Magic Properties of Solutions of the Diophantine Equation $\mathbf{A^4-B^4=C^4-D^4=E^4-F^4}$

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Suppose we have a primitive (i.e. with no common divisor of the terms being positive integers) solution of

$$A^4-B^4=C^4-D^4=E^4-F^4$$

ordered in such a way, that

$$A>B, \qquad C>D, \qquad E>F, \qquad A>C>E\,.$$

OBSERVATION 1: In each of the 3 pairs (A,B), (C,D), (E,F) both numbers have the same parity. There are 2 pairs of even numbers and 1 pair of odd numbers.

OBSERVATION 2: Well, those 4 even numbers are in fact divisible by 4.

Let

$$x_1=\frac{A+B}{2} \qquad y_1=\frac{A-B}{2}$$
$$x_2=\frac{C+D}{2} \qquad y_2=\frac{C-D}{2}$$

and forget E, F for the moment.

The two straight lines on the plane passing through points

$$P_1=\left(y_1^2,\, x_1y_1\right), \qquad Q_2=\left(-x_2^2,\, -x_2y_2\right)$$

and

$$P_2=\left(y_2^2,\, x_2y_2\right), \qquad Q_1=\left(-x_1^2,\, -x_1y_1\right)$$

intersect somewhere, let us call the intersection point P_3.

OBSERVATION 3: Then P_3 has integer coordinates, moreover it has a square of an integer as the first coordinate and

$$P_3=\left(y_3^2,\, x_3y_3\right)$$

for appropriate **integers** x_3, y_3.

OBSERVATION 4: Put

$$Q_3=\left(-x_3^2,\, -x_3y_3\right)\,.$$

Then P_1, P_2, Q_3 lie on a straight line as well as Q_1, Q_2, Q_3 do.

OBSERVATION 5: We have

$$x_1y_1\left(x_1^2+y_1^2\right)=x_2y_2\left(x_2^2+y_2^2\right)=x_3y_3\left(x_3^2+y_3^2\right)$$

and we can get back forgotten E and F by

$$E=x_3+y_3, \qquad F=\left|x_3-y_3\right|\,.$$

OBSERVATION 6: The equal products above are divisible by 480.

OBSERVATION 7: It follows easily from previous observations that among 6 numbers x_1, y_1, x_2, y_2, x_3, y_3 there is exactly one odd number. But the odd guy is always one of the x's and all y's are always even.

OBSERVATION 8: Let a power of a prime p^n be a common divisor of some x_j and y_j. Then the same power p^n is also a divisor of some other x_k and y_k. In the third pair x_ℓ, y_ℓ only one number can be divisible by p. If p is not of the form $4m+1$, the number divisible by p is y_ℓ.

For p of the form $4m+1$:
if $p=5$, then any of the numbers x_ℓ, y_ℓ can be divisible by 5;
if $p=17$, then in 2 known solutions y_ℓ or the factor $x_\ell^2+y_\ell^2$ is divisible by 17.

Let

$$r_0 = \mathrm{GCD}(x_1,x_2,x_3)$$
$$r_1 = \mathrm{GCD}(x_1,y_2,y_3)$$
$$r_2 = \mathrm{GCD}(y_1,x_2,y_3)$$
$$r_3 = \mathrm{GCD}(y_1,y_2,x_3)$$

OBSERVATION 9: We have

$$r_0 \equiv 1 \pmod 4,$$

but r_0 may be something like $33=3\cdot 11$.

Let

$$a_1 = x_1/(r_0\cdot r_1) \qquad b_1 = y_1/(r_2\cdot r_3)$$
$$a_2 = x_2/(r_0\cdot r_2) \qquad b_2 = y_2/(r_3\cdot r_1)$$
$$a_3 = x_3/(r_0\cdot r_3) \qquad b_3 = y_3/(r_1\cdot r_2)$$

and

$$z_{23} = \mathrm{GCD}(a_1, y_2+x_2 i, y_3+x_3 i)$$
$$z_{31} = \mathrm{GCD}(a_2, y_3+x_3 i, y_1+x_1 i)$$
$$z_{12} = \mathrm{GCD}(a_3, y_1+x_1 i, y_2+x_2 i)$$
$$z_{32} = \mathrm{GCD}(b_1, y_2+x_2 i, y_3-x_3 i)$$
$$z_{13} = \mathrm{GCD}(b_2, y_3+x_3 i, y_1-x_1 i)$$
$$z_{21} = \mathrm{GCD}(b_3, y_1+x_1 i, y_2-x_2 i)$$

where GCD is taken in Gaussian integers and leaves freedom of choosing a factor of ± 1 or $\pm i$.

OBSERVATION 10: We can choose z's in such a way that the following equalities hold

$$x_1 = r_0 r_1 z_{23} \overline{z_{23}}$$
$$y_1 = r_2 r_3 z_{32} \overline{z_{32}}$$
$$x_2 = r_0 r_2 z_{31} \overline{z_{31}}$$
$$y_2 = r_3 r_1 z_{13} \overline{z_{13}}$$
$$x_3 = r_0 r_3 z_{12} \overline{z_{12}}$$
$$y_3 = r_1 r_2 z_{21} \overline{z_{21}}$$

$$y_1 + x_1 i = z_{12} z_{21} z_{31} \overline{z_{13}}$$
$$y_2 + x_2 i = z_{23} z_{32} z_{12} \overline{z_{21}}$$
$$y_3 + x_3 i = z_{31} z_{13} z_{23} \overline{z_{32}}$$

OBSERVATION 11: The product

$$\left(x_1^2 + y_1^2\right)\left(x_2^2 + y_2^2\right)\left(x_3^2 + y_3^2\right)$$

is always a square and a bit more... If a prime p is dividing the product, then some 2 factors are divisible by the same power of p.

OBSERVATION 12: In some solutions $y_1 << x_1$, i.e. $A \approx B$, and at the same time $x_3 < y_3$ with $C = x_2 + y_2 \approx E = x_3 + y_3$.

OBSERVATION 13: In the pair (x_3, y_3) the greater term is y_3 about twice more often than x_3.

OBSERVATION 14: Let x_1, y_1, x_2, y_2 be **ANY** numbers satisfying

$$x_1 y_1 \left(x_1^2 + y_1^2\right) = x_2 y_2 \left(x_2^2 + y_2^2\right) = S\,.$$

Then by substituting

$$X_1 = x_1^2, \qquad Y_1 = x_1 y_1, \qquad X_2 = x_2^2, \qquad Y_2 = x_2 y_2$$

we get

$$\frac{Y_1(X_1^2 + Y_1^2)}{X_1} = S = \frac{Y_2(X_2^2 + Y_2^2)}{X_2}\,.$$

Therefore the cubic equation

$$y(x^2 + y^2) = Sx$$

has 2 integer solutions, namely (X_1, Y_1) and (X_2, Y_2), so it should have a 3rd rational solution on the straight line passing through the first 2 solutions.

This is the key observation, since it leads to

$$x_1 y_1 \left(x_1^2 + y_1^2\right) = x_2 y_2 \left(x_2^2 + y_2^2\right) = c^2 x_3 y_3 \left(x_3^2 + y_3^2\right)$$

with rational x_3, y_3 and a square free integer c. If $c = 1$ we get a solution to

$$x_1 y_1 \left(x_1^2 + y_1^2\right) = x_2 y_2 \left(x_2^2 + y_2^2\right) = x_3 y_3 \left(x_3^2 + y_3^2\right)\,.$$