathbf{H}_\alpha F = 0$. Then F is pluriharmonic and the conjugate function is the P_L-Poisson integral of a BMO function $\tilde{f}$ with $\|\tilde{f}\|_{\mathrm{BMO}} \le C\|f\|_{\mathrm{BMO}}$. Moreover, if $F = P_L f$, $f \in L^p$, $1 < p < \infty$, then $\tilde{F} = P_L \tilde{f}$ and $\|\tilde{f}\|_{L^p(V)} \le \|f\|_{L^p(V)}$ for a properly chosen $\tilde{f}$ (up to an additive constant).*

The strategy of the proof is as follows. First we show that $\mathbf{H}_\alpha F = 0$ implies $\mathrm{supp}\hat{f} \subset \bar{\Omega} \cup -\bar{\Omega}$ (section 4). Then using appropriate singular integral operators T_1, T_2 we obtain two BMO functions $T_1 f$, $T_2 f$ with $\mathrm{supp}\,\widehat{T_1 f} \subset \bar{\Omega}$, $\mathrm{supp}\,\widehat{T_2 f} \subset -\bar{\Omega}$ and such that

$$T_1 f + T_2 f = f \quad \text{as elements in BMO(V)}.$$

Then it amounts to prove that P_L-Poisson integrals of $T_1 f$, $T_2 f$ are holomorphic and antyholomorphic functions, respectively (Section 5).

A natural question arises to characterize the P_L-Poisson integrals F of BMO functions in terms of F not referring directly to its boundary values. In analogy to Poisson integrals of L^p functions a first guess could be:

F is P_L-Poisson integral of a BMO function f, if, and only if, $LF = 0$ and $\sup_{y\in\Omega} \|F_y\|_{\mathrm{BMO}}$, where $F_y(x) = F(x+iy)$

This however is not true, because adding to F a function $h(y)$, that is L-harmonic and constant on any slice $V + iy$ does not change the BMO-norm. It is easy to see that there exist such functions h that are not the P_L-Poisson integrals. Therefore a more appropriate characterization is the following:

THEOREM 3.13. *Assume $N(L) = V$. $LF = 0$ and $\sup_{y\in\Omega} \|F_y\|_{\mathrm{BMO}} < \infty$, if, and only if, $F(x+iy) = P_L f(x+iy) + h(y)$, $x \in V, y \in \Omega$ for a BMO function f and $Lh = 0$.*

We summarize all these in a slightly more general theorem:

[3] Theorem 3.3 in [BDH]. Although L is elliptic there, the result holds for a Hörmander type L with the same proof.

THEOREM 4.2. *Given a Hörmander type operator L, there is an operator $\mathbf{H}_\alpha$ such that if $LF=0$, $\mathbf{H}_\alpha F=0$ and $\sup_{y\in\Omega}\|F_y\|_{\mathrm{BMO}}<\infty$, then $F(x+iy)=P_Lf(x+iy)+h(y)$, P_Lf is pluriharmonic, $Lh=0$, $\mathbf{H}_\alpha h=0$ and the function conjugate to P_Lf is the P_L-Poisson integral of a BMO function and the same norm inequalities as in Theorem 4.3 hold.*

An example at the end shows that h does not have to be pluriharmonic (Section 6).

The organization of the paper is as follows: Preliminaries contain basic information about tube domains, the action of the group S, admissible operators and Poisson integrals. Section 3 is devoted to Poisson integrals of BMO functions. In section 4 we formulate the main results and we prove that (1.3) implies (1.2). In Section 5 we show that (1.2) implies pluriharmonicity together with norm inequalities for the boundary functions.

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2. Preliminaries

Symmetric tube domains. Let Ω be an irreducible symmetric cone in a Euclidean spaces $(V,\langle\cdot,\cdot\rangle)$ and let

$$\mathcal{D}=V+i\Omega\subset V^{\mathbb{C}}$$

be the corresponding tube domain. There is a solvable Lie group S acting simply transitively on $\mathcal{D}$. To construct S we consider the connected component G of the linear group $G(\Omega)=\{g\in GL(V):\ g(\Omega)=\Omega\}$. The Iwasawa decomposition of $G(\Omega)=S_0K$ yields a triangular group S_0 acting simply transitively on Ω. The action of S_0 extends to $\mathcal{D}$ by

$$s\circ(x+iy)=s\circ x+is\circ y,\quad x\in V,\ y\in\Omega,\ s\in S_0. \tag{2.1}$$

Moreover, V acts on $\mathcal{D}$ by translations

$$v\circ(x+iy)=v+x+iy,\quad v\in V. \tag{2.2}$$

These actions generate a solvable Lie group S that acts simply transitively on $\mathcal{D}$. The group $S=VS_0$ is a semidirect product of V and S_0:

$$(v,s)(v_1,s_1)=(v+s\circ v_1,ss_1),\quad v,v_1\in V,\ s,s_1\in S_0. \tag{2.3}$$

The group $S_0=N_0A$ is a semi-direct product of a nilpotent Lie group N_0 and $A=(\mathbb{R}^+)^r$. The Lie algebra $\mathcal{S}$ of S has the decomposition

$$\mathcal{S}=\mathcal{N}\oplus\mathcal{A},\quad \mathcal{N}=V\oplus\mathcal{N}_0 \tag{2.4}$$

$\mathcal{A}$ being the Lie algebra of A, $\mathcal{N}_0$ the Lie algebra of N_0 .

In virtue of (2.1) and (2.2) we may identify $\mathcal{D}$ with S. More precisely, let e be the stabilizer of K, $\mathbf{e} = ie$ and let

$$\theta : S \ni s \mapsto \theta(s) = s \circ \mathbf{e} \in \mathcal{D}. \tag{2.5}$$

Then θ is a diffeomorphism of S and $\mathcal{D}$. θ identifies also the spaces of smooth functions on S and $\mathcal{D}$. The Lie algebra $\mathcal{S}$ becomes then the tangent space $T_{\mathbf{e}}$ of $\mathcal{D}$ at $\mathbf{e}$. This allows to transport of the Bergmann metric g and the complex structure $\mathcal{J}$ from $\mathcal{D}$ to S, where they become left-invariant tensors.

The group S and its Lie algebra $\mathcal{S}$ can be also described in terms of the Jordan algebra structure of V [FK]. Since we will make no use of it in the paper, we refer the reader to e.g. [DHMP] and [BDH] for more details concerning symmetric tube domains in the framework adapted to what we need here.

Under identification (2.5) holomorphic functions on $\mathcal{D}$ are called holomorphic functions on S. A real left-invariant second order elliptic degenerate operator L is called *admissible* if L annihilates holomorphic functions. Admissible operators can be described more precisely in terms of $\mathcal{S}$ ([DHP], [DHMP], [BDH]). Namely, we choose a $g-$orthonormal basis $H_1,...,H_r$ in $\mathcal{A}$, we let $\lambda_1,...,\lambda_r$ be the dual basis in $\mathcal{A}^*$. It turns out that the spaces in the decomposition (2.4) are g-orthogonal and we let $\Lambda \subset \mathcal{A}^*$ be

$$\Lambda = \Big\{ \frac{\lambda_i+\lambda_j}{2},\ 1\le i\le j\le r,\quad \frac{\lambda_j-\lambda_i}{2},\ 1\le i<j\le r \Big\}.$$

Then V and $\mathcal{N}_0$ admit further orthogonal decompositions

$$V = \bigoplus_{1\le i\le j\le r} V_{ij}, \qquad \mathcal{N}_0 = \bigoplus_{1\le i<j\le r} \mathcal{N}_{ij}, \tag{2.6}$$

where

$$V_{ij} = \mathcal{N}_{\frac{\lambda_i+\lambda_j}{2}}, \quad \mathcal{N}_{ij} = \mathcal{N}_{\frac{\lambda_j-\lambda_i}{2}}$$

and for $\eta \in \Lambda$,

$$\mathcal{N}_\eta = \{X \in V \oplus \mathcal{N}_0 :\ [H,X] = \eta(H)X \ \text{ for every } H \in \mathcal{A}\}.$$

Moreover, $\dim V_{jj} = 1$ and $\dim V_{ij} = \dim \mathcal{N}_{ij} = d$. We denote by

$$\begin{aligned}
&X_j \ \text{ the orthonormal basis of } \ V_{jj},\ j=1,..,r\\
&X_{ij}^\alpha,\ \alpha=1,...,d,\ \text{ an orthonormal basis of } V_{ij},\ 1\le i<j\le r,\\
&Y_{ij}^\alpha,\ \alpha=1,...,d,\ \text{ an orthonormal basis of } \mathcal{N}_{ij},\ \ 1\le i<j\le r,
\end{aligned}$$

in such a way that

$$H_j = \mathcal{J}(X_j),\ Y_{ij}^\alpha = \mathcal{J}(X_{ij}^\alpha).$$

Let now

$$Z_j = X_j - iH_j, \quad j = 1, ..., r,$$
$$Z_{ij}^\alpha = X_{ij}^\alpha - iY_{ij}^\alpha, \quad 1 \le i < j \le r.$$

Then $\{Z_j : \ j = 1, ..., r\} \cup \{ \ Z_{ij}^\alpha : \ 1 \le i < j \le r, \ \alpha = 1, ..., d\}$ is an orthonormal basis of S-invariant holomorphic vector fields. Any admissible operator L is a linear combination of the operators

$$\Delta(Z, W) = Z\overline{W} - \nabla_Z \overline{W},$$

∇ being the Riemannian connection determined by g. ∇ can be easily calculated and

$$\begin{aligned} \Delta(Z_j, Z_j) &= X_j^2 + H_j^2 - H_j = \Delta_j, \\ \Delta(Z_{ij}^\alpha, Z_{ij}^\alpha) &= (X_{ij}^\alpha)^2 + (Y_{ij}^\alpha)^2 - H_j = \Delta_{ij}^\alpha \end{aligned} \tag{2.7}$$

(see e.g. [DHMP]).

Poisson boundaries. Given a Hörmander type admissible operator

$$L = \sum_{j=1}^m \ \mathrm{X}_j^2 + \ \mathrm{X}_0$$

we write $X_0 = Y + Z$, $Y \in \mathcal{A}$, $Z \in \mathcal{N}$ and we let

$$\begin{aligned} &\Lambda_0 = \{\eta \in \Lambda : \ \eta(Y) \ge 0\}, \\ &\mathcal{N}_0(L) = \bigoplus_{\eta \in \Lambda_0} \mathcal{N}_\eta, \ \ N_0(L) = \exp \mathcal{N}_0(L). \end{aligned} \tag{2.8}$$

The space

$$N(L) = \exp \mathcal{N} / N_0(L) = S/N_0(L)A$$

is the L-Poisson boundary (cf. [DH]). This means that the bounded L-harmonic functions are in one-one correspondence with the L^∞ functions on $N(L)$ via the following Poisson integral

$$F(w) = \int_{N(L)} f(w \bullet u)\nu(u)\, du, \quad w \in S, \tag{2.9}$$

where $u \mapsto w \bullet u$ is the action of $w \in S$ on $N(L)$.

For an admissible operator of Hörmander type its $\mathcal{A}$ component Y of the first order term is of the form

$$Y = \sum_{j=1}^r b_j H_j \quad \text{with} \ \ b_j < 0 \tag{2.10}$$

(see [DHP]). Therefore, by (2.8), $V \cap N_0(L) = \{0\}$. Thus it follows from the general theory developed in [DH] that V is also a boundary for L, i.e. there is a Poisson kernel P_L on V such that the functions

$$F(w) = \int_V f(w \bullet x) P_L(x)\, dx, \quad f \in L^\infty(V), \tag{2.11}$$

are bounded and L-harmonic. Here $x \mapsto w \bullet x$ is the action of S on $V = S/N_0A$. If $N_0(L) = N_0$ i.e. $N(L) = V$, then (2.11) gives all the bounded L harmonic functions.

For any admissible operator L of Hörmander type the kernel P_L has the following properties (see [DH])

$$P_L \in L^1(V) \cap L^\infty(V) \cap C^\infty(V),$$

$$P_L(x) > 0,$$

$$\text{there is } \varepsilon = \varepsilon(L) > 0 \ \text{ such that } \int_V |x|^\varepsilon P_L(x)\, dx < \infty. \tag{2.12}$$

3. Poisson integrals of BMO functions

It is convenient to rewrite (2.11) in a slightly different form to omit (,) in (2.3) and to put $(x, s) = xs$, $x \in V$, $s \in S_0$. Then the product of two elements of the group S takes form

$$ww_1 = xsx_1s_1 = xsx_1s^{-1}ss_1,$$

where $sx_1s^{-1} = s \circ x_1$ is the linear action (2.1) of S_0 on V. Therefore, the action S on $V = S/N_0A$ in (2.11) becomes

$$xs \bullet u = xsus^{-1} = x + s \circ u, \quad u \in V. \tag{3.1}$$

Let $\det s$ be the determinant of the linear transformation $u \mapsto s \circ u$ and let

$$P_s(x) = \det s^{-1} \check{P}_L(s^{-1}xs). \tag{3.2}$$

We define a function P on S by

$$P(xs) = P_s(x), \ x \in V, \ s \in S_0. \tag{3.3}$$

Then, by (3.1), the P_L-Poisson integral can be written as

$$\begin{aligned} P_L f(xs) = F(xs) &= \int_V f(xsus^{-1}) P_L(u)\, du \\ &= \int_V f(u) P_s(u^{-1}x)\, du = f *_V P_s(x) \\ &= \int_V f(u) P(u^{-1}xs)\, du, \end{aligned} \tag{3.4}$$

and it shows, in particular, that P is L-harmonic. (3.4) makes sense for $f \in \mathrm{BMO}(V)$. Indeed, if $f \in \mathrm{BMO}(V)$ then $g(u) = f(xs \bullet u)$ is a BMO-function, because the action of S on V is affine. Moreover, we show that in virtue of (2.12), the integral

$$\int_V f(u) P_L(u)\, du$$

is absolutely convergent. The proof uses standard techniques, but we include it for completeness.

LEMMA 3.5. *Let $g \in L^1(\mathbb{R}^k) \cap L^\infty(\mathbb{R}^k)$ and $\int_{\mathbb{R}^k} |g(u)||u|^\varepsilon du < \infty$ with an $\varepsilon > 0$. Then there is a constant $C > 0$ such that*

$$\Big| \int_{\mathbb{R}^k} g(u) f(u) du \Big| \le C \|f\|_{\mathrm{BMO}(\mathbb{R}^k)} + \|g\|_{L^1} |m_B(f)|,$$

where B is the unit ball in $\mathbb{R}^k$ centered at the origin and

$$m_B(f) = \frac{1}{|B|} \int_B f(u)\, du.$$

In particular the integral is absolutely convergent.

PROOF: Let us note that for every $\tau > 0$

$$\begin{aligned} \int_{2^{j+1}\tau^{-1}B \setminus 2^j \tau^{-1} B} |g(u)|\, du &\le 2^{-\varepsilon j} \tau^\varepsilon \int_{2^{j+1}\tau^{-1}B \setminus 2^j \tau^{-1} B} |g(u)||u|^\varepsilon du \\ &\le C 2^{-j\varepsilon} \tau^\varepsilon. \end{aligned} \tag{3.6}$$

Since $\int_{\mathbb{R}^k} |g(u)||m_B(f)|\, du \le \|g\|_{L^1} |m_B(f)|$, it is enough to prove that

$$\int_{\mathbb{R}^k} |g(u)||f(u) - m_B(f)| du \le C \|f\|_{\mathrm{BMO}}. \tag{3.7}$$

We have

$$\begin{aligned} \int_{\mathbb{R}^k} |g(u)||f(u) - m_B(f)|\, du \le & \sum_{j=0}^\infty \int_{2^{j+1}B \setminus 2^j B} |g(u)||f(u) - m_B(f)|\, du \\ & + \int_B |g(u)|\, |f(u) - m_B(f)|\, du. \end{aligned}$$

But

$$\begin{aligned} \int_B |g(u)||f(u) - m_B(f)|\, du &\le \|g\|_{L^\infty} \int_B |f(u) - m_B(f)|\, du \\ &\le \|g\|_{L^\infty} |B| \|f\|_{\mathrm{BMO}} \end{aligned}$$

and

$$
\begin{aligned}
&\int_{2^{j+1}B\setminus 2^j B} |g(u)|\,|f(u)-m_B(f)|\,du \\
(3.8)\qquad &\qquad \le \int_{2^{j+1}B\setminus 2^j B} |g(u)|\,|f(u)-m_{2^{j+1}B}(f)|du \\
&\qquad\quad + \int_{2^{j+1}B\setminus 2^j B} |g(u)| \sum_{i=0}^{j} |m_{2^{i+1}B}(f)-m_{2^i B}(f)|du.
\end{aligned}
$$

Since $|m_{2^{i+1}B}(f)-m_{2^i B}(f)| \le C\|f\|_{\mathrm{BMO}}$, by (3.6), the second summand on the right-hand side of (3.8) is smaller or equal than $C(j+1)2^{-j\varepsilon}\|f\|_{\mathrm{BMO}}$. To estimate the first summand we apply the Hölder inequality to obtain

$$
\begin{aligned}
&\int_{2^{j+1}B\setminus 2^j B} |g(u)|\,|f(u)-m_{2^{j+1}B}(f)|\,du \\
&\qquad \le \Big[\int_{2^{j+1}B\setminus 2^j B} |g(u)|^p\,du\Big]^{\frac{1}{p}} \Big[\int_{2^{j+1}B} |f(u)-m_{2^{j+1}B}(f)|^q\,du\Big]^{\frac{1}{q}} \\
&\qquad \le C\Big(\|g\|_{L^\infty}\Big)^{\frac{p-1}{p}} \Big(\int_{2^{j+1}B\setminus 2^j B} |g(u)|\,du\Big)^{\frac{1}{p}} |2^{j+1}B|^{\frac{1}{q}}\|f\|_{\mathrm{BMO}} \\
&\qquad \le C\big(\|g\|_{L^\infty}\big)^{\frac{p-1}{p}} 2^{-\frac{j\varepsilon}{p}} 2^{\frac{(j+1)k}{q}} \|f\|_{\mathrm{BMO}}
\end{aligned}
$$

Taking p close to 1, estimate (3.7) follows.

Corollary 3.9. *Under the assumptions of Lemma 3.5 there exists a constant $C>0$ such that*

$$
\Big|\int_{\mathbb{R}^k} g(x-u)f(u)du\Big| \le C\big(\|f\|_{\mathrm{BMO}} + |m_B({}_xf)|\big).
$$

Proof: By Lemma 3.5, we obtain

$$
\begin{aligned}
\Big|\int_{\mathbb{R}^k} g(x-u)f(u)\,du\Big| &= \Big|\int_{\mathbb{R}^k} g(-u)f(x+u)\,du\Big| \\
&\le C\big(\|{}_xf\|_{\mathrm{BMO}} + |m_B({}_xf)|\big) \\
&= C\big(\|f\|_{\mathrm{BMO}} + |m_B({}_xf)|\big).
\end{aligned}
$$

Let us remark that Lemma 3.5 yields a formula for the left invariant derivatives of $F=P_Lf$, $f\in \mathrm{BMO}(V)$. Let D be a left-invariant differential operator on S. Then

$$
DF(xs) = \int_V f(u)(DP)(u^{-1}xs)\,du, \qquad (3.10)
$$

where P is as in (3.3). To prove (3.10) we notice that by the Harnack inequality for L and left-invariance of L

$$\sup_{w_1 \in K} |(DP)(ww_1)| \leq C_D P(w), \tag{3.11}$$

for a compact set $K \subset S$ and for every $w \in S$. Then we use Lemma 3.5. Taking the partial Fourier transform of $P(xs)$ with respect to x, we see that for any $\xi \in V$, $P(\widehat{\xi}, s)$ as a function of $s \in S_0$ is annihilated by a hypoelliptic operator on S_0 (for details see 2.5 in [DHMP] or Lemma 5.4 in [BDH]). Hence

$$s \mapsto P(\widehat{\xi}, s)$$

is a smooth function. Moreover, for $\xi \in \Omega$ we have

$$P(\widehat{\xi}, s) = e^{-\langle \xi, s\circ e\rangle}. \tag{3.12}$$

Indeed, given $\xi \in \Omega$, $e^{-i\langle \xi, \bar{z}\rangle}$ is a bounded antyholomorphic function on $\mathcal{D}$ with the boundary value $e^{-i\langle \xi, x\rangle}$. Hence

$$e^{-i\langle \xi, -s\circ ie\rangle} = \int_V e^{-i\langle \xi, x\rangle} P_s(x)\, dx$$

and (3.12) follows.

The following is a characterization of harmonic functions with BMO boundary values.

THEOREM 3.13. *Assume that $N(L) = V$ and $LF = 0$. For $s \in S_0$, $x \in V$ let $F_s(x) = F(xs)$. Then the following conditions are equivalent:*

$$\sup_{s \in S_0} \|F_s\|_{\mathrm{BMO}} < \infty \tag{3.14}$$

and

there is $f \in BMO(V)$ and an L-harmonic function $h(xs) = h(s)$, $x \in V$, $s \in S_0$ such that

$$F_s(x) = f * P_s(x) + h(s). \tag{3.15}$$

Moreover, given F the presentation (3.15) is unique up to an additive constant.

We start with uniqueness. Suppose

$$f * P_s(x) + h(s) = f_1 * P_s(x) + h_1(s).$$

For every ϕ in the Hardy space $H^1(V)$ we then have

$$\phi * f * P_s(x) = \phi * f_1 * P_s(x)$$

and, since $\mathrm{BMO}(V)$ is the dual space to $H^1(V)$, $\phi * f$ and $\phi * f_1$ are in $L^\infty(V)$. Therefore, (2.11) implies $\phi * f = \phi * f_1$, whence $f = f_1$ as elements in BMO and the rest follows.

Now, since the P_L-Poisson integral (3.4) is absolutely convergent, for every ball B we have

$$\frac{1}{|B|}\Big(\int_B |f * P_s(x) - m_B(f * P_s)|\, dx\Big) \leq C\|f\|_{\mathrm{BMO}}. \tag{3.16}$$

Hence (3.15) implies (3.14). To prove the converse, we need the following lemma.

LEMMA 3.17. *Let* $H = \sum_{j=1}^r H_j$, $P_t = P_{\exp tH}$, *and let* $f \in BMO(V)$. *Then for every* $g \in H^1(V)$

$$\lim_{t\to-\infty} \langle f * P_t, g\rangle = \langle f, g\rangle.$$

Proof. In view of (3.16) it suffices to prove that

$$\lim_{t\to-\infty} \langle f * P_t, \phi\rangle = \langle f, \phi\rangle$$

for every $\phi \in C_c^\infty$, $\int \phi = 0$. Fix a ball B centered at the origin such that $\operatorname{supp}\phi \subset B$. Since

$$f_{|2B} \in L^p(V), \qquad 1 \le p < \infty,$$

and P_t is an approximate identity, while $t \to -\infty$, we have

$$\lim_{t\to-\infty} \langle f_{|2B} * P_t, \phi\rangle = \langle f_{|2B}, \phi\rangle = \langle f, \phi\rangle.$$

Now, we show that there is $\delta > 0$ such that for $x \in B$,

$$|f_{|(2B)^c}| * P_t(x) \le Ce^{\delta t}. \tag{3.18}$$

If $x \in B$, then

$$\begin{aligned} |f_{|(2B)^c}| * P_t(x) &\le \int_{B^c} |f(x-u)| P_t(u)\, du \\ &= \int_{B^c} |{}_x f(u)| \check{P}_t(u)\, du = I. \end{aligned}$$

We proceed as in the proof of Lemma 3.5. We have

$$I \le \int_{B^c} |{}_x f(u) - m_B({}_x f)| \check{P}_t(u)\, du + \int_{B^c} |m_B({}_x f)| \check{P}_t(u)\, du.$$

In view of (3.6), we have

$$\int_{B^c} |m_B({}_x f)| \check{P}_t(u)\, du \le Cm_{2B}(|f|) \int_{(e^{-t}B)^c} \check{P}(u)\, du \le Ce^{\varepsilon t} m_{2B}(|f|).$$

Then, as in (3.8) with f replaced by ${}_x f$ and g by $\check{P}_t$, respectively, by (3.6), we have

$$\begin{aligned} &\int_{2^{j+1}B\setminus 2^j B} \check{P}_t(u) \sum_{i=0}^{j} |m_{2^{i+1}B}({}_x f) - m_{2^i B}({}_x f)| du \\ &\le C(j+1)\|f\|_{\mathrm{BMO}} \int_{2^{j+1}e^{-t}B\setminus 2^j e^{-t}B} \check{P}(u)\, du \le C(j+1)2^{-j\varepsilon} e^{\varepsilon t} \|f\|_{\mathrm{BMO}} \end{aligned}$$

and

$$\int_{2^{j+1}B\setminus 2^jB} |_xf(u) - m_{2^{j+1}B}(_xf)||\check{P}_t(u)du \le C(\|P_t\|_{L^\infty})^{\frac{p-1}{p}} 2^{-\frac{j\varepsilon}{p}} e^{\frac{\varepsilon}{p}t} 2^{\frac{(j+1)Q}{q}} \|f\|_{\mathrm{BMO}}$$
$$\le Ce^{(\frac{\varepsilon}{p}-Q\frac{p-1}{p})t} 2^{-\frac{j\varepsilon}{p}+\frac{(j+1)Q}{q}} \|f\|_{\mathrm{BMO}},$$

where $Q = \dim V$. Now taking p close to 1, for $\delta = \frac{\varepsilon - Q(p-1)}{p} > 0$ and $\delta_1 = \frac{\varepsilon}{p} - \frac{Q}{q} > 0$, we have

$$\begin{aligned} I &\le Ce^{\varepsilon t} m_{2B}(|f|) + C\sum_{j=0}^{\infty}(e^{\varepsilon t}(j+1)2^{-j\varepsilon} + e^{\delta t}2^{-j\delta_1})\|f\|_{\mathrm{BMO}} \\ &\le Ce^{\delta t}(m_{2B}(|f|) + \|f\|_{\mathrm{BMO}}) \end{aligned}$$

and (3.18) follows.

To complete the proof of Theorem 3.13 we show that

(3.14) implies (3.15). Take $s = \exp tH$. There is a sequence $t_n \to -\infty$ and a function $f \in \mathrm{BMO}(V)$ such that for every $g \in H^1(V)$

$$\lim_{t_n \to -\infty} \langle F_{\exp t_n H}, g\rangle = \langle f, g\rangle \tag{3.19}$$

For a $\phi \in C_c^\infty$, $\int \phi = 0$, consider now

$$F_\phi(xs) = \phi * F_s(x).$$

F_ϕ is harmonic and bounded:

$$\|\phi * F_s\|_{L^\infty} \le \|\phi\|_{H^1}\|F_s\|_{\mathrm{BMO}}.$$

We are going to prove that

$$F_\phi(xs) = \phi * f * P_s(x).$$

Since $N(L) = V$, the Poisson integral (2.9) turns into (3.4) and yields the one-to-one correspondence between bounded L-harmonic functions F and $L^\infty(V)$. Moreover,

$$f(x) = *\text{weak} \lim_{t\to-\infty} F(x\exp[tH]).$$

Therefore it is enough to show that the boundary value of $F_\phi(xs) - \phi * f * P_s(x)$ is zero i.e. that the boundary value of F_ϕ is $\phi * f$.

Let $g \in L^1$ and consider

$$\langle \phi * F_{\exp t_n H} - \phi * f, g\rangle = \langle F_{\exp t_n H} - f, \check{\phi} * g\rangle.$$

Since $\check{\phi} * g \in H^1(V)$, by (3.19)

$$\lim_{t_n \to -\infty} \langle F_{\exp t_n H} - f, \check{\phi} * g \rangle = 0.$$

Therefore, for every $s \in S_0$,

$$\phi * F_s(x) = \phi * f * P_s(x)$$

and so

$$F_s = f * P_s \quad \text{as elements inBMO}(V).$$

Hence

$$F_s(x) - f * P_s(x) = h(s)$$

and since the function on the left hand side is L harmonic so is $h(s)$.

4. Pluriharmonicity

For $\alpha = (\alpha_1, ..., \alpha_r)$, $\alpha_j > 0$ let

$$\mathbf{H}_\alpha = \sum_{j=1}^{r} \alpha_j \Delta_j. \tag{4.1}$$

THEOREM 4.2. *Let L_0 be an admissible operator of Hörmander type and let $\mathbf{H}_\alpha$ be such that for $L = L_0 + \mathbf{H}_\alpha$ we have $N(L) = V$. Let F be a real valued function on $VS_0 = \mathcal{D}$ such that $LF = \mathbf{H}_\alpha F = 0$ and*

$$\sup_{s \in S_0} \|F_s\|_{\mathrm{BMO}} < \infty.$$

Then

$$F(xs) = G(xs) + h(s),$$

where G is a pluriharmonic function and the function h, independent of x, is annihilated by both L and $\mathbf{H}_\alpha$. $G = P_L g$ and its conjugate $\tilde{G} = P_L \tilde{g}$ are P_L-Poisson integrals of BMO functions. Moreover, $\|\tilde{g}\|_{\mathrm{BMO}(V)} \le C \|g\|_{\mathrm{BMO}(V)}$.

REMARKS. *Given a Hörmander type admissible L_0 there is always a $\mathbf{H}_\alpha$ such that $N(L_0 + \mathbf{H}_\alpha) = V$ (see [DHMP], [DHP]). As it is shown at the end of the article the function h is non-zero, in general.*

If F is already of the from $F = P_L f$, then α in $\mathbf{H}_\alpha$ may be arbitrary and the following theorem holds.

THEOREM 4.3. *Let L be an admissible operator of Hörmander type and let $\mathbf{H}_\alpha$ be as in (4.1). Assume that $F = P_L f$ with $f \in BMO(V)$ and*

$$\mathbf{H}_\alpha F = 0.$$

Then F is pluriharmonic and the conjugate function $\tilde{F}$ is the P_L-Poisson integral of a BMO function $\tilde{f}$. Moreover, $\|\tilde{f}\|_{\mathrm{BMO}(V)} \leq C\|f\|_{\mathrm{BMO}(V)}$.

If $f \in L^p(V)$, $1 < p \leq \infty$, then $\|\tilde{f}\|_{L^p(V)} \leq C\|f\|_{L^p(V)}$ for a properly chosen $\tilde{f}$ (up to a positive constant)[4].

The only difference in proofs of the above two theorems is that for the first one we need the conclusion of Theorem 3.13. The rest is the same.

Proof of Theorem 4.2 Suppose F satisfies the assumptions of Theorem 4.2. By Theorem 3.13,

$$F(xs) = f * P_s(x) + h(s) \tag{4.4}$$

for a BMO function f. The first step in the proof of Theorem 4.2 is to show that

$$\mathrm{supp}\hat{f} \subset \overline{\Omega} \cup -\overline{\Omega}. \tag{4.5}$$

We let $\psi \in \mathcal{S}(V)$ be such that

$$\widehat{\psi}(\xi) = 0 \iff \xi = 0. \tag{4.6}$$

Set $f_1 = \psi * f$, $F_1(xs) = f_1 * P_s(x)$. By (3.10) and (4.4),

$$\mathbf{H}_\alpha F_1(xs) = \int_V f_1(u)(\mathbf{H}_\alpha P)(u^{-1}xs)du = \psi * (\mathbf{H}_\alpha F)(\,\cdot\, s)(x) = 0,$$

and, by (3.11), $x \to \mathbf{H}_\alpha P(xs)$ belongs to $L^1(V)$. Since $\psi \in H^1(V)$, f_1 is a bounded function. Therefore the Wiener's theorem ([Ru], Theorem 9.3) implies that for every $s \in S_0$,

$$\mathrm{supp}\hat{f}_1 \subset \{\xi : (\mathbf{H}_\alpha P)(\hat{\xi}, s) = 0\}$$

Fix now $\xi \in \mathrm{supp}\hat{f}_1$. We then have

$$\Big(\forall_{s\in S_0}\Big) \quad \Big((\mathbf{H}_\alpha P)(\hat{\xi}, s) = 0\Big). \tag{4.7}$$

On the Fourier transform side for $s = na$, $n \in N_0, a \in A$, (4.7) reads

$$\sum_{j=1}^{r} \alpha_j(-a_j^2 Q_j^2 + a_j^2 \partial_{a_j}^2)P(\hat{\xi}, na) = 0,$$

[4]When $f \in L^p(V)$, $1 < p \leq \infty$, pluriharmonicity follows from the results of [BDH], but the norm inequalities do not.

where

$$na = n \prod_{j=1}^{r} \exp((\log a_j)H_j), \qquad n \in N_0,\ a_j > 0, \tag{4.8}$$

are coordinates in $S_0 = NA$ and

$$Q_j = Q_j(n, \xi)$$

is a polynomial in n and ξ (see [DHMP], Theorem 2.20). But, in view of (3.2),

$$|P(\hat{\xi}, na)| \leq \|P_L\|_{L^1(V)} = 1,$$

so $P(\hat{\xi}, na)$ must be of the form

$$P(\hat{\xi}, na) = g(n) \exp\Big(- \sum_{j=1}^{r} a_j |Q_j(n, \xi)|\Big).$$

(See [DHMP], Lemma 2.16). Moreover, $g(n)$ is a constant and so

$$P(\hat{\xi}, na) = c \exp\Big(- \sum_{j=1}^{r} a_j |Q_j(n, \xi)|\Big). \tag{4.9}$$

It was proved in [BDH], Lemma 5.7 and Corollary 5.9 that whenever $\xi \notin \overline{\Omega} \cup -\overline{\Omega}$, (4.9) contradicts smoothness of $s \to P(\hat{\xi}, s)$. Hence $\operatorname{supp} \hat{f_1} = \operatorname{supp} \widehat{\psi * f} \subset \overline{\Omega} \cup -\overline{\Omega}$, and, by (4.6), we obtain (4.5).

Notice that if $f \in L^p(V)$, $1 < p < \infty$, then convolving $F = P_L f$ on V from the left by a C_c^∞ function ϕ we obtain the Poisson integral of a bounded function an so $\operatorname{supp}\hat{\phi}\hat{f} \subset \overline{\Omega} \cup -\overline{\Omega}$. Hence $\operatorname{supp}\hat{f} \subset \overline{\Omega} \cup -\overline{\Omega}$. As we will see in Section 5, (4.5) implies pluriharmonicity. Therefore, granted $F = P_L f$, $f \in L^p(V)$, $1 < p < \infty$ or $f \in \mathrm{BMO}(V)$, $\mathbf{H}_\alpha F = 0$ is equivalent to (4.5).

The rest of the argument is contained in Section 5, where the following theorem is proved:

THEOREM 4.10. *Assume that f is a real valued BMO function and*

$$\mathit{supp}\widehat{f} \subset \overline{\Omega} \cup -\overline{\Omega}.$$

Then for any admissible operator L of Hörmander type the function

$$F(xs) = P_L f(xs) = f * P_s(x)$$

is pluriharmonic and its conjugate function $\tilde{F}$ is a P_L-Poisson integral of a BMO function $\tilde{f}$. Moreover, $\tilde{f} = Tf$, where T is an operator that maps all $L^p(V) \mapsto L^p(V)$, $1 < p < \infty$, and $BMO(V) \mapsto BMO(V)$ boundedly.

5. Proof of the Theorem 4.10

Let Σ be the unit sphere in V. For a regular cone Ω we have

$$(\overline{\Omega} \cap \Sigma) \cap (-\overline{\Omega} \cap \Sigma) = \emptyset$$

and so there is a smooth function m_1 on Σ such that

$$m_1(\xi) = \begin{cases} 1 & \text{for } \xi \in \text{ neighbourhood of } \overline{\Omega} \cap \Sigma \text{ in } \Sigma \\ 0 & \text{for } \xi \in -\overline{\Omega} \cap \Sigma. \end{cases} \tag{5.1}$$

We extend m_1 to V by

$$m_1(\lambda\xi) = m_1(\xi), \quad \xi \in \Sigma, \ \lambda > 0,$$

to obtain a smooth homogeneous function on $V \setminus \{0\}$. Let $m_2(\xi) = m_1(-\xi)$, $\xi \in V$.

We define two multiplier operators T_1 and T_2 by

$$\widehat{T_j f} = m_j \widehat{f}, \quad f \in L^2(V), \ j = 1, 2.$$

Clearly

$$\langle T_1 f, g\rangle = \langle f, T_2 g\rangle \ \text{ for } f, g \in L^2(V), \tag{5.2}$$

where $\langle f, g\rangle = \int_V f(x)g(x)\,dx$. Moreover, by Theorem 4 in [St] III. § 3.2,

$$T_j : H^1(V) \to H^1(V) \text{ and } T_j : L^p(V) \to L^p(V), \ 1 < p < \infty \tag{5.3}$$

boundedly.

In view of (5.2) and (5.3), T_j may be extended to BMO(V), by

$$\begin{aligned} \langle T_1 f, g\rangle &= \langle f, T_2 g\rangle, \\ \langle T_2 f, g\rangle &= \langle f, T_1 g\rangle, \end{aligned} \tag{5.4}$$

where $g \in H^1$, $f \in$ BMO (see [St] IV. § 4.1). Since T_j is defined up to an additive constant, we pick it up in the way that the integral of $T_j f$ over the unit ball is 0.

Let us note that Theorem 4.10, and, consequently Theorem 4.2 will be proved if we show

THEOREM 5.5. *Let f be a real valued BMO function, $\operatorname{supp}\widehat{f} \subset \overline{\Omega} \cup -\overline{\Omega}$, then*

$$F_1(xs) = T_1 f * P_s(x) \quad \text{is holomorphic,} \tag{5.6}$$

$$F_2(xs) = T_2 f * P_s(x) \quad \text{is antiholomorphic,} \tag{5.7}$$

$$F_2 = \overline{F_1}, \tag{5.8}$$

$$\text{there is a constant } c \in \mathbb{R} \text{ such that } F = F_1 + F_2 + c, \tag{5.9}$$

$$\text{the conjugate function to } F \text{ is given by } \tilde{F} = \frac{1}{i}(F_1 - F_2). \tag{5.10}$$

We need a few lemmas:

LEMMA 5.11. *Assume that $\operatorname{supp}\widehat{f}$ is compact and contained in $\overline{\Omega} \cup -\overline{\Omega}$. Then $\operatorname{supp}\widehat{T_1 f} \subset \overline{\Omega}$.*

PROOF: Let $\varphi \in C_c^\infty(V)$, $\operatorname{supp}\varphi \subset V \setminus \overline{\Omega}$. In particular, $\widehat{\varphi} \in H^1$. By (5.4) we have

$$\langle \widehat{T_1 f}, \varphi \rangle = \langle T_1 f, \widehat{\varphi} \rangle = \langle f, T_2 \widehat{\varphi} \rangle.$$

Notice that

$$\operatorname{supp}\widehat{f} \cap (-\operatorname{supp}\widehat{T_2\widehat{\varphi}}) = \emptyset. \tag{5.12}$$

Indeed,

$$\widehat{T_2\widehat{\varphi}} = m_2 \varphi\check{},$$

where $\varphi\check{}(x) = \varphi(x^{-1})$.

But $m_2\varphi\check{} \in C_c^\infty(V)$ and $(\operatorname{supp} m_2\varphi\check{}) \cap (\overline{\Omega} \cup -\overline{\Omega}) = \emptyset$, hence (5.12) follows. Therefore,

$$\langle f, T_2\widehat{\varphi} \rangle = 0.$$

LEMMA 5.13. *Let $f \in BMO$, $g \in H^1$. Assume that $\operatorname{supp}\widehat{f}$ is compact and $\operatorname{supp}\widehat{f} \cap (-\operatorname{supp}\widehat{g}) = \emptyset$. Then $\langle f, g \rangle = 0$.*

PROOF: Take a sequence of g_n that tends to g in H^1 and such that $\check{g_n} \in C_c^\infty(V)$. Let ψ be a Schwartz function such that $\widehat{\psi} = 1$ on $-\operatorname{supp}\widehat{f}$ and $\widehat{\psi} = 0$ on $\operatorname{supp}\widehat{g}$. Then

$$g_n * \psi \to g * \psi \quad \text{in } H^1$$

and

$$g * \psi = 0.$$

We have

$$\langle f, g \rangle = \lim_{n\to\infty} \langle f, g_n \rangle = \lim_{n\to\infty} \langle f, g_n - g_n * \psi \rangle.$$

But $(g_n - g_n * \psi)\check{} \in C_c^\infty$ and $(g_n - g_n * \psi)\check{} = 0$ on $\operatorname{supp}\widehat{f}$. Hence

$$\langle f, g_n - g_n * \varphi \rangle = \langle f, \big((g_n - g_n * \psi)\check{}\big)\widehat{} \rangle = 0.$$

Lemma 5.14. *Assume that* $\operatorname{supp}\widehat{f}$ *is compact and contained in* $\overline{\Omega} \cup -\overline{\Omega}$. *Then* $(T_1 + T_2)f = f$ *in BMO.*

Proof: Let $g \in H^1(V)$ be such that $0 \notin \operatorname{supp}\widehat{g}$. Then

$$\operatorname{supp}\widehat{f} \cap -\operatorname{supp}\widehat{(T_1 + T_2 - I)g} = \emptyset.$$

Indeed, $\widehat{(T_1 + T_2 - I)g} = (m_1 + m_2 - 1)\widehat{g}$. By (5.1) there is a neighborhood of $\overline{\Omega} \cup -\overline{\Omega}$ such that $(m_1 + m_2 - 1)\widehat{g}_{|U} = 0$. Therefore, by Lemma 5.13,

$$\langle (T_1 + T_2 - I)f, g\rangle = \langle f, (T_2 + T_1 - I)g\rangle = 0.$$

Since $\{g \in H^1(V) : 0 \notin \operatorname{supp}\widehat{g}\}$ is a dense set in $H^1(V)$, the conclusion follows.

Lemma 5.15. *Let f be a real valued BMO function. Then* $T_2 f = \overline{T_1 f}$ *as functions in BMO.*

Proof: A simple calculation shows that for a test function $\varphi \in C_c^\infty(V)$ we have $T_2\varphi = \overline{T_1\overline{\varphi}}$. Assume now $\widehat{\varphi}(0) = 0$. Then

$$\begin{aligned}\langle T_2 f, \varphi\rangle &= \langle f, T_1\varphi\rangle = \langle f, \overline{T_2\overline{\varphi}}\rangle = \overline{\langle f, T_2\overline{\varphi}\rangle} \\ &= \overline{\langle T_1 f, \overline{\varphi}\rangle} = \langle \overline{T_1 f}, \varphi\rangle.\end{aligned}$$

Now, to complete the proof of Theorem (5.5) we have to prove the main point: $F(xs) = T_1 f * P_s(x)$ is a holomorphic function. This follows from the following proposition.

Proposition 5.16. *Assume that $g \in BMO$ and* $\operatorname{supp}\widehat{g} \subset \overline{\Omega}$. *Then*

$$h(xs) = g * P_s(x).$$

is holomorphic.

Proof: Consider $\lambda \in \Omega$ and $g_\lambda(x) = e^{i\langle\lambda,x\rangle} g(x)$. Then

$$\operatorname{supp}\widehat{g_\lambda} \subset \lambda + \overline{\Omega} \subset \Omega.$$

Is is enough to prove that $h_\lambda(xs) = g_\lambda * P_s(x)$ is holomorphic because we then take $\lambda = \frac{1}{n}e$ and observe that $h_\lambda \to h$ as distributions.

Let now G_λ be a function on $\mathcal{D}$ determined by $G_\lambda \circ \theta = h_\lambda$. In view of (2.5)

$$G_\lambda(x + iy) = g_\lambda * P_s(x),$$

where $y = s \circ e$. We consider the partial Fourier transform of G_λ along the variable x. For a system of coordinates $x_1, ..., x_Q$ in V and the corresponding system of coordinates $z_1, ..., z_Q$ in $V^{\mathbb{C}}$ we have

$$\begin{aligned}(\partial_{x_j} G_\lambda)(\widehat{\xi}, y) &= i\xi_j G_\lambda(\widehat{\xi}, y) \\ (\partial_{y_j} G_\lambda)(\widehat{\xi}, y) &= \partial_{y_j} G_\lambda(\widehat{\xi}, y)\end{aligned},$$

where $G_\lambda(\widehat{\xi}, y)$, $(\partial_{x_j} G_\lambda)(\widehat{\xi}, y)$, $(\partial_{y_j} G_\lambda)(\widehat{\xi}, y)$ denote the distributions on $\mathcal{D}$ that are the partial Fourier transforms along the x variable of the functions $G_\lambda(x+iy)$, $\partial_{x_j} G_\lambda(x+iy)$, $\partial_{y_j} G_\lambda(x+iy)$, respectively. Therefore to show that $\partial_{\overline{z}_j} G_\lambda = 0$, $j = 1, ..., Q$, it suffices to prove that

$$\xi_j G_\lambda(\widehat{\xi}, y) + \partial_{y_j} G_\lambda(\widehat{\xi}, y) = 0. \tag{5.17}$$

Now we observe that due to the assumptions on the support of $\widehat{g}_\lambda$ and (3.12)

$$G_\lambda(\widehat{\xi}, y) = \widehat{g_\lambda}(\xi)\widehat{P_s}(\xi) = \widehat{g_\lambda}(\xi) e^{-\langle \xi, y\rangle} \tag{5.18}$$

whence (5.17) follows (here $\widehat{g_\lambda}(\xi)$ is understood as a distribution). To finish it amounts to show the first equality in (5.18) i.e. that

$$\widehat{g_\lambda * P_s}(\xi) = \widehat{g_\lambda}(\xi)\widehat{P_s}(\xi). \tag{5.19}$$

The proof of (5.19) is based on the following two facts:

$$\operatorname{supp} \widehat{g_\lambda} \subset \Omega,$$

$$\int |g_\lambda(x-y)| P_s(y)\, dy < \infty. \tag{5.20}$$

Let now φ be a Schwartz function. If $\operatorname{supp} \widehat{\varphi} \cap \operatorname{supp} \widehat{g_\lambda} = \emptyset$ then (5.20) and the Fubini theorem imply

$$\langle \widehat{g_\lambda * P_s}, \widehat{\varphi} \rangle = 0.$$

Therefore, we may restrict our attention to $\widehat{\varphi} \in C_c^\infty(\Omega)$. But then

$$\begin{aligned} \langle \widehat{g_\lambda * P_s}, \widehat{\varphi} \rangle &= \langle g_\lambda * P_s, \varphi^{\sim} \rangle \\ &= \langle g_\lambda, \varphi^{\sim} * P_s^{\sim} \rangle = \langle g_\lambda, (\varphi * P_s)^{\sim} \rangle. \end{aligned} \tag{5.21}$$

Since $\widehat{P_s}(\xi) = e^{-\langle \xi, y\rangle}$, $\xi \in \Omega$, the function $\widehat{\varphi}(\xi)\widehat{P_s}(\xi)$ belongs to $C_c^\infty(\Omega)$, so $\varphi * P_s$ is a Schwartz function. Consequently, we may write

$$\begin{aligned} \langle g_\lambda, (\varphi * P_s)^{\sim} \rangle &= \langle \widehat{g}_\lambda, \widehat{\varphi * P_s} \rangle \\ &= \langle \widehat{g}_\lambda, \widehat{\varphi}\widehat{P_s} \rangle = \langle \widehat{g}_\lambda \widehat{P_s}, \widehat{\varphi} \rangle. \end{aligned} \tag{5.22}$$

Now (5.21) and (5.22) imply (5.19).

Proof of Theorem 5.5: Let us note that (5.6)–(5.10) follow immediately from Proposition 5.16 and Lemmas 5.14 and 5.15 provided $\operatorname{supp} f$ is compact. The constant c in (5.9) is real, because, in fact, $T_2 f = \overline{T_1 f}$ pointwise according to our convention $\int_B T_j f = 0$, B being the unit ball in V.

To complete the proof of Theorem 5.5 for arbitrary $f \in \mathrm{BMO}$, we may assume that $m_B(f) = 0$. Let $\widehat{\varphi}$ be a real valued $C_c^\infty(V)$ function such that $\widehat{\varphi}(\xi) = 1$ for ξ in the unit ball in V, $\widehat{\varphi}(-\xi) = \widehat{\varphi}(\xi)$ and let $\varphi_n(x) = n^k \varphi(nx)$. Then $f_n = \varphi_n * f$ is a real valued BMO function with compactly supported Fourier transform $\widehat{f_n}$ contained in $\overline{\Omega} \cup -\overline{\Omega}$. Moreover,

$$T_j f_n = \varphi_n * T_j f - m_B(\varphi_n * T_j f) \tag{5.23}$$

pointwise. Indeed, if $g \in H^1$ then $T_j(\varphi_n * g) = \varphi_n * T_j g$ and so (5.23) follows. Now set

$$F_{j,n}(xs) = T_j f_n * P_s(x) = \varphi_n * T_j f * P_s(x) - m_B(\varphi_n * T_j f).$$

Then $F_{1,n}$ is holomorphic, $F_{2,n}$ is antiholomorphic, $F_{2,n} = \overline{F_{1,n}}$, and

$$F_n = \varphi_n * f * P_s(x) = F_{1,n}(xs) + F_{2,n}(xs) + c_n$$

pointwise. But all the functions above have pointwise limit when $n \to \infty$, hence $\lim_{n\to\infty} c_n = c$ and

$$f * P_s(x) = T_1 f * P_s(x) + T_2 f * P_s(x) + c,$$

which finishes the proof.

6. Example

For the domain $\mathcal{D}$ over the cone of 2×2 real symmetric positive definite matrices we are going to show a family of functions h satisfying

$$Lh = 0 \qquad \mathbf{H}_\alpha h = 0 \tag{6.1}$$

that are not pluriharmonic. Let

$$h(xna) = a_1^{r_1} a_2^{r_2}$$

in coordinates (4.8). Then, by (2.7),

$$\begin{aligned} \Delta_j h =& ((a_j \partial_{a_j})^2 - (a_j \partial_{a_j}))h = r_j(r_j - 1)h, \ j = 1, 2 \\ \Delta_{12} h =& - a_j \partial_{a_j} h = -r_2 h \end{aligned}$$

Given $L = \beta_1 \Delta_1 + \beta_2 \Delta_2 + \beta_3 \Delta_{12}$, $\beta_j > 0$, we going to find $\mathbf{H}_\alpha = \Delta_1 + \beta \Delta_2$, $\beta > 0$, and r_1, r_2 different from 0 and 1 such that (6.1) holds i.e.

$$\begin{aligned} r_1(r_1 - 1) + \beta r_2(r_2 - 1) =& 0 \\ \beta_1 r_1(r_1 - 1) + \beta_2 r_2(r_2 - 1) - \beta_3 r_2 =& 0. \end{aligned}$$

Let

$$r = r_1(r_1 - 1). \tag{6.2}$$

Notice that if $r \geq -\frac{1}{4}$ then r_1 satisfying (6.2) exists. Therefore we solve

$$\begin{aligned} r + \beta r_2(r_2 - 1) =& 0 \\ \beta_1 r + \beta_2 r_2(r_2 - 1) - \beta_3 r_2 =& 0. \end{aligned} \tag{6.3}$$

for $r_2 \neq 0, 1$. (6.3) is equivalent to

$$\begin{aligned} r =& -\beta r_2(r_2 - 1) \\ (\beta_2 - \beta_1\beta)(r_2 - 1) =& \beta_3 \\ r_2 \neq& 0 \end{aligned}$$

or

$$\begin{aligned} r_2 &= \frac{\beta_3}{\beta_2 - \beta_1\beta} + 1 \\ r &= -\beta\frac{(\beta_3 + \beta_2 - \beta_1\beta)\beta_3}{(\beta_2 - \beta_1\beta)^2} \\ r_2 &\neq 0. \end{aligned}$$

Taking β sufficiently close to 0, we can make $r_2 > 1$ and $-\frac{1}{4} < r < 0$.

References

[BBDHPT] A. Bonami, D. Buraczewski, E. Damek, A. Hulanicki, R. Penney, B. Trojan, *Hua system and pluriharmonicity for symmetric irreducible Siegel domains of type II*, Journal of Functional Analysis, 2002.

[BDH] D. Buraczewski, E. Damek, A. Hulanicki, *Bounded pluriharmonic functions on symmetric irreducible Siegel domains*, Mathematische Zeitschrift, 2002.

[DH] E. Damek, A. Hulanicki, *Boundaries for left-invariant subelliptic operators on semidirect products of nilpotent and abelian groups*, J.Reine Angew. Math. 411 (1990), 1-38.

[DHMP] E.Damek, A. Hulanicki, D. Müller, M. Peloso, *Pluriharmonic H^2 functions on symmetric irreducible Siegel domains*, Geom. and Funct. Analysis 10, 2000, 1090-1117.

[DHP] E. Damek, A. Hulanicki, R. Penney, *Hua operators on bounded homogeneous domains in and alternative reproducing kernels for holomorphic functions*, Jour. Func. Analysis, 151 (1), (1997), 77-120.

[FK] J. Faraut, A. Korányi, *Analysis On Symmetric Cones*, Oxford Math. Monographs, Oxford Sc.Publ. Calderon Press (1994).

[F] H. Furstenberg, *A Poisson formula for semi-simple Lie groups*, Annals of Math. 77 (1963), 335 - 386.

[G] Y. Guivarc'h, *Sur la reprsentation integrale des fonctions harmoniques et des fonctions propres positives dans un espace riemannien symetrique*, Bull. Sci. Math. (2) 108 (1984), no. 4, 373–392.

[Ra] A. Raugi, *Fonctions harmoniques sur les groupes localement compactà base dénombrable*, Bull. Soc. Math. France, Mémoire 54 (1977), 5-118.

[Ru] W. Rudin, *Functional Analysis*, McGraw-HillBook Co., New York - Düsseldorf - Johannesburg, 1973.

[St] E.Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton 1993.

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