

**A Collection of
Numerical Solutions of
Multigrade Equations
Related to the
Prouhet-Tarry-Escott
Problem**

by Jarosław Wróblewski

jwr@math.uni.wroc.pl

<http://www.math.uni.wroc.pl/~jwr/PTE>

Version 1, December 4, 2009

Notes

I am interested only in solutions leading to symmetric PTE solutions, hence I require a multigrade to be satisfied by all exponents of the same parity up to a certain level.

Pure product solution comes from multiplying polynomials of the form

$$x^p - x^{-p}$$

and then extracting solutions in the well known way.

For even powers there is the same number of terms on both sides (I do not discard zero terms).

For odd powers numbers of left and right terms may differ.

Solution code is composed from:
 the largest power
 total number of terms
 the largest term

The best results known to me at the moment

Max exponent	Number of terms
8	10
9	12
10	12
11	20
12	28
13	26
14	30
15	34
16	42
17	48
18	64
19	74
20	88
21	92
22	92

Solution code: **8.10.313**

Powers: 2, 4, 6, 8.

Number of terms: **10**

Number of left terms: **5**

Number of right terms: **5**

Left terms:

313, 301, 188, 100, 99

Right terms:

308, 307, 180, 131, 71

Remarks:

Discovered by Peter Borwein, Petr Lisonek and Colin Percival (2002).

Solution code: **8.10.515**

Powers: 2, 4, 6, 8.

Number of terms: **10**

Number of left terms: **5**

Number of right terms: **5**

Left terms:

515, 452, 366, 189, 103

Right terms:

508, 471, 331, 245, 18

Remarks:

Discovered by Peter Borwein, Petr Lisonek and Colin Percival (2002).

Used by Jarosław Wróblewski (November 27, 2009) to produce solution **9.12.1293**.

Solution code: **9.12.323**

Powers: 1, 3, 5, 7, 9.

Number of terms: **12**

Number of left terms: **6**

Number of right terms: **6**

Left terms:

323, 289, 269, 173, 91, 7

Right terms:

313, 311, 247, 193, 59, 29

Remarks:

Discovered by Chen Shuwen (2000).

Solution code: **9.12.407**

Powers: 1, 3, 5, 7, 9.

Number of terms: **12**

Number of left terms: **6**

Number of right terms: **6**

Left terms:

407, 347, 341, 181, 163, 23

Right terms:

403, 371, 311, 221, 119, 37

Remarks:

Discovered by Jarosław Wróblewski (November 28, 2009).

Solution code: **9.12.463**

Powers: 1, 3, 5, 7, 9.

Number of terms: **12**

Number of left terms: **6**

Number of right terms: **6**

Left terms:

463, 391, 335, 217, 161, 43

Right terms:

461, 403, 287, 283, 91, 85

Remarks:

Discovered by Jarosław Wróblewski (November 28, 2009).

Solution code: **9.12.1293**

Powers: 1, 3, 5, 7, 9.

Number of terms: **12**

Number of left terms: **6**

Number of right terms: **6**

Left terms:

1293, 1167, 995, 679, 399, 57

Right terms:

1279, 1205, 925, 767, 299, 115

Remarks:

Constructed by Jarosław Wróblewski (November 27, 2009) from solution **8.10.515**.

Solution code: **10.12.151**

Powers: 2, 4, 6, 8, 10.

Number of terms: **12**

Number of left terms: **6**

Number of right terms: **6**

Left terms:

151, 140, 127, 86, 61, 22

Right terms:

148, 146, 121, 94, 47, 35

Remarks:

Discovered by Nuutti Kuosa (1999) using a computer program written by Jean-Charles Meyrignac, as a single-grade solution to power 10. Four days later Chen Shuwen noticed it was in fact a multigrade.

Solution code: **10.12.1511**

Powers: 2, 4, 6, 8, 10.

Number of terms: **12**

Number of left terms: **6**

Number of right terms: **6**

Left terms:

1511, 1138, 1075, 700, 622, 107

Right terms:

1510, 1180, 953, 886, 413, 293

Remarks:

The smallest solution of the infinite family of solutions constructed in:

Ajai Choudhry, Jarosław Wróblewski, *Ideal Solutions of the Tarry-Escott Problem of degree eleven with applications to Sums of Thirteenth Powers*, Hardy-Ramanujan Journal, Vol. 31 (2008) pp. 1-13

Solution code: **10.12.2058**

Powers: 2, 4, 6, 8, 10.

Number of terms: **12**

Number of left terms: **6**

Number of right terms: **6**

Left terms:

2058, 1896, 1618, 1109, 891, 257

Right terms:

2037, 1947, 1514, 1294, 639, 472

Remarks:

Discovered by David Broadhurst (2007):

D. Broadhurst, *A Chinese Prouhet-Tarry-Escott solution*,

<http://physics.open.ac.uk/~dbroadhu/cpte.pdf>

The second known solution.

It was later used by Ajai Choudhry and Jarosław Wróblewski to produce an infinite family of solutions:

Ajai Choudhry, Jarosław Wróblewski, *Ideal Solutions of the Tarry-Escott Problem of degree eleven with applications to Sums of Thirteenth Powers*, Hardy-Ramanujan Journal, Vol. 31 (2008) pp. 1-13

Solution code: **10.14.68**

Powers: 2, 4, 6, 8, 10.

Number of terms: **14**

Number of left terms: **7**

Number of right terms: **7**

Left terms:

68, 61, 55, 32, 31, 28, 1

Right terms:

67, 64, 49, 44, 23, 20, 17

Solution code: **10.14.400**

Powers: 2, 4, 6, 8, 10.

Number of terms: **14**

Number of left terms: **7**

Number of right terms: **7**

Left terms:

400, 365, 359, 254, 242, 89, 35

Right terms:

395, 383, 341, 271, 230, 70, 64

Remarks:

Constructed by Tito Piezas and Jarosław Wróblewski (November 2009).

Solution code: **10.14.4139**

Powers: 2, 4, 6, 8, 10.

Number of terms: 14

Number of left terms: 7

Number of right terms: 7

Left terms:

4139, 3812, 3691, 2545, 2468, 979, 448

Right terms:

4111, 3923, 3580, 2684, 2357, 896, 587

Remarks:

Constructed by Tito Piezas and Jarosław Wróblewski (November 2009).

Solution code: **10.16.93**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**

Number of left terms: **8**

Number of right terms: **8**

Left terms:

93, 87, 72, 52, 44, 41, 29, 1

Right terms:

92, 89, 67, 61, 39, 36, 33, 8

Solution code: **10.16.113**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**

Number of left terms: **8**

Number of right terms: **8**

Left terms:

113, 103, 91, 64, 59, 34, 10, 2

Right terms:

112, 106, 85, 74, 53, 26, 23, 1

Solution code: **10.16.132**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**

Number of left terms: **8**

Number of right terms: **8**

Left terms:

132, 117, 103, 71, 61, 59, 56, 12

Right terms:

131, 121, 92, 84, 72, 43, 39, 37

Solution code: **10.16.155**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**

Number of left terms: **8**

Number of right terms: **8**

Left terms:

155, 135, 133, 97, 93, 93, 55, 17

Right terms:

153, 145, 115, 107, 105, 83, 43, 33

Solution code: **10.16.172**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**

Number of left terms: **8**

Number of right terms: **8**

Left terms:

172, 159, 149, 135, 106, 78, 43, 14

Right terms:

169, 166, 140, 138, 111, 74, 37, 27

Solution code: **10.16.173**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**

Number of left terms: **8**

Number of right terms: **8**

Left terms:

173, 156, 143, 104, 92, 91, 39, 13

Right terms:

168, 167, 123, 116, 113, 61, 44, 29

Solution code: **10.16.188**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**

Number of left terms: **8**

Number of right terms: **8**

Left terms:

188, 168, 166, 125, 114, 83, 39, 25

Right terms:

183, 182, 151, 131, 120, 62, 60, 19

Solution code: **10.16.193**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**

Number of left terms: **8**

Number of right terms: **8**

Left terms:

193, 179, 164, 100, 75, 72, 51, 31

Right terms:

191, 184, 159, 109, 68, 60, 53, 45

Remarks:

By a proper sign changes we can make the solution work for powers 1 and 3:

Left terms:

193, -179, 164, -100, 75, 72, 51, -31

Right terms:

191, -184, 159, 109, -68, -60, 53, 45

Solution code: **10.16.2567**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**

Number of left terms: **8**

Number of right terms: **8**

Left terms:

2567, 2339, 2283, 1544, 1426, 710, 479, 237

Right terms:

2536, 2449, 2173, 1654, 1347, 631, 510, 347

Remarks:

Constructed by Tito Piezas and Jarosław Wróblewski (November 2009).

Solution code: **11.20.107**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**

Number of left terms: **10**

Number of right terms: **10**

Left terms:

107, 101, 86, 78, 66, 55, 43, 25, 19, 13

Right terms:

106, 103, 81, 79, 73, 50, 38, 30, 27, 6

Remarks:

Derived from solution **10.16.93**.

Solution code: **11.20.139**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**

Number of left terms: **11**

Number of right terms: **9**

Left terms:

139, 125, 125, 113, 95, 85, 67, 65, 31, 5, 1

Right terms:

137, 133, 119, 107, 101, 91, 61, 53, 49

Remarks:

Derived from solution **10.14.68**.

Solution code: **11.20.178**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**

Number of left terms: **11**

Number of right terms: **9**

Left terms:

178, 167, 154, 119, 94, 88, 67, 49, 20, 8, 5

Right terms:

175, 173, 148, 124, 100, 74, 62, 59, 34

Remarks:

Derived from solution **10.12.151**.

Solution code: **11.20.199**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**

Number of left terms: **11**

Number of right terms: **9**

Left terms:

199, 182, 169, 141, 118, 97, 90, 39, 35, 18, 3

Right terms:

194, 193, 149, 147, 139, 78, 70, 66, 55

Remarks:

Derived from solution **10.16.173**.

Solution code: **11.20.327**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**

Number of left terms: **10**

Number of right terms: **10**

Left terms:

327, 305, 279, 271, 217, 197, 163, 69, 69, 45

Right terms:

321, 317, 277, 255, 229, 213, 119, 97, 95, 19

Remarks:

Derived from solution **10.12.151**.

Solution code: **11.20.329**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**

Number of left terms: **10**

Number of right terms: **10**

Left terms:

329, 307, 281, 265, 199, 161, 149, 71, 67, 43

Right terms:

323, 319, 275, 253, 227, 145, 121, 97, 95, 17

Remarks:

Derived from solution **10.12.151**.

Solution code: **11.20.431**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**

Number of left terms: **10**

Number of right terms: **10**

Left terms:

431, 409, 383, 301, 251, 167, 163, 113, 85, 7

Right terms:

425, 421, 371, 317, 223, 199, 151, 125, 43, 35

Remarks:

Derived from solution **10.12.151**.

Solution code: **11.20.569**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**

Number of left terms: **10**

Number of right terms: **10**

Left terms:

569, 547, 521, 439, 389, 311, 223, 145, 95, 29

Right terms:

563, 559, 509, 455, 361, 337, 197, 173, 79, 35

Remarks:

Derived from solution **10.12.151**.

Solution code: **11.20.3615**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**

Number of left terms: **10**

Number of right terms: **10**

Left terms:

3615, 2869, 2743, 2427, 1993, 1837, 1767, 1313, 379, 233

Right terms:

3613, 2953, 2499, 2429, 2365, 1683, 1557, 1419, 651, 7

Remarks:

Derived from solution **10.12.1511**.

Solution code: **11.20.5155**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**

Number of left terms: **9**

Number of right terms: **11**

Left terms:

5155, 4409, 4283, 3533, 3377, 2347, 1919, 887, 733

Right terms:

5153, 4493, 4039, 3905, 2959, 2719, 1547, 1307, 361, 143, 17

Remarks:

Derived from solution **10.12.1511**.

Solution code: **11.20.6269**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**

Number of left terms: **10**

Number of right terms: **10**

Left terms:

6269, 5945, 5389, 4371, 3935, 2667, 1921, 1741, 435, 371

Right terms:

6227, 6047, 5181, 4741, 3431, 3097, 1963, 1209, 1083, 65

Remarks:

Derived from solution **10.12.2058**.

Solution code: **11.20.6625**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**

Number of left terms: **11**

Number of right terms: **9**

Left terms:

6625, 6301, 5745, 4727, 4291, 3023, 1995, 1385, 519, 291, 79

Right terms:

6583, 6403, 5537, 5097, 3787, 3453, 1607, 1283, 1231

Remarks:

Derived from solution **10.12.2058**.

Solution code: **11.22.65**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **22**

Number of left terms: **11**

Number of right terms: **11**

Left terms:

65, 60, 59, 47, 45, 40, 30, 18, 16, 10, 4

Right terms:

64, 63, 56, 49, 43, 42, 27, 21, 14, 13, 2

Pure product of:

1, 2, 3, 4, 5, 7, 9, 11, 13, 16, 17, 19, 23.

Solution code: **12.28.81**

Powers: 2, 4, 6, 8, 10, 12.

Number of terms: **28**

Number of left terms: **14**

Number of right terms: **14**

Left terms:

81, 74, 73, 72, 59, 52, 50, 48, 44, 30, 23, 15, 14, 1

Right terms:

80, 78, 71, 69, 62, 54, 49, 47, 40, 33, 25, 18, 4, 4

Pure product of:

1, 3, 4, 5, 6, 7, 10, 11, 13, 16, 17, 19, 23, 27.

Solution code: **12.28.82**

Powers: 2, 4, 6, 8, 10, 12.

Number of terms: **28**

Number of left terms: **14**

Number of right terms: **14**

Left terms:

82, 75, 74, 73, 60, 51, 49, 47, 33, 24, 21, 15, 12, 6

Right terms:

81, 79, 72, 70, 63, 54, 45, 44, 36, 27, 18, 17, 9, 5

Pure product of:

1, 3, 4, 5, 6, 7, 10, 11, 13, 16, 17, 19, 23, 29.

Solution code: **12.28.169**

Powers: 2, 4, 6, 8, 10, 12.

Number of terms: **28**

Number of left terms: **14**

Number of right terms: **14**

Left terms:

169, 151, 149, 145, 105, 101, 99, 95, 71, 63, 51, 47, 43, 23

Right terms:

167, 161, 139, 135, 133, 89, 85, 83, 81, 79, 61, 39, 33, 25

Pure product of:

1, 4, 5, 6, 7, 8, 9, 11, 13, 15, 17, 19, 23, 31.

Solution code: **13.26.173**

Powers: 1, 3, 5, 7, 9, 11, 13.

Number of terms: **26**

Number of left terms: **13**

Number of right terms: **13**

Left terms:

173, 159, 157, 131, 129, 107, 103, 79, 75, 51, 25, 9, 1

Right terms:

171, 167, 141, 139, 137, 97, 93, 89, 85, 43, 19, 15, 3

Pure product of:

1, 3, 4, 5, 6, 7, 9, 10, 11, 13, 16, 17, 19, 23, 29.

Remarks:

Published by L.J. Lander (1973), *Mathematics of Computation* 27 (122), 1973, p. 397

Solution code: **14.30.153**

Powers: 2, 4, 6, 8, 10, 12, 14.

Number of terms: **30**

Number of left terms: **15**

Number of right terms: **15**

Left terms:

153, 143, 139, 115, 111, 111, 103, 89, 61, 59, 57, 53, 47, 5, 3

Right terms:

151, 149, 127, 123, 123, 99, 95, 93, 73, 65, 51, 43, 41, 23, 1

Pure product of:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 15, 17, 19, 23.

Solution code: **15.34.189**

Powers: 1, 3, 5, 7, 9, 11, 13, 15.

Number of terms: **34**

Number of left terms: **18**

Number of right terms: **16**

Left terms:

189, 179, 175, 173, 159, 149, 129, 127, 97, 85, 55, 41, 31, 25, 17, 11, 3, 1

Right terms:

187, 185, 171, 169, 161, 155, 121, 119, 117, 69, 63, 39, 37, 21, 19, 13

Pure product of:

1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 14, 16, 17, 19, 23, 29.

Solution code: **16.42.95**

Powers: 2, 4, 6, 8, 10, 12, 14, 16.

Number of terms: **42**

Number of left terms: **21**

Number of right terms: **21**

Left terms:

95, 90, 88, 77, 75, 71, 70, 59, 49, 46, 45, 37, 34, 28, 17, 16, 16, 13, 13, 12, 9

Right terms:

94, 93, 82, 81, 79, 68, 67, 56, 55, 53, 35, 32, 31, 29, 27, 26, 20, 7, 6, 5, 0

Pure product of:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 16, 17, 19, 22, 23.

Solution code: **17.48.337**

Powers: 1, 3, 5, 7, 9, 11, 13, 15, 17.

Number of terms: **48**

Number of left terms: **24**

Number of right terms: **24**

Left terms:

337, 329, 315, 287, 287, 285, 273, 245, 243, 221, 219, 189, 179, 153, 123, 119, 95, 93, 87, 85, 45, 31, 21, 1

Right terms:

335, 333, 305, 303, 291, 269, 267, 261, 239, 225, 205, 201, 173, 157, 117, 113, 107, 103, 83, 73, 47, 39, 9, 7

Pure product of:

1, 2, 3, 5, 7, 8, 9, 10, 11, 12, 13, 17, 19, 23, 29, 31, 41, 43, 53.

Solution code: **18.64.185**

Powers: 2, 4, 6, 8, 10, 12, 14, 16, 18.

Number of terms: **64**

Number of left terms: **32**

Number of right terms: **32**

Left terms:

185, 181, 174, 160, 160, 159, 153, 139, 138, 135, 118, 114, 111, 96, 86, 84, 76, 64, 63, 62, 59, 42, 39, 37, 35, 32, 25, 17, 13, 12, 7, 3

Right terms:

184, 183, 169, 168, 162, 152, 148, 147, 141, 127, 120, 119, 105, 95, 94, 75, 73, 73, 68, 58, 53, 45, 43, 36, 31, 30, 26, 21, 16, 6, 6, 1

Pure product of:

1, 2, 3, 5, 7, 8, 9, 10, 11, 12, 13, 17, 19, 23, 29, 31, 33, 41, 43, 53.

Solution code: **19.74.287**

Powers: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19.

Number of terms: **74**

Number of left terms: **37**

Number of right terms: **37**

Left terms:

287, 277, 273, 251, 247, 239, 235, 221, 199, 193, 183, 181, 179, 171, 145, 141, 135, 131, 129, 129, 125, 123, 105, 105, 99, 93, 77, 75, 65, 63, 55, 51, 37, 35, 25, 9, 7

Right terms:

285, 283, 261, 259, 255, 231, 227, 225, 207, 191, 189, 185, 167, 155, 151, 149, 139, 137, 137, 133, 113, 111, 109, 97, 97, 91, 83, 81, 79, 53, 49, 43, 43, 39, 23, 17, 1

Pure product of:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 16, 17, 19, 22, 23, 25, 31, 41.

Solution code: **20.88.129**

Powers: 2, 4, 6, 8, 10, 12, 14, 16, 18, 20.

Number of terms: **88**

Number of left terms: **44**

Number of right terms: **44**

Left terms:

129, 126, 123, 122, 118, 113, 113, 111, 102, 97, 92, 91, 90, 89, 86, 79, 76, 70, 70, 70, 65,
59, 59, 53, 52, 49, 44, 43, 37, 37, 33, 32, 31, 24, 17, 16, 16, 15, 12, 11, 9, 5, 4, 0

Right terms:

128, 128, 121, 121, 119, 117, 112, 108, 101, 99, 95, 93, 87, 84, 84, 82, 80, 73, 67, 66, 62,
61, 60, 60, 47, 46, 45, 41, 40, 39, 34, 34, 25, 23, 20, 19, 14, 14, 13, 13, 7, 2, 1, 1

Pure product of:

1, 1, 2, 3, 4, 5, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 17, 19, 23, 25, 29, 31.

Solution code: **21.92.317**

Powers: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21.

Number of terms: **92**

Number of left terms: **46**

Number of right terms: **46**

Left terms:

317, 311, 307, 303, 285, 285, 275, 265, 261, 255, 243, 241, 241, 203, 199, 199, 189, 189, 175, 165, 157, 153, 147, 133, 119, 113, 109, 107, 97, 83, 77, 73, 71, 67, 67, 49, 43, 35, 31, 29, 25, 25, 19, 11, 1, 1

Right terms:

315, 315, 305, 301, 291, 283, 273, 263, 257, 257, 249, 249, 227, 211, 205, 187, 187, 185, 181, 169, 163, 145, 141, 135, 127, 117, 103, 99, 93, 91, 85, 79, 69, 59, 55, 51, 47, 45, 37, 23, 21, 17, 13, 13, 9, 3

Pure product of:

1, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 17, 19, 23, 25, 27, 29, 31, 37.

Solution code: **22.92.285**

Powers: 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22.

Number of terms: **92**

Number of left terms: **46**

Number of right terms: **46**

Left terms:

285, 279, 275, 269, 259, 259, 253, 249, 237, 233, 217, 203, 203, 201, 199, 199, 193, 185, 177, 167, 163, 159, 151, 151, 151, 133, 131, 117, 111, 107, 101, 95, 91, 91, 81, 81, 77, 77, 65, 59, 51, 45, 41, 41, 5, 1

Right terms:

283, 283, 273, 265, 261, 261, 257, 247, 231, 231, 221, 215, 209, 197, 191, 189, 189, 179, 179, 179, 175, 157, 155, 139, 139, 139, 137, 115, 109, 105, 105, 103, 89, 89, 83, 79, 73, 73, 67, 63, 61, 39, 33, 31, 21, 19

Pure product of:

1, 1, 2, 3, 4, 5, 6, 7, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19, 22, 23, 29, 31.