A Collection of Numerical Solutions of Multigrade Equations Related to the Prouhet-Tarry-Escott Problem

by Jarosław Wróblewski

jwr@math.uni.wroc.pl

http://www.math.uni.wroc.pl/∼jwr/PTE

Version 7, December 11, 2009

Notes

I am interested only in solutions leading to symmetric PTE solutions, hence I require a multigrade to be satisfied by all exponents of the same parity up to a certain level.

Pure product solution comes from multiplying polynomials of the form

$$x^p - x^{-p}$$

and then extracting solutions in the well known way.

For even powers there is the same number of terms on both sides (I do not discard zero terms).

For odd powers numbers of left and right terms may differ.

Solution code is composed from:
the largest power
total number of terms
the largest term
the second, third ... largest terms of any side (if needed)

The best results known to me at the moment

Max exponent	Number of terms	Ideal	Known/Ideal
8	10	10	1.000
9	12	11	1.091
10	12	12	1.000
11	20	13	1.538
12	26	14	1.857
13	26	15	1.733
14	30	16	1.875
15	34	17	2.000
16	42	18	2.333
17	48	19	2.526
18	58	20	2.900
19	65	21	3.095
20	70	22	3.182

For solutions of higher degree see **Version 5** and/or the last page of this document and/or the following paper:
Mihai Cipu, *Upper bounds for norms of products of binomials.* LMS Journal of Computation and Mathematics, 7 (2004), pp. 37-49

Solution code: **8.10.313**

Powers: 2, 4, 6, 8.

Number of terms: **10**
Number of left terms: **5**
Number of right terms: **5**
Left terms:
313, 301, 188, 100, 99
Right terms:
308, 307, 180, 131, 71

Remarks:
Discovered by Peter Borwein, Petr Lisonek and Colin Percival (2002).

Solution code: **8.10.515**

Powers: 2, 4, 6, 8.

Number of terms: **10**
Number of left terms: **5**
Number of right terms: **5**
Left terms:
515, 452, 366, 189, 103
Right terms:
508, 471, 331, 245, 18

Remarks:
Discovered by Peter Borwein, Petr Lisonek and Colin Percival (2002).
Used by Jarosław Wróblewski (November 27, 2009) to produce solution **9.12.1293**.

Solution code: **8.10.23750**

Powers: 2, 4, 6, 8.

Number of terms: **10**
Number of left terms: **5**
Number of right terms: **5**
Left terms:
23750, 20667, 20449, 11857, 436
Right terms:
23738, 20885, 20231, 11881, 12

Remarks:
The smallest member of family of solutions discovered by A. Letac in 1940's.

Solution code	Right terms	Left terms
8.12.36	36, 31, 30, 17, 7, 1	35, 34, 27, 19, 4, 3
8.12.62	62, 54, 47, 35, 27, 9	61, 57, 42, 37, 30, 1
8.12.66	66, 59, 57, 55, 28, 16	64, 62, 60, 49, 33, 11
8.12.71	71, 63, 50, 23, 18, 9	69, 67, 42, 37, 6, 5
8.12.74	74, 67, 47, 46, 27, 15	73, 69, 45, 41, 38, 2
8.12.109	109, 97, 91, 88, 54, 30	107, 99, 98, 74, 65, 24
8.12.111	111, 86, 81, 28, 25, 22	110, 94, 63, 57, 4, 1
8.12.113	113, 97, 89, 52, 31, 10	109, 107, 74, 67, 20, 13
8.12.114	114, 97, 89, 70, 43, 24	111, 106, 75, 73, 56, 2
8.12.119.115	119, 97, 94, 92, 45, 30	115, 111, 90, 74, 68, 7
8.12.119.118	119, 94, 85, 43, 38, 37	118, 101, 67, 61, 50, 7
8.12.137	137, 120, 110, 73, 57, 3	135, 127, 97, 88, 45, 18
8.12.139	139, 124, 115, 89, 42, 18	135, 133, 106, 93, 46, 4
8.12.151	151, 139, 117, 58, 55, 28	149, 143, 113, 70, 37, 36
8.12.158	158, 137, 126, 125, 76, 55	154, 148, 127, 106, 95, 45
8.12.163	163, 145, 138, 103, 56, 14	161, 152, 131, 105, 58, 2
8.12.167	167, 150, 122, 89, 71, 47	163, 158, 109, 93, 85, 34
8.12.179.172	179, 151, 150, 108, 67, 38	172, 171, 123, 122, 74, 25
8.12.179.173	179, 149, 139, 65, 63, 57	173, 167, 111, 91, 81, 5
8.12.179.178	179, 142, 132, 83, 59, 35	178, 151, 111, 103, 55, 28
8.12.195	195, 169, 148, 98, 71, 42	190, 182, 127, 111, 84, 13
8.12.211	211, 165, 155, 59, 54, 44	209, 180, 121, 111, 31, 10
8.12.212	212, 189, 167, 114, 83, 10	206, 202, 148, 133, 75, 9
8.12.222	222, 182, 164, 59, 55, 41	220, 193, 146, 97, 26, 21
8.12.229	229, 215, 166, 118, 63, 48	224, 222, 162, 113, 89, 5
8.12.237	237, 206, 195, 88, 49, 41	231, 223, 179, 104, 50, 3
8.12.243	243, 219, 178, 112, 71, 50	240, 226, 167, 126, 67, 43
8.12.245	245, 213, 202, 141, 101, 22	239, 231, 178, 158, 97, 15
8.12.265.256	265, 240, 204, 179, 73, 67	256, 255, 197, 172, 111, 5
8.12.265.257	265, 229, 223, 133, 72, 4	257, 252, 200, 149, 43, 41
8.12.265.264	265, 229, 219, 108, 88, 6	264, 236, 211, 122, 57, 45
8.12.267	267, 245, 177, 104, 79, 4	265, 249, 168, 124, 61, 13
8.12.282.277	282, 237, 223, 122, 107, 53	277, 257, 197, 138, 118, 3
8.12.282.278	282, 229, 215, 107, 54, 47	278, 250, 177, 149, 37, 9
8.12.295	295, 216, 203, 106, 91, 78	294, 232, 169, 126, 125, 13

Remarks:

Powers: 2, 4, 6, 8 with 6+6=12 terms.

Results of a selective search by Jarosław Wróblewski (December 2009).

Solution code	Right terms	Left terms
8.12.303	303, 265, 227, 119, 81, 19	291, 289, 191, 167, 45, 7
8.12.325	325, 277, 271, 201, 131, 57	317, 305, 233, 219, 139, 39
8.12.326	326, 273, 243, 227, 112, 89	317, 301, 208, 207, 186, 7
8.12.347	347, 289, 246, 112, 97, 54	343, 306, 194, 192, 43, 41
8.12.358	358, 329, 323, 281, 75, 12	357, 335, 317, 282, 76, 1
8.12.362	362, 295, 291, 163, 118, 81	353, 333, 233, 205, 134, 6
8.12.365	365, 305, 291, 219, 139, 79	355, 339, 241, 229, 181, 15
8.12.370	370, 311, 252, 201, 101, 61	369, 316, 241, 205, 123, 14
8.12.375	375, 325, 266, 184, 162, 61	371, 338, 230, 215, 171, 24
8.12.389	389, 354, 329, 281, 80, 50	379, 375, 304, 295, 94, 14
8.12.392	392, 328, 308, 241, 125, 109	385, 356, 277, 224, 197, 32
8.12.405	405, 362, 351, 205, 162, 49	393, 390, 331, 210, 167, 29
8.12.407.398.383	407, 365, 284, 242, 163, 79	398, 383, 251, 233, 220, 11
8.12.407.398.385	407, 363, 328, 265, 98, 70	398, 385, 293, 287, 120, 22
8.12.407.405	407, 386, 249, 129, 105, 2	405, 389, 243, 154, 74, 27
8.12.412	412, 358, 337, 161, 154, 27	407, 378, 314, 203, 92, 71
8.12.417	417, 347, 277, 184, 169, 64	416, 353, 248, 233, 139, 69
8.12.418	418, 368, 335, 237, 149, 66	402, 401, 302, 253, 165, 16
8.12.421	421, 357, 343, 274, 76, 62	419, 372, 323, 281, 98, 14
8.12.430	430, 351, 344, 203, 179, 82	424, 386, 283, 259, 162, 65
8.12.438	438, 389, 335, 256, 122, 97	430, 409, 302, 277, 151, 48
8.12.449	449, 397, 381, 374, 300, 28	436, 431, 363, 357, 316, 10
8.12.470	470, 396, 382, 257, 183, 61	465, 423, 349, 268, 194, 22
8.12.471	471, 407, 379, 295, 163, 105	453, 449, 335, 303, 209, 35
8.12.509	509, 429, 420, 248, 172, 111	495, 477, 364, 284, 179, 72
8.12.513	513, 410, 336, 218, 109, 77	512, 418, 315, 241, 123, 14
8.12.531	531, 471, 467, 269, 67, 7	523, 501, 441, 277, 41, 37
8.12.538	538, 472, 415, 321, 137, 114	529, 498, 361, 358, 177, 40

Remarks:

Powers: 2, 4, 6, 8 with 6+6=12 terms.

Results of a selective search by Jarosław Wróblewski (December 2009).

Solution code: **8.12.541**

Powers: 2, 4, 6, 8.

Number of terms: **12**
Number of left terms: **6**
Number of right terms: **6**
Left terms:
541, 503, 339, 176, 140, 73
Right terms:
532, 517, 305, 251, 96, 31

Remarks:
A member of family of solutions discovered by Jarosław Wróblewski (November 2009).
Left side terms:
2 a + 5 b + d
2 a + 5 b - d
5 a - 2 b + c
5 a - 2 b - c
4 a + 6 b
6 a - 4 b
Right side terms:
5 a + 2 b + c
5 a + 2 b - c
-2 a + 5 b + d
-2 a + 5 b - d
6 a + 4 b
-4 a + 6 b
Assume:

$$c^2 = p*a^2 + q*b^2$$
$$d^2 = p*b^2 + q*a^2$$
$$p = -11/5$$
$$q = 64/5$$

This solution is obtained with
a = 118
b = 89
c = 266
d = 401

Solution code	Right terms	Left terms
8.12.562	562, 487, 466, 439, 233, 225	549, 523, 470, 373, 326, 163
8.12.575	575, 497, 357, 216, 181, 76	573, 504, 323, 281, 140, 71
8.12.583	583, 501, 427, 205, 113, 109	569, 539, 347, 317, 67, 15
8.12.596	596, 446, 445, 243, 242, 111	594, 485, 354, 353, 149, 148
8.12.797	797, 703, 635, 471, 252, 136	771, 760, 567, 508, 289, 37
8.12.890	890, 732, 653, 470, 233, 213	883, 772, 555, 535, 318, 62

Remarks:

Powers: 2, 4, 6, 8 with 6+6=12 terms.

Results of a selective search by Jarosław Wróblewski (December 2009).

Solution code: **8.12.55292**

Powers: 2, 4, 6, 8.

Number of terms: **12**
Number of left terms: **6**
Number of right terms: **6**
Left terms:
55292, 50841, 50712, 41050, 23681, 11369
Right terms:
54280, 54151, 45759, 44489, 20902, 14148

Remarks:
A member of family of solutions discovered by Jarosław Wróblewski (November 2009).
Left side terms:
a + 10 b + d
a + 10 b - d
10 a - b + c
10 a - b - c
a + 11 b
11 a - b
Right side terms:
10 a + b + c
10 a + b - c
-a + 10 b + d
-a + 10 b - d
11 a + b
-a + 11 b
Assume:

$$c^2 = p*a^2 + q*b^2$$
$$d^2 = p*b^2 + q*a^2$$
$$p = -27/5$$
$$q = 248/5$$

This solution is obtained with
a = 9533
b = 3439
c = 9791
d = 66661

Solution code: **9.12.323**

Powers: 1, 3, 5, 7, 9.

Number of terms: **12**
Number of left terms: **6**
Number of right terms: **6**
Left terms:
323, 289, 269, 173, 91, 7
Right terms:
313, 311, 247, 193, 59, 29

Remarks:
Discovered by Chen Shuwen (2000).

Solution code: **9.12.407**

Powers: 1, 3, 5, 7, 9.

Number of terms: **12**
Number of left terms: **6**
Number of right terms: **6**
Left terms:
407, 347, 341, 181, 163, 23
Right terms:
403, 371, 311, 221, 119, 37

Remarks:
Discovered by Jarosław Wróblewski (November 28, 2009).

Solution code: **9.12.463**

Powers: 1, 3, 5, 7, 9.

Number of terms: **12**
Number of left terms: **6**
Number of right terms: **6**
Left terms:
463, 391, 335, 217, 161, 43
Right terms:
461, 403, 287, 283, 91, 85

Remarks:
Discovered by Jarosław Wróblewski (November 28, 2009).

Solution code: **9.12.1293**

Powers: 1, 3, 5, 7, 9.

Number of terms: **12**
Number of left terms: **6**
Number of right terms: **6**
Left terms:
1293, 1167, 995, 679, 399, 57
Right terms:
1279, 1205, 925, 767, 299, 115

Remarks:
Constructed by Jarosław Wróblewski (November 27, 2009) from solution **8.10.515**.

Solution code: **10.12.151**

Powers: 2, 4, 6, 8, 10.

Number of terms: **12**
Number of left terms: **6**
Number of right terms: **6**
Left terms:
151, 140, 127, 86, 61, 22
Right terms:
148, 146, 121, 94, 47, 35

Remarks:
Discovered by Nuutti Kuosa (1999) using a computer program written by Jean-Charles Meyrignac, as a single-grade solution to power 10. Four days later Chen Shuwen noticed it was in fact a multigrade.

Solution code: **10.12.1511**

Powers: 2, 4, 6, 8, 10.

Number of terms: **12**
Number of left terms: **6**
Number of right terms: **6**
Left terms:
1511, 1138, 1075, 700, 622, 107
Right terms:
1510, 1180, 953, 886, 413, 293

Remarks:
The smallest solution of the infinite family of solutions constructed in:

Ajai Choudhry, Jarosław Wróblewski, *Ideal Solutions of the Tarry-Escott Problem of degree eleven with applications to Sums of Thirteenth Powers*, Hardy-Ramanujan Journal, Vol. 31 (2008) pp. 1-13

The above paper is available at:
`http://www.nias.res.in/hrj/contentsvol31.htm`

Solution code: **10.12.2058**

Powers: 2, 4, 6, 8, 10.

Number of terms: **12**
Number of left terms: **6**
Number of right terms: **6**
Left terms:
2058, 1896, 1618, 1109, 891, 257
Right terms:
2037, 1947, 1514, 1294, 639, 472

Remarks:
Discovered by David Broadhurst (2007):
D. Broadhurst, *A Chinese Prouhet-Tarry-Escott solution*,
`http://physics.open.ac.uk/~dbroadhu/cpte.pdf`

The second known solution.
It was later used by Ajai Choudhry and Jarosław Wróblewski to produce an infinite family of solutions:

Ajai Choudhry, Jarosław Wróblewski, *Ideal Solutions of the Tarry-Escott Problem of degree eleven with applications to Sums of Thirteenth Powers*, Hardy-Ramanujan Journal, Vol. 31 (2008) pp. 1-13

The above paper is available at:
`http://www.nias.res.in/hrj/contentsvol31.htm`

Solution code: **10.12.14770**

Powers: 2, 4, 6, 8, 10.

Number of terms: **12**
Number of left terms: **6**
Number of right terms: **6**
Left terms:
14770, 12638, 11632, 7115, 7043, 929
Right terms:
14693, 13165, 10112, 9718, 4054, 3455

Remarks:
3rd smallest solution of the infinite family of solutions constructed in:

Ajai Choudhry, Jarosław Wróblewski, *Ideal Solutions of the Tarry-Escott Problem of degree eleven with applications to Sums of Thirteenth Powers*, Hardy-Ramanujan Journal, Vol. 31 (2008) pp. 1-13

The above paper is available at:
`http://www.nias.res.in/hrj/contentsvol31.htm`

Solution code: **10.12.23742**

Powers: 2, 4, 6, 8, 10.

Number of terms: **12**
Number of left terms: **6**
Number of right terms: **6**
Left terms:
23742, 18687, 18372, 12734, 9611, 349
Right terms:
23708, 19653, 16426, 14714, 7713, 3309

Remarks:
4th smallest solution of the infinite family of solutions constructed in:

Ajai Choudhry, Jarosław Wróblewski, *Ideal Solutions of the Tarry-Escott Problem of degree eleven with applications to Sums of Thirteenth Powers*, Hardy-Ramanujan Journal, Vol. 31 (2008) pp. 1-13

The above paper is available at:
`http://www.nias.res.in/hrj/contentsvol31.htm`

Solution code: **10.14.68**

Powers: 2, 4, 6, 8, 10.

Number of terms: **14**
Number of left terms: **7**
Number of right terms: **7**
Left terms:
68, 61, 55, 32, 31, 28, 1
Right terms:
67, 64, 49, 44, 23, 20, 17

Solution code: **10.14.400**

Powers: 2, 4, 6, 8, 10.

Number of terms: **14**
Number of left terms: **7**
Number of right terms: **7**
Left terms:
400, 365, 359, 254, 242, 89, 35
Right terms:
395, 383, 341, 271, 230, 70, 64

Remarks:
Constructed by Tito Piezas and Jarosław Wróblewski (November 2009).

Solution code: **10.14.4139**

Powers: 2, 4, 6, 8, 10.

Number of terms: **14**
Number of left terms: **7**
Number of right terms: **7**
Left terms:
4139, 3812, 3691, 2545, 2468, 979, 448
Right terms:
4111, 3923, 3580, 2684, 2357, 896, 587

Remarks:
Constructed by Tito Piezas and Jarosław Wróblewski (November 2009).

Solution code: **10.14.12689**

Powers: 2, 4, 6, 8, 10.

Number of terms: **14**
Number of left terms: **7**
Number of right terms: **7**
Left terms:
12689, 10560, 9236, 6755, 5745, 3767, 1324
Right terms:
12676, 10711, 8733, 7585, 5280, 3164, 2305

Remarks:
Constructed by Tito Piezas and Jarosław Wróblewski (November 2009).

Solution code: **10.16.93**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**
Number of left terms: **8**
Number of right terms: **8**
Left terms:
93, 87, 72, 52, 44, 41, 29, 1
Right terms:
92, 89, 67, 61, 39, 36, 33, 8

Solution code: **10.16.113**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**
Number of left terms: **8**
Number of right terms: **8**
Left terms:
113, 103, 91, 64, 59, 34, 10, 2
Right terms:
112, 106, 85, 74, 53, 26, 23, 1

Solution code: **10.16.132**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**
Number of left terms: **8**
Number of right terms: **8**
Left terms:
132, 117, 103, 71, 61, 59, 56, 12
Right terms:
131, 121, 92, 84, 72, 43, 39, 37

Solution code: **10.16.155**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**
Number of left terms: **8**
Number of right terms: **8**
Left terms:
155, 135, 133, 97, 93, 93, 55, 17
Right terms:
153, 145, 115, 107, 105, 83, 43, 33

Solution code: **10.16.172**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**
Number of left terms: **8**
Number of right terms: **8**
Left terms:
172, 159, 149, 135, 106, 78, 43, 14
Right terms:
169, 166, 140, 138, 111, 74, 37, 27

Solution code: **10.16.173**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**
Number of left terms: **8**
Number of right terms: **8**
Left terms:
173, 156, 143, 104, 92, 91, 39, 13
Right terms:
168, 167, 123, 116, 113, 61, 44, 29

Solution code: **10.16.188**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**
Number of left terms: **8**
Number of right terms: **8**
Left terms:
188, 168, 166, 125, 114, 83, 39, 25
Right terms:
183, 182, 151, 131, 120, 62, 60, 19

Solution code: **10.16.193**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**
Number of left terms: **8**
Number of right terms: **8**
Left terms:
193, 179, 164, 100, 75, 72, 51, 31
Right terms:
191, 184, 159, 109, 68, 60, 53, 45

Remarks:
By a proper sign changes we can make the solution work for powers 1 and 3:
Left terms:
193, -179, 164, -100, 75, 72, 51, -31
Right terms:
191, -184, 159, 109, -68, -60, 53, 45

Solution code: **10.16.275**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**
Number of left terms: **8**
Number of right terms: **8**
Left terms:
275, 239, 211, 158, 100, 98, 2, 1
Right terms:
274, 245, 188, 185, 89, 86, 37, 22

Remarks:
Derived from solution **9.12.463**.

Solution code: **10.16.2567**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**
Number of left terms: **8**
Number of right terms: **8**
Left terms:
2567, 2339, 2283, 1544, 1426, 710, 479, 237
Right terms:
2536, 2449, 2173, 1654, 1347, 631, 510, 347

Remarks:
Constructed by Tito Piezas and Jarosław Wróblewski (November 2009).

Solution code: **11.20.107**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**
Number of left terms: **10**
Number of right terms: **10**
Left terms:
107, 101, 86, 78, 66, 55, 43, 25, 19, 13
Right terms:
106, 103, 81, 79, 73, 50, 38, 30, 27, 6

Remarks:
Derived from solution **10.16.93**.

Solution code: **11.20.139**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**
Number of left terms: **11**
Number of right terms: **9**
Left terms:
139, 125, 125, 113, 95, 85, 67, 65, 31, 5, 1
Right terms:
137, 133, 119, 107, 101, 91, 61, 53, 49

Remarks:
Derived from solution **10.14.68**.

Solution code: **11.20.178**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**
Number of left terms: **11**
Number of right terms: **9**
Left terms:
178, 167, 154, 119, 94, 88, 67, 49, 20, 8, 5
Right terms:
175, 173, 148, 124, 100, 74, 62, 59, 34

Remarks:
Derived from solution **10.12.151**.

Solution code: **11.20.199**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**
Number of left terms: **11**
Number of right terms: **9**
Left terms:
199, 182, 169, 141, 118, 97, 90, 39, 35, 18, 3
Right terms:
194, 193, 149, 147, 139, 78, 70, 66, 55

Remarks:
Derived from solution **10.16.173**.

Solution code: **11.20.327**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**
Number of left terms: **10**
Number of right terms: **10**
Left terms:
327, 305, 279, 271, 217, 197, 163, 69, 69, 45
Right terms:
321, 317, 277, 255, 229, 213, 119, 97, 95, 19

Remarks:
Derived from solution **10.12.151**.

Solution code: **11.20.329**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**
Number of left terms: **10**
Number of right terms: **10**
Left terms:
329, 307, 281, 265, 199, 161, 149, 71, 67, 43
Right terms:
323, 319, 275, 253, 227, 145, 121, 97, 95, 17

Remarks:
Derived from solution **10.12.151**.

Solution code: **11.20.431**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**
Number of left terms: **10**
Number of right terms: **10**
Left terms:
431, 409, 383, 301, 251, 167, 163, 113, 85, 7
Right terms:
425, 421, 371, 317, 223, 199, 151, 125, 43, 35

Remarks:
Derived from solution **10.12.151**.

Solution code: **11.20.569**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**
Number of left terms: **10**
Number of right terms: **10**
Left terms:
569, 547, 521, 439, 389, 311, 223, 145, 95, 29
Right terms:
563, 559, 509, 455, 361, 337, 197, 173, 79, 35

Remarks:
Derived from solution **10.12.151**.

Solution code: **11.20.3615**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**
Number of left terms: **10**
Number of right terms: **10**
Left terms:
3615, 2869, 2743, 2427, 1993, 1837, 1767, 1313, 379, 233
Right terms:
3613, 2953, 2499, 2429, 2365, 1683, 1557, 1419, 651, 7

Remarks:
Derived from solution **10.12.1511**.

Solution code: **11.20.5155**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**
Number of left terms: **9**
Number of right terms: **11**
Left terms:
5155, 4409, 4283, 3533, 3377, 2347, 1919, 887, 733
Right terms:
5153, 4493, 4039, 3905, 2959, 2719, 1547, 1307, 361, 143, 17

Remarks:
Derived from solution **10.12.1511**.

Solution code: **11.20.6269**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**
Number of left terms: **10**
Number of right terms: **10**
Left terms:
6269, 5945, 5389, 4371, 3935, 2667, 1921, 1741, 435, 371
Right terms:
6227, 6047, 5181, 4741, 3431, 3097, 1963, 1209, 1083, 65

Remarks:
Derived from solution **10.12.2058**.

Solution code: **11.20.6625**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**
Number of left terms: **11**
Number of right terms: **9**
Left terms:
6625, 6301, 5745, 4727, 4291, 3023, 1995, 1385, 519, 291, 79
Right terms:
6583, 6403, 5537, 5097, 3787, 3453, 1607, 1283, 1231

Remarks:
Derived from solution **10.12.2058**.

Solution code: **11.20.43107**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**
Number of left terms: **9**
Number of right terms: **11**
Left terms:
43107, 38843, 36831, 27797, 27653, 15819, 15425, 12763, 5869
Right terms:
42953, 39897, 33791, 33003, 21675, 20477, 15973, 9697, 5459, 663, 519

Remarks:
Derived from solution **10.12.14770**.

Solution code: **11.20.48287**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**
Number of left terms: **11**
Number of right terms: **9**
Left terms:
48287, 44023, 42011, 32977, 32833, 20605, 16889, 7583, 4661, 1477, 689
Right terms:
48133, 45077, 38971, 38183, 26855, 25657, 11837, 10793, 6529

Remarks:
Derived from solution **10.12.14770**.

Solution code: **11.20.65507**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**
Number of left terms: **9**
Number of right terms: **11**
Left terms:
65507, 55397, 54767, 43491, 37245, 29393, 21283, 17325, 14829
Right terms:
65439, 57329, 50875, 47451, 33449, 29461, 24641, 19351, 7445, 2597, 1199

Remarks:
Derived from solution **10.12.23742**.

Solution code: **11.20.78905**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**
Number of left terms: **10**
Number of right terms: **10**
Left terms:
78905, 68795, 68165, 56889, 50643, 32119, 30723, 12199, 7885, 1431
Right terms:
78837, 70727, 64273, 60849, 46847, 38039, 24803, 16063, 5323, 1993

Remarks:
Derived from solution **10.12.23742**.

Solution code: **11.22.65**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **22**
Number of left terms: **11**
Number of right terms: **11**
Left terms:
65, 60, 59, 47, 45, 40, 30, 18, 16, 10, 4
Right terms:
64, 63, 56, 49, 43, 42, 27, 21, 14, 13, 2

Pure product of:
1, 2, 3, 4, 5, 7, 9, 11, 13, 16, 17, 19, 23.

Solution code: **12.26.79**

Powers: 2, 4, 6, 8, 10, 12.

Number of terms: **26**
Number of left terms: **13**
Number of right terms: **13**
Left terms:
79, 76, 72, 69, 58, 53, 48, 44, 26, 21, 17, 14, 11
Right terms:
78, 77, 74, 64, 61, 54, 51, 33, 31, 28, 19, 8, 4

Pure product of:
1, 2, 5, 6, 7, 8, 9, 11, 13, 15, 17, 19, 20, 25.

Remarks:
This solution is taken from the following paper:
Mihai Cipu, *Upper bounds for norms of products of binomials*. LMS Journal of Computation and Mathematics, 7 (2004), pp. 37-49

Solution code: **12.28.81**

Powers: 2, 4, 6, 8, 10, 12.

Number of terms: **28**
Number of left terms: **14**
Number of right terms: **14**
Left terms:
81, 74, 73, 72, 59, 52, 50, 48, 44, 30, 23, 15, 14, 1
Right terms:
80, 78, 71, 69, 62, 54, 49, 47, 40, 33, 25, 18, 4, 4

Pure product of:
1, 3, 4, 5, 6, 7, 10, 11, 13, 16, 17, 19, 23, 27.

Solution code: **12.28.82**

Powers: 2, 4, 6, 8, 10, 12.

Number of terms: **28**
Number of left terms: **14**
Number of right terms: **14**
Left terms:
82, 75, 74, 73, 60, 51, 49, 47, 33, 24, 21, 15, 12, 6
Right terms:
81, 79, 72, 70, 63, 54, 45, 44, 36, 27, 18, 17, 9, 5

Pure product of:
1, 3, 4, 5, 6, 7, 10, 11, 13, 16, 17, 19, 23, 29.

Solution code: **12.28.119**

Powers: 2, 4, 6, 8, 10, 12.

Number of terms: **28**
Number of left terms: **14**
Number of right terms: **14**
Left terms:
119, 109, 105, 103, 81, 81, 67, 65, 57, 29, 27, 19, 19, 11
Right terms:
117, 115, 101, 99, 89, 79, 73, 51, 49, 45, 41, 7, 3, 1

Solution code: **12.28.169**

Powers: 2, 4, 6, 8, 10, 12.

Number of terms: **28**
Number of left terms: **14**
Number of right terms: **14**
Left terms:
169, 151, 149, 145, 105, 101, 99, 95, 71, 63, 51, 47, 43, 23
Right terms:
167, 161, 139, 135, 133, 89, 85, 83, 81, 79, 61, 39, 33, 25

Pure product of:
1, 4, 5, 6, 7, 8, 9, 11, 13, 15, 17, 19, 23, 31.

Solution code: **13.26.173**

Powers: 1, 3, 5, 7, 9, 11, 13.

Number of terms: **26**
Number of left terms: **13**
Number of right terms: **13**
Left terms:
173, 159, 157, 131, 129, 107, 103, 79, 75, 51, 25, 9, 1
Right terms:
171, 167, 141, 139, 137, 97, 93, 89, 85, 43, 19, 15, 3

Pure product of:
1, 3, 4, 5, 6, 7, 9, 10, 11, 13, 16, 17, 19, 23, 29.

Remarks:
Published by L.J. Lander (1973), *Mathematics of Computation 27 (122), 1973, p. 397*

Solution code: **13.28.191**

Powers: 1, 3, 5, 7, 9, 11, 13.

Number of terms: **28**
Number of left terms: **14**
Number of right terms: **14**
Left terms:
191, 177, 175, 173, 147, 131, 129, 125, 113, 69, 57, 51, 27, 7
Right terms:
189, 185, 171, 167, 153, 137, 123, 119, 115, 71, 67, 37, 21, 17

Pure product of:
1, 3, 4, 5, 6, 7, 10, 11, 13, 16, 17, 19, 23, 27, 29.

Solution code: **13.30.69**

Powers: 1, 3, 5, 7, 9, 11, 13.

Number of terms: **30**
Number of left terms: **14**
Number of right terms: **16**
Left terms:
69, 64, 62, 61, 50, 48, 41, 40, 38, 27, 19, 16, 15, 15
Right terms:
68, 67, 60, 59, 54, 46, 45, 34, 32, 31, 31, 10, 9, 8, 6, 5

Pure product of:
1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 15, 16, 17, 19.

Solution code: **13.30.129**

Powers: 1, 3, 5, 7, 9, 11, 13.

Number of terms: **30**
Number of left terms: **13**
Number of right terms: **17**

Left terms:
129, 119, 115, 89, 87, 85, 83, 47, 43, 41, 37, 25, 23

Right terms:
127, 125, 103, 101, 95, 81, 63, 59, 51, 49, 19, 17, 9, 9, 7, 5, 3

Pure product of:
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 17, 19.

Solution code: **14.30.153**

Powers: 2, 4, 6, 8, 10, 12, 14.

Number of terms: **30**
Number of left terms: **15**
Number of right terms: **15**
Left terms:
153, 143, 139, 115, 111, 111, 103, 89, 61, 59, 57, 53, 47, 5, 3
Right terms:
151, 149, 127, 123, 123, 99, 95, 93, 73, 65, 51, 43, 41, 23, 1

Pure product of:
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 15, 17, 19, 23.

Solution code: **14.32.187**

Powers: 2, 4, 6, 8, 10, 12, 14.

Number of terms: **32**
Number of left terms: **16**
Number of right terms: **16**
Left terms:
187, 173, 171, 157, 127, 125, 123, 121, 83, 79, 75, 71, 39, 23, 15, 1
Right terms:
185, 181, 159, 155, 151, 115, 111, 107, 103, 99, 61, 57, 37, 33, 17, 13

Pure product of:
1, 3, 4, 5, 6, 7, 9, 10, 11, 13, 14, 16, 17, 19, 23, 29.

Solution code: **15.34.189**

Powers: 1, 3, 5, 7, 9, 11, 13, 15.

Number of terms: **34**
Number of left terms: **18**
Number of right terms: **16**

Left terms:
189, 179, 175, 173, 159, 149, 129, 127, 97, 85, 55, 41, 31, 25, 17, 11, 3, 1

Right terms:
187, 185, 171, 169, 161, 155, 121, 119, 117, 69, 63, 39, 37, 21, 19, 13

Pure product of:
1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 14, 16, 17, 19, 23, 29.

Solution code: **16.42.95**

Powers: 2, 4, 6, 8, 10, 12, 14, 16.

Number of terms: **42**
Number of left terms: **21**
Number of right terms: **21**

Left terms:

95, 90, 88, 77, 75, 71, 70, 59, 49, 46, 45, 37, 34, 28, 17, 16, 16, 13, 13, 12, 9

Right terms:

94, 93, 82, 81, 79, 68, 67, 56, 55, 53, 35, 32, 31, 29, 27, 26, 20, 7, 6, 5, 0

Pure product of:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 16, 17, 19, 22, 23.

Solution code: **17.48.337**

Powers: 1, 3, 5, 7, 9, 11, 13, 15, 17.

Number of terms: **48**
Number of left terms: **24**
Number of right terms: **24**

Left terms:
337, 329, 315, 287, 287, 285, 273, 245, 243, 221, 219, 189, 179, 153, 123, 119, 95, 93, 87, 85, 45, 31, 21, 1

Right terms:
335, 333, 305, 303, 291, 269, 267, 261, 239, 225, 205, 201, 173, 157, 117, 113, 107, 103, 83, 73, 47, 39, 9, 7

Pure product of:
1, 2, 3, 5, 7, 8, 9, 10, 11, 12, 13, 17, 19, 23, 29, 31, 41, 43, 53.

Solution code: **18.58.257**

Powers: 2, 4, 6, 8, 10, 12, 14, 16, 18.

Number of terms: **58**
Number of left terms: **29**
Number of right terms: **29**

Left terms:
257, 247, 243, 219, 215, 213, 211, 191, 185, 177, 155, 147, 143, 141, 127, 119, 103, 97, 77, 73, 71, 69, 69, 45, 45, 21, 13, 11, 3

Right terms:
255, 253, 231, 227, 227, 203, 197, 197, 195, 169, 159, 157, 139, 131, 123, 121, 111, 93, 87, 81, 65, 63, 57, 53, 35, 29, 23, 9, 5

Pure product of:
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 15, 17, 19, 21, 23, 25, 27, 31.

Remarks:
My original own search gave 64 terms as the best result.

This solution is taken from *The Prouhet-Tarry-Escott Problem Revisited* by Peter Borwein and Colin Ignalls (1993).

Solution code: **19.65.143**

Powers: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19.

Number of terms: **65**
Number of left terms: **33**
Number of right terms: **32**

Left terms:
143, 138, 136, 124, 122, 121, 120, 110, 103, 101, 99, 88, 87, 86, 85, 84, 83, 66, 65, 64, 53, 51, 51, 50, 49, 37, 25, 18, 16, 14, 13, 9, 4

Right terms:
142, 141, 130, 128, 128, 116, 114, 113, 109, 95, 94, 93, 93, 91, 81, 80, 76, 75, 61, 59, 58, 57, 56, 45, 43, 34, 24, 22, 20, 19, 10, 8

Pure product of:
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31.

Remarks:
My original own search gave 74 terms as the best result.

This solution is taken from *The Prouhet-Tarry-Escott Problem Revisited* by Peter Borwein and Colin Ignalls (1993).

Solution code: **20.70.173**

Powers: 2, 4, 6, 8, 10, 12, 14, 16, 18, 20.

Number of terms: **70**
Number of left terms: **35**
Number of right terms: **35**

Left terms:
173, 168, 166, 163, 156, 154, 153, 134, 127, 125, 115, 106, 96, 95, 86, 86, 84, 74, 72, 70, 67, 60, 56, 46, 45, 36, 27, 26, 25, 24, 17, 15, 13, 7, 3

Right terms:
172, 171, 162, 161, 160, 158, 148, 136, 126, 123, 119, 101, 100, 91, 89, 88, 79, 78, 75, 69, 65, 59, 53, 49, 41, 40, 30, 30, 21, 20, 18, 18, 8, 6, 2

Pure product of:
1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 37.

Remarks:
My original own search gave 88 terms as the best result.

This solution is taken from *The Prouhet-Tarry-Escott Problem Revisited* by Peter Borwein and Colin Ignalls (1993).

Pure product polynomials of a special kind.

Here are examples of polynomials of the form

$$\prod_{i=1}^{k}\prod_{p=1}^{a_i}\left(x^p - x^{-p}\right)$$

with

$$n = a_1 + \ldots + a_k$$

and relatively small (the best I could find) half of the sum of absolute values of coefficients (denoted by H).

H seems to fit quite well the estimate

$$E(n) = n^{(\log n)/e}$$

with *log* being base e logarithm.

Although they lead to multigrade equations for exponents up to $n-2$ with total of H terms, multigrade solutions obtained that way aren't very interesting, due to a huge number of terms.

n	H	$a_1,\ldots,a_k$	$H/E(n)$
100	5622	91, 8, 1	2.2993
200	58148	185, 13, 2	1.90327
300	221438	275, 21, 3, 1	1.40442
400	713774	370, 25, 4, 1	1.313
500	1685194	465, 28, 6, 1	1.13813
600	3442128	551, 40, 7, 2	0.997715
700	7730080	639, 50, 8, 3	1.07517
800	11745536	733, 55, 8, 3, 1	0.852695
900	20192374	825, 61, 10, 3, 1	0.817166
1000	35087104	919, 65, 12, 3, 1	0.834611
1100	52615906	1013, 71, 12, 3, 1	0.768469
1200	77839804	1106, 79, 10, 4, 1	0.724091
1300	117903318	1195, 85, 15, 4, 1	0.7207
1400	188230374	1291, 89, 15, 4, 1	0.776696
1500	253188340	1374, 104, 16, 4, 2	0.722005
1600	367054114	1442, 125, 24, 6, 2, 1	0.738491
1700	429572776	1538, 132, 21, 6, 2, 1	0.621076
1800	552094724	1634, 136, 21, 6, 2, 1	0.583101
1900	709247742	1738, 132, 21, 6, 2, 1	0.555347
2000	948582974	1823, 147, 21, 6, 2, 1	0.558063
3000	11686822018	2736, 215, 36, 8, 4, 1	0.670286
4000	51936520048	3654, 275, 49, 15, 4, 2, 1	0.530677
5000	198767088094	4555, 357, 63, 16, 6, 2, 1	0.510929
6000	847295849480	5444, 437, 82, 23, 8, 3, 2, 1	0.686346
7000	2212575241804	6358, 508, 93, 26, 9, 3, 2, 1	0.66237
8000	5482592103696	7230, 617, 106, 30, 10, 4, 2, 1	0.683236