Gaussian Integers

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Let a, b be relatively prime positive integers.

Consider prime factorization of the Gaussian integer a^2+b^2i. We allow as prime factors number $1+i$ and Gaussian primes of the form $p_j=x\pm yi$ with x, y being positive integers, x odd, y even. Since odd factors of a^4+b^4 are $1 \pmod 8$, only Gaussian primes with y divisible by 4 may appear.

If a is odd and b even, then

$$a^2+b^2i=\pm p_1p_2\dots p_n\,.$$

If moreover

$$a^4+b^4=c^4+d^4\,,$$

then

$$c^2+d^2i=\pm q_1q_2\dots q_n$$

with $q_j=p_j$ or $q_j=\overline{p_j}$. In any case $q_j\equiv p_j \pmod 8$ and

$$a^2+b^2i\equiv\pm\left(c^2+d^2i\right) \pmod 8\,,$$

which gives $b\equiv d \pmod 4$. This yields

$$b^4-d^4\equiv 0 \pmod{256}$$

and

$$a^4-c^4=(a-c)(a+c)\left(a^2+c^2\right)\equiv 0 \pmod{256}\,.$$

Therefore $a+c$ or $a-c$ is divisible by 64.

If both a and b are odd, then

$$a^2+b^2i=\pm(1+i)p_1p_2\dots p_n\,.$$

If moreover

$$17\left(a^4+b^4\right)=c^4+d^4$$

with odd c, d, then

$$c^2+d^2i=\pm(1+i)(1\pm 4i)q_1q_2\dots q_n\,.$$

This gives

$$c^2+d^2i\equiv\pm(1\pm 4i)\left(a^2+b^2i\right) \pmod 8$$

and since a,b,c,d are all odd, we get

$$1+i\equiv\pm(1\pm 4i)(1+i)=1\mp 4+i(1\pm 4) \pmod 8\,.$$

This is a contradiction with the assumption $17(a^4+b^4)=c^4+d^4$ may have an odd solution.

Now, suppose we have

$$73\left(a^4+b^4\right)=c^4+d^4\,.$$

Then

$$a^2+b^2i=\pm(1+i)p_1p_2\dots p_n$$

and

$$c^2+d^2i=\pm(1+i)(3\pm 8i)q_1q_2\dots q_n$$

in case of a,c odd and b,d even and

$$a^2+b^2i=\pm p_1p_2\dots p_n$$
$$c^2+d^2i=\pm(3\pm 8i)q_1q_2\dots q_n$$

in case of all a,b,c,d odd. The $\pm$ signs in both equations are independent.

In any case

$$c^2+d^2i\equiv\pm(3\pm 8i)\left(a^2+b^2i\right)\pmod 8$$

which can be simplified to

$$c^2+d^2i\equiv\pm 3\left(a^2+b^2i\right)\pmod 8.$$

But a^2+b^2i and c^2+d^2i can be one of the following numbers $\pmod 8$: $1+i$, 1, $1+4i$ and $3(a^2+b^2i)\pmod 8$ is one of the numbers $3+3i$, 3, $3+4i$.

We got contradiction with assumption $73(a^4+b^4)=c^4+d^4$ may have a solution.