A Collection of Numerical Solutions of Multigrade Equations Related to the Prouhet-Tarry-Escott Problem

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Notes

I am interested only in solutions leading to symmetric PTE solutions, hence I require a multigrade to be satisfied by all exponents of the same parity up to a certain level.

Pure product solution comes from multiplying polynomials of the form

$$x^p - x^{-p}$$

and then extracting solutions in the well known way.

For even powers there is the same number of terms on both sides (I do not discard zero terms).

For odd powers numbers of left and right terms may differ.

Solution code is composed from:
the largest power
total number of terms
the largest term
the second, third ... largest terms of any side (if needed)

The best results known to me at the moment

Max exponent	Number of terms	Ideal	Known/Ideal
8	10	10	1.000
9	12	11	1.091
10	12	12	1.000
11	20	13	1.538
12	26	14	1.857
13	26	15	1.733
14	30	16	1.875
15	34	17	2.000
16	42	18	2.333
17	48	19	2.526
18	58	20	2.900
19	65	21	3.095
20	70	22	3.182

For solutions of higher degree see **Version 5** and/or the last page of this document and/or the following paper:
Mihai Cipu, *Upper bounds for norms of products of binomials.* LMS Journal of Computation and Mathematics, 7 (2004), pp. 37-49

Solution code: **8.10.313**

Powers: 2, 4, 6, 8.

Number of terms: **10**
Number of left terms: **5**
Number of right terms: **5**
Left terms:
313, 301, 188, 100, 99
Right terms:
308, 307, 180, 131, 71

Remarks:
Discovered by Peter Borwein, Petr Lisonek and Colin Percival (2002).

Solution code: **8.10.515**

Powers: 2, 4, 6, 8.

Number of terms: **10**
Number of left terms: **5**
Number of right terms: **5**
Left terms:
515, 452, 366, 189, 103
Right terms:
508, 471, 331, 245, 18

Remarks:
Discovered by Peter Borwein, Petr Lisonek and Colin Percival (2002).
Used by Jarosław Wróblewski (November 27, 2009) to produce solution **9.12.1293**.

Solution code: **8.10.23750**

Powers: 2, 4, 6, 8.

Number of terms: **10**
Number of left terms: **5**
Number of right terms: **5**
Left terms:
23750, 20667, 20449, 11857, 436
Right terms:
23738, 20885, 20231, 11881, 12

Remarks:
The smallest member of family of solutions discovered by A. Letac in 1940's.

Solution code	Right terms	Left terms
8.12.36	36, 31, 30, 17, 7, 1	35, 34, 27, 19, 4, 3
8.12.62	62, 54, 47, 35, 27, 9	61, 57, 42, 37, 30, 1
8.12.66	66, 59, 57, 55, 28, 16	64, 62, 60, 49, 33, 11
8.12.71	71, 63, 50, 23, 18, 9	69, 67, 42, 37, 6, 5
8.12.74	74, 67, 47, 46, 27, 15	73, 69, 45, 41, 38, 2
8.12.109	109, 97, 91, 88, 54, 30	107, 99, 98, 74, 65, 24
8.12.111	111, 86, 81, 28, 25, 22	110, 94, 63, 57, 4, 1
8.12.113	113, 97, 89, 52, 31, 10	109, 107, 74, 67, 20, 13
8.12.114	114, 97, 89, 70, 43, 24	111, 106, 75, 73, 56, 2
8.12.119.115	119, 97, 94, 92, 45, 30	115, 111, 90, 74, 68, 7
8.12.119.118	119, 94, 85, 43, 38, 37	118, 101, 67, 61, 50, 7
8.12.137	137, 120, 110, 73, 57, 3	135, 127, 97, 88, 45, 18
8.12.139	139, 124, 115, 89, 42, 18	135, 133, 106, 93, 46, 4
8.12.151	151, 139, 117, 58, 55, 28	149, 143, 113, 70, 37, 36
8.12.158	158, 137, 126, 125, 76, 55	154, 148, 127, 106, 95, 45
8.12.163	163, 145, 138, 103, 56, 14	161, 152, 131, 105, 58, 2
8.12.167	167, 150, 122, 89, 71, 47	163, 158, 109, 93, 85, 34
8.12.179.172	179, 151, 150, 108, 67, 38	172, 171, 123, 122, 74, 25
8.12.179.173	179, 149, 139, 65, 63, 57	173, 167, 111, 91, 81, 5
8.12.179.178	179, 142, 132, 83, 59, 35	178, 151, 111, 103, 55, 28
8.12.195	195, 169, 148, 98, 71, 42	190, 182, 127, 111, 84, 13
8.12.211	211, 165, 155, 59, 54, 44	209, 180, 121, 111, 31, 10
8.12.212	212, 189, 167, 114, 83, 10	206, 202, 148, 133, 75, 9
8.12.222	222, 182, 164, 59, 55, 41	220, 193, 146, 97, 26, 21
8.12.229	229, 215, 166, 118, 63, 48	224, 222, 162, 113, 89, 5
8.12.237	237, 206, 195, 88, 49, 41	231, 223, 179, 104, 50, 3
8.12.243	243, 219, 178, 112, 71, 50	240, 226, 167, 126, 67, 43
8.12.245	245, 213, 202, 141, 101, 22	239, 231, 178, 158, 97, 15
8.12.265.256	265, 240, 204, 179, 73, 67	256, 255, 197, 172, 111, 5
8.12.265.257	265, 229, 223, 133, 72, 4	257, 252, 200, 149, 43, 41
8.12.265.264	265, 229, 219, 108, 88, 6	264, 236, 211, 122, 57, 45
8.12.267	267, 245, 177, 104, 79, 4	265, 249, 168, 124, 61, 13
8.12.282.277	282, 237, 223, 122, 107, 53	277, 257, 197, 138, 118, 3
8.12.282.278	282, 229, 215, 107, 54, 47	278, 250, 177, 149, 37, 9
8.12.295	295, 216, 203, 106, 91, 78	294, 232, 169, 126, 125, 13

Remarks:

Powers: 2, 4, 6, 8 with 6+6=12 terms.
Results of a selective search by Jarosław Wróblewski (December 2009).

Solution code	Right terms	Left terms
8.12.303	303, 265, 227, 119, 81, 19	291, 289, 191, 167, 45, 7
8.12.325	325, 277, 271, 201, 131, 57	317, 305, 233, 219, 139, 39
8.12.326	326, 273, 243, 227, 112, 89	317, 301, 208, 207, 186, 7
8.12.347	347, 289, 246, 112, 97, 54	343, 306, 194, 192, 43, 41
8.12.358	358, 329, 323, 281, 75, 12	357, 335, 317, 282, 76, 1
8.12.362	362, 295, 291, 163, 118, 81	353, 333, 233, 205, 134, 6
8.12.365	365, 305, 291, 219, 139, 79	355, 339, 241, 229, 181, 15
8.12.370	370, 311, 252, 201, 101, 61	369, 316, 241, 205, 123, 14
8.12.375	375, 325, 266, 184, 162, 61	371, 338, 230, 215, 171, 24
8.12.389	389, 354, 329, 281, 80, 50	379, 375, 304, 295, 94, 14
8.12.392	392, 328, 308, 241, 125, 109	385, 356, 277, 224, 197, 32
8.12.405	405, 362, 351, 205, 162, 49	393, 390, 331, 210, 167, 29
8.12.407.398.383	407, 365, 284, 242, 163, 79	398, 383, 251, 233, 220, 11
8.12.407.398.385	407, 363, 328, 265, 98, 70	398, 385, 293, 287, 120, 22
8.12.407.405	407, 386, 249, 129, 105, 2	405, 389, 243, 154, 74, 27
8.12.412	412, 358, 337, 161, 154, 27	407, 378, 314, 203, 92, 71
8.12.417	417, 347, 277, 184, 169, 64	416, 353, 248, 233, 139, 69
8.12.418	418, 368, 335, 237, 149, 66	402, 401, 302, 253, 165, 16
8.12.421	421, 357, 343, 274, 76, 62	419, 372, 323, 281, 98, 14
8.12.430	430, 351, 344, 203, 179, 82	424, 386, 283, 259, 162, 65
8.12.438	438, 389, 335, 256, 122, 97	430, 409, 302, 277, 151, 48
8.12.449	449, 397, 381, 374, 300, 28	436, 431, 363, 357, 316, 10
8.12.470	470, 396, 382, 257, 183, 61	465, 423, 349, 268, 194, 22
8.12.471	471, 407, 379, 295, 163, 105	453, 449, 335, 303, 209, 35
8.12.509	509, 429, 420, 248, 172, 111	495, 477, 364, 284, 179, 72
8.12.513	513, 410, 336, 218, 109, 77	512, 418, 315, 241, 123, 14
8.12.531	531, 471, 467, 269, 67, 7	523, 501, 441, 277, 41, 37
8.12.538	538, 472, 415, 321, 137, 114	529, 498, 361, 358, 177, 40

Remarks:

Powers: 2, 4, 6, 8 with 6+6=12 terms.

Results of a selective search by Jarosław Wróblewski (December 2009).

Solution code: **8.12.541**

Powers: 2, 4, 6, 8.

Number of terms: **12**
Number of left terms: **6**
Number of right terms: **6**
Left terms:
541, 503, 339, 176, 140, 73
Right terms:
532, 517, 305, 251, 96, 31

Remarks:
A member of family of solutions discovered by Jarosław Wróblewski (November 2009).
Left side terms:
2 a + 5 b + d
2 a + 5 b - d
5 a - 2 b + c
5 a - 2 b - c
4 a + 6 b
6 a - 4 b
Right side terms:
5 a + 2 b + c
5 a + 2 b - c
-2 a + 5 b + d
-2 a + 5 b - d
6 a + 4 b
-4 a + 6 b
Assume:

$$c^2 = p * a^2 + q * b^2$$

$$d^2 = p * b^2 + q * a^2$$

$$p = -11/5$$

$$q = 64/5$$

This solution is obtained with
a = 118
b = 89
c = 266
d = 401

Solution code	Right terms	Left terms
8.12.562	562, 487, 466, 439, 233, 225	549, 523, 470, 373, 326, 163
8.12.575	575, 497, 357, 216, 181, 76	573, 504, 323, 281, 140, 71
8.12.583	583, 501, 427, 205, 113, 109	569, 539, 347, 317, 67, 15
8.12.596	596, 446, 445, 243, 242, 111	594, 485, 354, 353, 149, 148
8.12.797	797, 703, 635, 471, 252, 136	771, 760, 567, 508, 289, 37
8.12.890	890, 732, 653, 470, 233, 213	883, 772, 555, 535, 318, 62

Remarks:

Powers: 2, 4, 6, 8 with 6+6=12 terms.
Results of a selective search by Jarosław Wróblewski (December 2009).

See the following pages for numerical analysis of **8.12.596**, **8.12.797** and **8.12.890**.

Numerical analysis of 8.12.596

Performed by Jarosław Wróblewski (December 12, 2009).

Analysis of possible linear constrains in solution **8.12.596** together with the equation for power 2 led me to the following substitutions:

Left side terms:
a + 2 b + c
2 c + 3 x
c + 3 x
b + c
b
a

Right side terms:
2 c + 4 x
2 b + c
a + b + c
a + b
c + x
x

with (a,b,c,x)=(111,242,1,148) in case of solution **8.12.596**.

While after this substitution the multigrade equation **is satisfied for power 2**, equation for **power 4** is quadratic in a and can be solved as such.

Equations for **powers 6 and 8** happen to have common quadratic factor and will be both satisfied if

$$3b^2+3bc-2c^2-8cx-8x^2=0\,.$$

This can be solved for x.

Value of x is rational iff

$$6b(b+c) \tag{8.12.596-1}$$

is a square and value of a is rational iff

$$2\left(17b^2+17bc-4c^2\right) \tag{8.12.596-2}$$

is a square.

With $b=2s^2$ and $c=3t^2-b$ the expression $(8.12.596-1)$ is a square, while the expression $(8.12.596-2)$, after dividing by 4, takes form

$$-8s^4+75s^2t^2-18t^4\,.$$

In the analized numerical solution we have $s=11$ and $t=9$.

For (s,t,a,b,c,x) = (9, 17, -738, 162, 705, -582) we get solution **8.12.347**.

For (s,t,a,b,c,x) = (123, 209, -138504, 30258, 100785, -88953) we get
55358, 46168, 43681, 21763, 10086, 7599
53767, 51414, 36082, 29651, 3944, 2487

For (s,t,a,b,c,x) = (113, 137, -91739, 25538, 30769, -38606) we get
92886, 81845, 66201, 38606, 35432, 7837
91739, 85049, 56307, 54280, 25538, 9894
For (s,t,a,b,c,x) = (659, 361, -1409216, 868562, -477599, -118049) we get
1427394, 1259525, 1018253, 595648, 540654, 118049
1409216, 1309345, 868562, 831746, 390963, 149691
For (s,t,a,b,c,x) = (907, 417, -2175549, 1645298, -1123631, -5513) we get
2269314, 2166965, 1653882, 1129144, 530251, 5513
2263801, 2175549, 1645298, 1140170, 521667, 8584
For (s,t,a,b,c,x) = (849, 1697, -5789991, 1441602, 7197825, -5760042) we get
3360767, 2879809, 1929997, 1430346, 961492, 480534
3360343, 2881506, 1920014, 1449463, 949812, 479261

The above is with the exact notations I have been discovering the family of solutions. A nicer form can be obtained by substitutions:

$$b = A - B$$

$$c = 2B$$

$$X^2 = 3(A^2 - B^2)/8 \qquad (8.12.596-3)$$

$$Y^2 = (17A^2 - 33B^2)/8 \qquad (8.12.596-4)$$

$$a = -A + Y$$

$$x = -B + X$$

Now with $(8.12.596-3)$ and $(8.12.596-4)$ satisfied, we get the following solution for **powers 2,4,6,8**:

Left side terms:
A + Y
B + 3 X
-B + 3 X
A + B
A - B
-A + Y

Right side terms:
4 X
2 A
B + Y
-B + Y
B + X
-B + X

Numerical analysis of 8.12.797

Performed by Jarosław Wróblewski (December 13, 2009).

Analysis of possible linear constrains in solution **8.12.797** led me to the following substitutions:

Left terms:
2 a
2 c
2 b
4 n
m - n
m + n

Right terms:
a + b + c
a - b + c
a + b - c
a - b - c
m - 3 n
m + 3 n

This already works for **power 2**.

To get it working for **powers 4,6** we need any 2 of the following 3 equalities satisfied (any equality follows from the remaining two):

$$a^4-2a^2b^2+b^4-2a^2c^2-2b^2c^2+c^4=(-a-b+c)(a-b+c)(-a+b+c)(a+b+c)=$$
$$=8(m-n)n^2(m+n)=8m^2n^2-8n^4$$

$$2\left(a^2+b^2+c^2\right)=m^2+11n^2$$

$$8\left(a^4+b^4+c^4\right)=m^4+54m^2n^2+89n^4$$

Then **power 8** is for free.

Numerical solutions.

(a, b, c, m, n) = (30, 31, 7, 16, 18) leads to
36, 31, 30, 17, 7, 1
35, 34, 27, 19, 4, 3

(a, b, c, m, n) = (42, 139, 89, 239, 9) leads to
139, 124, 115, 89, 42, 18
135, 133, 106, 93, 46, 4

(a, b, c, m, n) = (56, 163, 103, 283, 7) leads to
163, 145, 138, 103, 56, 14
161, 152, 131, 105, 58, 2

(a, b, c, m, n) = (98, 195, 71, 317, 21) leads to
195, 169, 148, 98, 71, 42
190, 182, 127, 111, 84, 13

(a, b, c, m, n) = (10, 209, 111, 152, 90) leads to
211, 165, 155, 59, 54, 44
209, 180, 121, 111, 31, 10

(a, b, c, m, n) = (294, 125, 13, 401, 63) leads to
295, 216, 203, 106, 91, 78
294, 232, 169, 126, 125, 13

(a, b, c, m, n) = (192, 343, 43, 235, 153) leads to
347, 289, 246, 112, 97, 54
343, 306, 194, 192, 43, 41

(a, b, c, m, n) = (358, 281, 75, 652, 6) leads to
358, 329, 323, 281, 75, 12
357, 335, 317, 282, 76, 1

(a, b, c, m, n) = (80, 389, 281, 683, 25) leads to
389, 354, 329, 281, 80, 50
379, 375, 304, 295, 94, 14

(a, b, c, m, n) = (98, 407, 265, 691, 35) leads to
407, 363, 328, 265, 98, 70
398, 385, 293, 287, 120, 22

(a, b, c, m, n) = (248, 353, 233, 70, 208) leads to
417, 347, 277, 184, 169, 64
416, 353, 248, 233, 139, 69

(a, b, c, m, n) = (418, 237, 149, 703, 33) leads to
418, 368, 335, 237, 149, 66
402, 401, 302, 253, 165, 16

(a, b, c, m, n) = (14, 419, 281, 695, 49) leads to
421, 357, 343, 274, 76, 62
419, 372, 323, 281, 98, 14

(a, b, c, m, n) = (538, 321, 137, 887, 57) leads to
538, 472, 415, 321, 137, 114
529, 498, 361, 358, 177, 40

(a, b, c, m, n) = (140, 573, 281, 394, 252) leads to
575, 497, 357, 216, 181, 76
573, 504, 323, 281, 140, 71

(a, b, c, m, n) = (436, 575, 93, 1022, 36) leads to
575, 529, 493, 436, 93, 72
565, 552, 459, 457, 116, 23

(a, b, c, m, n) = (354, 485, 353, 1, 297) leads to
596, 446, 445, 243, 242, 111
594, 485, 354, 353, 149, 148

(a, b, c, m, n) = (358, 615, 53, 964, 90) leads to
617, 513, 460, 347, 155, 102

615, 527, 437, 358, 180, 53

(a, b, c, m, n) = (82, 749, 347, 1073, 143) leads to
751, 589, 507, 322, 242, 160
749, 608, 465, 347, 286, 82

(a, b, c, m, n) = (252, 797, 471, 1338, 68) leads to
797, 703, 635, 471, 252, 136
771, 760, 567, 508, 289, 37

(a, b, c, m, n) = (664, 865, 41, 502, 440) leads to
911, 785, 744, 409, 121, 80
880, 865, 664, 471, 41, 31

(a, b, c, m, n) = (872, 769, 281, 394, 496) leads to
992, 872, 769, 445, 281, 51
961, 941, 680, 547, 192, 89

(a, b, c, m, n) = (176, 989, 705, 563, 495) leads to
1024, 935, 759, 461, 230, 54
990, 989, 705, 529, 176, 34

(a, b, c, m, n) = (426, 1097, 313, 1592, 198) leads to
1097, 895, 697, 426, 396, 313
1093, 918, 605, 499, 492, 179

(a, b, c, m, n) = (892, 1355, 259, 2281, 133) leads to
1355, 1207, 1074, 892, 266, 259
1340, 1253, 994, 941, 361, 102

(a, b, c, m, n) = (1356, 785, 401, 2243, 135) leads to
1356, 1189, 1054, 785, 401, 270
1324, 1271, 919, 870, 486, 85

(a, b, c, m, n) = (146, 1423, 645, 2017, 279) leads to
1427, 1107, 961, 590, 462, 316
1423, 1148, 869, 645, 558, 146

(a, b, c, m, n) = (794, 1493, 121, 1076, 646) leads to
1507, 1204, 1083, 431, 410, 289
1493, 1292, 861, 794, 215, 121

(a, b, c, m, n) = (14, 1599, 1249, 925, 819) leads to
1691, 1431, 1417, 766, 182, 168
1638, 1599, 1249, 872, 53, 14

(a, b, c, m, n) = (1226, 1823, 125, 1132, 874) leads to
1877, 1587, 1462, 745, 361, 236
1823, 1748, 1226, 1003, 129, 125

(a, b, c, m, n) = (1850, 1803, 291, 1008, 1066) leads to
2132, 1850, 1803, 1037, 291, 29
2103, 1972, 1681, 1095, 169, 122

(a, b, c, m, n) = (2160, 1401, 553, 3678, 176) leads to
2160, 1927, 1751, 1401, 553, 352
2103, 2057, 1575, 1504, 656, 103

(a, b, c, m, n) = (130, 2203, 917, 1864, 850) leads to

2207, 1625, 1495, 708, 578, 343
2203, 1700, 1357, 917, 507, 130

(a, b, c, m, n) = (1172, 2215, 1031, 3829, 47) leads to
2215, 1938, 1891, 1172, 1031, 94
2209, 1985, 1844, 1178, 1037, 6

(a, b, c, m, n) = (1528, 2411, 807, 4177, 117) leads to
2411, 2147, 2030, 1528, 807, 234
2373, 2264, 1913, 1566, 845, 38

Solutions added on December 14, 2009

(a, b, c ,m, n) = (2548, 1565, 941, 4423, 91) leads to
2548, 2257, 2166, 1565, 941, 182
2527, 2348, 2075, 1586, 962, 21

(a, b, c ,m, n) = (2582, 1903, 3, 1535, 1287) leads to
2698, 2244, 2241, 1163, 341, 338
2582, 2574, 1903, 1411, 124, 3

(a, b, c ,m, n) = (1558, 2729, 615, 4384, 342) leads to
2729, 2363, 2021, 1558, 684, 615
2705, 2451, 1836, 1679, 893, 278

(a, b, c ,m, n) = (2264, 3169, 631, 5530, 224) leads to
3169, 2877, 2653, 2264, 631, 448
3101, 3032, 2429, 2401, 768, 137

(a, b, c ,m, n) = (3126, 2681, 41, 1760, 1674) leads to
3391, 2924, 2883, 1631, 243, 202
3348, 3126, 2681, 1717, 43, 41

(a, b, c ,m, n) = (1772, 3745, 1443, 6046, 420) leads to
3745, 3233, 2813, 1772, 1443, 840
3653, 3480, 2393, 2037, 1708, 265

(a, b, c ,m, n) = (648, 3871, 1047, 4038, 1232) leads to
3871, 2635, 2464, 1403, 1047, 648
3867, 2783, 2135, 1736, 1088, 171

(a, b, c ,m, n) = (4004, 3491, 1137, 1906, 2244) leads to
4488, 4004, 3491, 2075, 1137, 169
4319, 4316, 3179, 2413, 825, 312

(a, b, c ,m, n) = (1474, 4535, 1741, 3283, 1925) leads to
4535, 3850, 2604, 1741, 1474, 679
4529, 3875, 2401, 2134, 1246, 660

(a, b, c ,m, n) = (2188, 4679, 1813, 7538, 532) leads to
4679, 4035, 3503, 2188, 1813, 1064
4567, 4340, 2971, 2527, 2152, 339

(a, b, c ,m, n) = (4900, 3589, 793, 8578, 364) leads to
4900, 4471, 4107, 3589, 793, 728
4835, 4641, 3848, 3743, 1052, 259

(a, b, c ,m, n) = (5212, 2253, 2145, 8334, 620) leads to
5212, 4477, 3857, 2253, 2145, 1240

5097, 4805, 3237, 2660, 2552, 407
(a, b, c ,m, n) = (5344, 1969, 93, 6781, 1311) leads to
5357, 3703, 3610, 1734, 1641, 1424
5344, 4046, 2735, 2622, 1969, 93
(a, b, c ,m, n) = (5410, 2751, 719, 8093, 915) leads to
5419, 4440, 3721, 2674, 1689, 970
5410, 4504, 3589, 2751, 1830, 719
(a, b, c ,m, n) = (1378, 5443, 3623, 9355, 403) leads to
5443, 4879, 4476, 3623, 1378, 806
5282, 5222, 4073, 3844, 1599, 221
(a, b, c ,m, n) = (514, 5649, 3301, 8875, 819) leads to
5666, 4732, 4218, 3209, 1431, 917
5649, 4847, 4028, 3301, 1638, 514
(a, b, c ,m, n) = (5692, 1783, 299, 6142, 1748) leads to
5693, 3887, 3588, 2104, 1805, 449
5692, 3945, 3496, 2197, 1783, 299
(a, b, c ,m, n) = (5384, 4285, 581, 3022, 2800) leads to
5711, 5125, 4544, 2689, 840, 259
5600, 5384, 4285, 2911, 581, 111
(a, b, c ,m, n) = (5922, 3043, 2395, 9871, 495) leads to
5922, 5183, 4688, 3043, 2395, 990
5680, 5678, 4193, 3285, 2637, 242
(a, b, c ,m, n) = (4820, 6211, 949, 11137, 355) leads to
6211, 5746, 5391, 4820, 949, 710
6101, 5990, 5041, 5036, 1170, 221
(a, b, c ,m, n) = (6980, 4161, 1651, 11438, 780) leads to
6980, 6109, 5329, 4161, 1651, 1560
6889, 6396, 4745, 4549, 2235, 584
(a, b, c ,m, n) = (6332, 4685, 3329, 2002, 3596) leads to
7192, 6332, 4685, 3329, 2799, 797
7173, 6395, 4393, 3844, 2488, 841
(a, b, c ,m, n) = (4670, 7447, 2723, 13001, 175) leads to
7447, 6588, 6413, 4670, 2723, 350
7420, 6763, 6238, 4697, 2750, 27
(a, b, c ,m, n) = (6606, 5591, 3219, 2100, 3886) leads to
7772, 6606, 5591, 3219, 2993, 893
7708, 6879, 4779, 4489, 2117, 1102
(a, b, c ,m, n) = (7826, 7039, 695, 14912, 130) leads to
7826, 7521, 7391, 7039, 695, 260
7780, 7651, 7261, 7085, 741, 46
(a, b, c ,m, n) = (7276, 6947, 2585, 3178, 4324) leads to
8648, 7276, 6947, 3751, 2585, 573
8404, 8075, 5819, 4897, 1457, 1128
(a, b, c ,m, n) = (7634, 5579, 4673, 1756, 4466) leads to

8943, 7577, 5821, 4270, 3364, 1309
8932, 7634, 5579, 4673, 3111, 1355
(a, b, c ,m, n) = (3152, 8869, 6731, 4006, 4784) leads to
9568, 8869, 6731, 4395, 3152, 389
9376, 9179, 6224, 5173, 2645, 507
(a, b, c ,m, n) = (3320, 9947, 3901, 14878, 1624) leads to
9947, 8251, 6627, 3901, 3320, 3248
9875, 8584, 5264, 5003, 4683, 1363
(a, b, c ,m, n) = (6958, 9883, 13, 6025, 4823) leads to
10247, 8427, 8414, 4222, 1469, 1456
9883, 9646, 6958, 5424, 601, 13
(a, b, c ,m, n) = (2680, 10569, 4639, 15658, 1800) leads to
10569, 8729, 6929, 4639, 3600, 2680
10529, 8944, 6264, 5129, 4305, 1625
(a, b, c ,m, n) = (1532, 11673, 5339, 9058, 4788) leads to
11711, 9272, 7740, 3933, 2653, 2401
11673, 9576, 6923, 5339, 2135, 1532
(a, b, c ,m, n) = (4448, 11857, 463, 15230, 2848) leads to
11887, 8384, 7921, 3936, 3473, 3343
11857, 9039, 6191, 5696, 4448, 463
(a, b, c ,m, n) = (6756, 9839, 7415, 917, 5985) leads to
12005, 9436, 8519, 5249, 4590, 2166
11970, 9839, 7415, 6756, 3451, 2534
(a, b, c ,m, n) = (12190, 12189, 137, 7036, 7038) leads to
14076, 12190, 12189, 7037, 137, 1
14075, 12258, 12121, 7039, 69, 68

Numerical analysis of 8.12.890

Performed by Jarosław Wróblewski (December 14, 2009).

Analysis of possible linear constrains in solution **8.12.890** led me to the following substitutions:

Left side terms:
a + 8 b + c
-a - 8 b + c
2 a - b + d
-2 a + b + d
3 a - 2 b + c
-3 a + 2 b + c

Right side terms:
a - 8 b + c
-a + 8 b + c
2 a + b + d
-2 a - b + d
3 a + 2 b + c
-3 a - 2 b + c

This solution is obtained with
a=335/2
b=151/2
c=237/2
d=945/2

Already works for **power 2**.

To get it working for **power 4** we need

$$-9a^2+81b^2+c^2-d^2=0$$

To get it working for **power 6** we need

$$5a^2-32b^2+3c^2=0$$

Then **power 8** is a free gift from nature.

Equivalently, $96b^2-15a^2$ and $825b^2-96a^2$ must be squares, equal to $9c^2$ and $9d^2$ respectively.

(a, b, c, d) = (29, 62, 199, 586) gives
362, 295, 291, 163, 118, 81
353, 333, 233, 205, 134, 6

(a, b, c, d) = (202, 83, 74, 443) gives
470, 396, 382, 257, 183, 61
465, 423, 349, 268, 194, 22

(a, b, c, d) = (335, 151, 237, 945) gives
890, 732, 653, 470, 233, 213
883, 772, 555, 535, 318, 62

Solution code: **8.12.55292**

Powers: 2, 4, 6, 8.

Number of terms: **12**
Number of left terms: **6**
Number of right terms: **6**
Left terms:
55292, 50841, 50712, 41050, 23681, 11369
Right terms:
54280, 54151, 45759, 44489, 20902, 14148

Remarks:
A member of family of solutions discovered by Jarosław Wróblewski (November 2009).
Left side terms:
a + 10 b + d
a + 10 b - d
10 a - b + c
10 a - b - c
a + 11 b
11 a - b
Right side terms:
10 a + b + c
10 a + b - c
-a + 10 b + d
-a + 10 b - d
11 a + b
-a + 11 b
Assume:

$$c^2 = p*a^2 + q*b^2$$

$$d^2 = p*b^2 + q*a^2$$

$$p = -27/5$$

$$q = 248/5$$

This solution is obtained with
a = 9533
b = 3439
c = 9791
d = 66661

Solution code: **9.12.323**

Powers: 1, 3, 5, 7, 9.

Number of terms: **12**
Number of left terms: **6**
Number of right terms: **6**
Left terms:
323, 289, 269, 173, 91, 7
Right terms:
313, 311, 247, 193, 59, 29

Remarks:
Discovered by Chen Shuwen (2000).

Solution code: **9.12.407**

Powers: 1, 3, 5, 7, 9.

Number of terms: **12**
Number of left terms: **6**
Number of right terms: **6**
Left terms:
407, 347, 341, 181, 163, 23
Right terms:
403, 371, 311, 221, 119, 37

Remarks:
Discovered by Jarosław Wróblewski (November 28, 2009).

Solution code: **9.12.463**

Powers: 1, 3, 5, 7, 9.

Number of terms: **12**
Number of left terms: **6**
Number of right terms: **6**
Left terms:
463, 391, 335, 217, 161, 43
Right terms:
461, 403, 287, 283, 91, 85

Remarks:
Discovered by Jarosław Wróblewski (November 28, 2009).

Solution code: **9.12.1293**

Powers: 1, 3, 5, 7, 9.

Number of terms: **12**
Number of left terms: **6**
Number of right terms: **6**
Left terms:
1293, 1167, 995, 679, 399, 57
Right terms:
1279, 1205, 925, 767, 299, 115

Remarks:
Constructed by Jarosław Wróblewski (November 27, 2009) from solution **8.10.515**.

Solution code: **10.12.151**

Powers: 2, 4, 6, 8, 10.

Number of terms: **12**
Number of left terms: **6**
Number of right terms: **6**
Left terms:
151, 140, 127, 86, 61, 22
Right terms:
148, 146, 121, 94, 47, 35

Remarks:
Discovered by Nuutti Kuosa (1999) using a computer program written by Jean-Charles Meyrignac, as a single-grade solution to power 10. Four days later Chen Shuwen noticed it was in fact a multigrade.

Solution code: **10.12.1511**

Powers: 2, 4, 6, 8, 10.

Number of terms: **12**
Number of left terms: **6**
Number of right terms: **6**
Left terms:
1511, 1138, 1075, 700, 622, 107
Right terms:
1510, 1180, 953, 886, 413, 293

Remarks:
The smallest solution of the infinite family of solutions constructed in:

Ajai Choudhry, Jarosław Wróblewski, *Ideal Solutions of the Tarry-Escott Problem of degree eleven with applications to Sums of Thirteenth Powers*, Hardy-Ramanujan Journal, Vol. 31 (2008) pp. 1-13

The above paper is available at:
`http://www.nias.res.in/hrj/contentsvol31.htm`

Solution code: **10.12.2058**

Powers: 2, 4, 6, 8, 10.

Number of terms: **12**
Number of left terms: **6**
Number of right terms: **6**
Left terms:
2058, 1896, 1618, 1109, 891, 257
Right terms:
2037, 1947, 1514, 1294, 639, 472

Remarks:
Discovered by David Broadhurst (2007):
D. Broadhurst, *A Chinese Prouhet-Tarry-Escott solution*,
`http://physics.open.ac.uk/~dbroadhu/cpte.pdf`

The second known solution.
It was later used by Ajai Choudhry and Jarosław Wróblewski to produce an infinite family of solutions:

Ajai Choudhry, Jarosław Wróblewski, *Ideal Solutions of the Tarry-Escott Problem of degree eleven with applications to Sums of Thirteenth Powers*, Hardy-Ramanujan Journal, Vol. 31 (2008) pp. 1-13

The above paper is available at:
`http://www.nias.res.in/hrj/contentsvol31.htm`

Solution code: **10.12.14770**

Powers: 2, 4, 6, 8, 10.

Number of terms: **12**
Number of left terms: **6**
Number of right terms: **6**
Left terms:
14770, 12638, 11632, 7115, 7043, 929
Right terms:
14693, 13165, 10112, 9718, 4054, 3455

Remarks:
3rd smallest solution of the infinite family of solutions constructed in:

Ajai Choudhry, Jarosław Wróblewski, *Ideal Solutions of the Tarry-Escott Problem of degree eleven with applications to Sums of Thirteenth Powers*, Hardy-Ramanujan Journal, Vol. 31 (2008) pp. 1-13

The above paper is available at:
`http://www.nias.res.in/hrj/contentsvol31.htm`

Solution code: **10.12.23742**

Powers: 2, 4, 6, 8, 10.

Number of terms: **12**
Number of left terms: **6**
Number of right terms: **6**
Left terms:
23742, 18687, 18372, 12734, 9611, 349
Right terms:
23708, 19653, 16426, 14714, 7713, 3309

Remarks:
4th smallest solution of the infinite family of solutions constructed in:

Ajai Choudhry, Jarosław Wróblewski, *Ideal Solutions of the Tarry-Escott Problem of degree eleven with applications to Sums of Thirteenth Powers*, Hardy-Ramanujan Journal, Vol. 31 (2008) pp. 1-13

The above paper is available at:
`http://www.nias.res.in/hrj/contentsvol31.htm`

Solution code: **10.14.68**

Powers: 2, 4, 6, 8, 10.

Number of terms: **14**
Number of left terms: **7**
Number of right terms: **7**
Left terms:
68, 61, 55, 32, 31, 28, 1
Right terms:
67, 64, 49, 44, 23, 20, 17

Remarks:
Found by direct search (Jarosław Wróblewski, November 2009) - see solution **10.14.400** for details of a family it belong to.

Solution code: **10.14.400**

Powers: 2, 4, 6, 8, 10.

Number of terms: **14**
Number of left terms: **7**
Number of right terms: **7**
Left terms:
400, 365, 359, 254, 242, 89, 35
Right terms:
395, 383, 341, 271, 230, 70, 64

Remarks:
Constructed by Tito Piezas and Jarosław Wróblewski (November 2009):
Left terms:
-15 a - 4 e
15 a - 4 e
8 e
-3 a - 6 y
3 a - 6 y
-12 e + 6 z
12 e + 6 z
Right terms:
-16 e
-12 a + 8 e
12 a + 8 e
-9 a - 6 y
9 a - 6 y
-3 a + 6 z
3 a + 6 z
Constrains:

$$y^2 = a^2 + 52e^2/9$$

$$z^2 = 4a^2 + 4e^2/9$$

The above is as it has appeared in the original construction - it can be simplified by dividing corresponding coefficients by a common factor.

Solutions:
(a,e,y,z) = (3,12,29,10) leads to solution **10.14.68**
(a,e,y,z) = (612,105,662,1226) leads to this solution
(a,e,y,z) = (783,168,881,1570) leads to **10.14.4139**
(a,e,y,z) = (989,1980,4861,2378) leads to **10.14.12676**
(a,e,y,z) = (4876,2079,6982,9850) leads to
3502, 3394, 2701, 2355, 1423, 1136, 693
3574, 3131, 3072, 1853, 1745, 1386, 83

(a,e,y,z) = (13231,10440,28369,27362) leads to
96484, 80075, 69969, 52235, 43507, 27840, 12964
96431, 80764, 67955, 55680, 41493, 25084, 17045

(a,e,y,z) = (2941,44460,106909,30218) leads to
238276, 216759, 210877, 118560, 117404, 73985, 44575
237120, 222641, 204995, 130324, 106796, 63377, 57495

Solution code: **10.14.4139**

Powers: 2, 4, 6, 8, 10.

Number of terms: **14**
Number of left terms: **7**
Number of right terms: **7**
Left terms:
4139, 3812, 3691, 2545, 2468, 979, 448
Right terms:
4111, 3923, 3580, 2684, 2357, 896, 587

Remarks:
Constructed by Tito Piezas and Jarosław Wróblewski (November 2009) - see solution **10.14.400** for details.

Solution code: **10.14.12689**

Powers: 2, 4, 6, 8, 10.

Number of terms: **14**
Number of left terms: **7**
Number of right terms: **7**
Left terms:
12689, 10560, 9236, 6755, 5745, 3767, 1324
Right terms:
12676, 10711, 8733, 7585, 5280, 3164, 2305

Remarks:
Constructed by Tito Piezas and Jarosław Wróblewski (November 2009) - see solution **10.14.400** for details.

Solution code: **10.16.93**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**
Number of left terms: **8**
Number of right terms: **8**
Left terms:
93, 87, 72, 52, 44, 41, 29, 1
Right terms:
92, 89, 67, 61, 39, 36, 33, 8

Solution code: **10.16.113**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**
Number of left terms: **8**
Number of right terms: **8**
Left terms:
113, 103, 91, 64, 59, 34, 10, 2
Right terms:
112, 106, 85, 74, 53, 26, 23, 1

Solution code: **10.16.132**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**
Number of left terms: **8**
Number of right terms: **8**

Left terms:

132, 117, 103, 71, 61, 59, 56, 12

Right terms:

131, 121, 92, 84, 72, 43, 39, 37

Solution code: **10.16.155**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**
Number of left terms: **8**
Number of right terms: **8**
Left terms:
155, 135, 133, 97, 93, 93, 55, 17
Right terms:
153, 145, 115, 107, 105, 83, 43, 33

Solution code: **10.16.172**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**
Number of left terms: **8**
Number of right terms: **8**
Left terms:
172, 159, 149, 135, 106, 78, 43, 14
Right terms:
169, 166, 140, 138, 111, 74, 37, 27

Solution code: **10.16.173**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**
Number of left terms: **8**
Number of right terms: **8**
Left terms:
173, 156, 143, 104, 92, 91, 39, 13
Right terms:
168, 167, 123, 116, 113, 61, 44, 29

Solution code: **10.16.188**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**
Number of left terms: **8**
Number of right terms: **8**
Left terms:
188, 168, 166, 125, 114, 83, 39, 25
Right terms:
183, 182, 151, 131, 120, 62, 60, 19

Solution code: **10.16.193**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**
Number of left terms: **8**
Number of right terms: **8**
Left terms:
193, 179, 164, 100, 75, 72, 51, 31
Right terms:
191, 184, 159, 109, 68, 60, 53, 45

Remarks:
By a proper sign changes we can make the solution work for powers 1 and 3:
Left terms:
193, -179, 164, -100, 75, 72, 51, -31
Right terms:
191, -184, 159, 109, -68, -60, 53, 45

Solution code: **10.16.275**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**
Number of left terms: **8**
Number of right terms: **8**
Left terms:
275, 239, 211, 158, 100, 98, 2, 1
Right terms:
274, 245, 188, 185, 89, 86, 37, 22

Remarks:
Derived from solution **9.12.463**.

Solution code: **10.16.2567**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**
Number of left terms: **8**
Number of right terms: **8**
Left terms:
2567, 2339, 2283, 1544, 1426, 710, 479, 237
Right terms:
2536, 2449, 2173, 1654, 1347, 631, 510, 347

Remarks:
Constructed by Tito Piezas and Jarosław Wróblewski (November 2009).

Solution code: **11.20.107**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**
Number of left terms: **10**
Number of right terms: **10**
Left terms:
107, 101, 86, 78, 66, 55, 43, 25, 19, 13
Right terms:
106, 103, 81, 79, 73, 50, 38, 30, 27, 6

Remarks:
Derived from solution **10.16.93**.

Solution code: **11.20.139**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**
Number of left terms: **11**
Number of right terms: **9**
Left terms:
139, 125, 125, 113, 95, 85, 67, 65, 31, 5, 1
Right terms:
137, 133, 119, 107, 101, 91, 61, 53, 49

Remarks:
Derived from solution **10.14.68**.

Solution code: **11.20.178**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**
Number of left terms: **11**
Number of right terms: **9**
Left terms:
178, 167, 154, 119, 94, 88, 67, 49, 20, 8, 5
Right terms:
175, 173, 148, 124, 100, 74, 62, 59, 34

Remarks:
Derived from solution **10.12.151**.

Solution code: **11.20.199**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**
Number of left terms: **11**
Number of right terms: **9**

Left terms:

199, 182, 169, 141, 118, 97, 90, 39, 35, 18, 3

Right terms:

194, 193, 149, 147, 139, 78, 70, 66, 55

Remarks:

Derived from solution **10.16.173**.

Solution code: **11.20.327**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**
Number of left terms: **10**
Number of right terms: **10**

Left terms:

327, 305, 279, 271, 217, 197, 163, 69, 69, 45

Right terms:

321, 317, 277, 255, 229, 213, 119, 97, 95, 19

Remarks:

Derived from solution **10.12.151**.

Solution code: **11.20.329**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**
Number of left terms: **10**
Number of right terms: **10**
Left terms:
329, 307, 281, 265, 199, 161, 149, 71, 67, 43
Right terms:
323, 319, 275, 253, 227, 145, 121, 97, 95, 17

Remarks:
Derived from solution **10.12.151**.

Solution code: **11.20.431**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**
Number of left terms: **10**
Number of right terms: **10**
Left terms:
431, 409, 383, 301, 251, 167, 163, 113, 85, 7
Right terms:
425, 421, 371, 317, 223, 199, 151, 125, 43, 35

Remarks:
Derived from solution **10.12.151**.

Solution code: **11.20.569**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**
Number of left terms: **10**
Number of right terms: **10**
Left terms:
569, 547, 521, 439, 389, 311, 223, 145, 95, 29
Right terms:
563, 559, 509, 455, 361, 337, 197, 173, 79, 35

Remarks:
Derived from solution **10.12.151**.

Solution code: **11.20.3615**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**
Number of left terms: **10**
Number of right terms: **10**

Left terms:

3615, 2869, 2743, 2427, 1993, 1837, 1767, 1313, 379, 233

Right terms:

3613, 2953, 2499, 2429, 2365, 1683, 1557, 1419, 651, 7

Remarks:

Derived from solution **10.12.1511**.

Solution code: **11.20.5155**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**
Number of left terms: **9**
Number of right terms: **11**
Left terms:
5155, 4409, 4283, 3533, 3377, 2347, 1919, 887, 733
Right terms:
5153, 4493, 4039, 3905, 2959, 2719, 1547, 1307, 361, 143, 17

Remarks:
Derived from solution **10.12.1511**.

Solution code: **11.20.6269**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**
Number of left terms: **10**
Number of right terms: **10**
Left terms:
6269, 5945, 5389, 4371, 3935, 2667, 1921, 1741, 435, 371
Right terms:
6227, 6047, 5181, 4741, 3431, 3097, 1963, 1209, 1083, 65

Remarks:
Derived from solution **10.12.2058**.

Solution code: **11.20.6625**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**
Number of left terms: **11**
Number of right terms: **9**

Left terms:

6625, 6301, 5745, 4727, 4291, 3023, 1995, 1385, 519, 291, 79

Right terms:

6583, 6403, 5537, 5097, 3787, 3453, 1607, 1283, 1231

Remarks:

Derived from solution **10.12.2058**.

Solution code: **11.20.43107**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**
Number of left terms: **9**
Number of right terms: **11**
Left terms:
43107, 38843, 36831, 27797, 27653, 15819, 15425, 12763, 5869
Right terms:
42953, 39897, 33791, 33003, 21675, 20477, 15973, 9697, 5459, 663, 519

Remarks:
Derived from solution **10.12.14770**.

Solution code: **11.20.48287**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**
Number of left terms: **11**
Number of right terms: **9**
Left terms:
48287, 44023, 42011, 32977, 32833, 20605, 16889, 7583, 4661, 1477, 689
Right terms:
48133, 45077, 38971, 38183, 26855, 25657, 11837, 10793, 6529

Remarks:
Derived from solution **10.12.14770**.

Solution code: **11.20.65507**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**
Number of left terms: **9**
Number of right terms: **11**

Left terms:
65507, 55397, 54767, 43491, 37245, 29393, 21283, 17325, 14829

Right terms:
65439, 57329, 50875, 47451, 33449, 29461, 24641, 19351, 7445, 2597, 1199

Remarks:
Derived from solution **10.12.23742**.

Solution code: **11.20.78905**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**
Number of left terms: **10**
Number of right terms: **10**
Left terms:
78905, 68795, 68165, 56889, 50643, 32119, 30723, 12199, 7885, 1431
Right terms:
78837, 70727, 64273, 60849, 46847, 38039, 24803, 16063, 5323, 1993

Remarks:
Derived from solution **10.12.23742**.

Solution code: **11.22.65**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **22**
Number of left terms: **11**
Number of right terms: **11**
Left terms:
65, 60, 59, 47, 45, 40, 30, 18, 16, 10, 4
Right terms:
64, 63, 56, 49, 43, 42, 27, 21, 14, 13, 2

Pure product of:
1, 2, 3, 4, 5, 7, 9, 11, 13, 16, 17, 19, 23.

Solution code: **12.26.79**

Powers: 2, 4, 6, 8, 10, 12.

Number of terms: **26**
Number of left terms: **13**
Number of right terms: **13**
Left terms:
79, 76, 72, 69, 58, 53, 48, 44, 26, 21, 17, 14, 11
Right terms:
78, 77, 74, 64, 61, 54, 51, 33, 31, 28, 19, 8, 4

Pure product of:
1, 2, 5, 6, 7, 8, 9, 11, 13, 15, 17, 19, 20, 25.

Remarks:
This solution is taken from the following paper:
Mihai Cipu, *Upper bounds for norms of products of binomials*. LMS Journal of Computation and Mathematics, 7 (2004), pp. 37-49

Solution code: **12.28.81**

Powers: 2, 4, 6, 8, 10, 12.

Number of terms: **28**
Number of left terms: **14**
Number of right terms: **14**
Left terms:
81, 74, 73, 72, 59, 52, 50, 48, 44, 30, 23, 15, 14, 1
Right terms:
80, 78, 71, 69, 62, 54, 49, 47, 40, 33, 25, 18, 4, 4

Pure product of:
1, 3, 4, 5, 6, 7, 10, 11, 13, 16, 17, 19, 23, 27.

Solution code: **12.28.82**

Powers: 2, 4, 6, 8, 10, 12.

Number of terms: **28**
Number of left terms: **14**
Number of right terms: **14**
Left terms:
82, 75, 74, 73, 60, 51, 49, 47, 33, 24, 21, 15, 12, 6
Right terms:
81, 79, 72, 70, 63, 54, 45, 44, 36, 27, 18, 17, 9, 5

Pure product of:
1, 3, 4, 5, 6, 7, 10, 11, 13, 16, 17, 19, 23, 29.

Solution code: **12.28.119**

Powers: 2, 4, 6, 8, 10, 12.

Number of terms: **28**
Number of left terms: **14**
Number of right terms: **14**
Left terms:
119, 109, 105, 103, 81, 81, 67, 65, 57, 29, 27, 19, 19, 11
Right terms:
117, 115, 101, 99, 89, 79, 73, 51, 49, 45, 41, 7, 3, 1

Solution code: **12.28.169**

Powers: 2, 4, 6, 8, 10, 12.

Number of terms: **28**
Number of left terms: **14**
Number of right terms: **14**
Left terms:
169, 151, 149, 145, 105, 101, 99, 95, 71, 63, 51, 47, 43, 23
Right terms:
167, 161, 139, 135, 133, 89, 85, 83, 81, 79, 61, 39, 33, 25

Pure product of:
1, 4, 5, 6, 7, 8, 9, 11, 13, 15, 17, 19, 23, 31.

Solution code: **13.26.173**

Powers: 1, 3, 5, 7, 9, 11, 13.

Number of terms: **26**
Number of left terms: **13**
Number of right terms: **13**
Left terms:
173, 159, 157, 131, 129, 107, 103, 79, 75, 51, 25, 9, 1
Right terms:
171, 167, 141, 139, 137, 97, 93, 89, 85, 43, 19, 15, 3

Pure product of:
1, 3, 4, 5, 6, 7, 9, 10, 11, 13, 16, 17, 19, 23, 29.

Remarks:
Published by L.J. Lander (1973), *Mathematics of Computation 27 (122), 1973, p. 397*

Solution code: **13.28.191**

Powers: 1, 3, 5, 7, 9, 11, 13.

Number of terms: **28**
Number of left terms: **14**
Number of right terms: **14**
Left terms:
191, 177, 175, 173, 147, 131, 129, 125, 113, 69, 57, 51, 27, 7
Right terms:
189, 185, 171, 167, 153, 137, 123, 119, 115, 71, 67, 37, 21, 17

Pure product of:
1, 3, 4, 5, 6, 7, 10, 11, 13, 16, 17, 19, 23, 27, 29.

Solution code: **13.30.69**

Powers: 1, 3, 5, 7, 9, 11, 13.

Number of terms: **30**
Number of left terms: **14**
Number of right terms: **16**
Left terms:
69, 64, 62, 61, 50, 48, 41, 40, 38, 27, 19, 16, 15, 15
Right terms:
68, 67, 60, 59, 54, 46, 45, 34, 32, 31, 31, 10, 9, 8, 6, 5

Pure product of:
1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 15, 16, 17, 19.

Solution code: **13.30.129**

Powers: 1, 3, 5, 7, 9, 11, 13.

Number of terms: **30**
Number of left terms: **13**
Number of right terms: **17**
Left terms:
129, 119, 115, 89, 87, 85, 83, 47, 43, 41, 37, 25, 23
Right terms:
127, 125, 103, 101, 95, 81, 63, 59, 51, 49, 19, 17, 9, 9, 7, 5, 3

Pure product of:
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 17, 19.

Solution code: **14.30.153**

Powers: 2, 4, 6, 8, 10, 12, 14.

Number of terms: **30**
Number of left terms: **15**
Number of right terms: **15**
Left terms:
153, 143, 139, 115, 111, 111, 103, 89, 61, 59, 57, 53, 47, 5, 3
Right terms:
151, 149, 127, 123, 123, 99, 95, 93, 73, 65, 51, 43, 41, 23, 1

Pure product of:
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 15, 17, 19, 23.

Solution code: **14.32.187**

Powers: 2, 4, 6, 8, 10, 12, 14.

Number of terms: **32**
Number of left terms: **16**
Number of right terms: **16**
Left terms:
187, 173, 171, 157, 127, 125, 123, 121, 83, 79, 75, 71, 39, 23, 15, 1
Right terms:
185, 181, 159, 155, 151, 115, 111, 107, 103, 99, 61, 57, 37, 33, 17, 13

Pure product of:
1, 3, 4, 5, 6, 7, 9, 10, 11, 13, 14, 16, 17, 19, 23, 29.

Solution code: **15.34.189**

Powers: 1, 3, 5, 7, 9, 11, 13, 15.

Number of terms: **34**
Number of left terms: **18**
Number of right terms: **16**
Left terms:
189, 179, 175, 173, 159, 149, 129, 127, 97, 85, 55, 41, 31, 25, 17, 11, 3, 1
Right terms:
187, 185, 171, 169, 161, 155, 121, 119, 117, 69, 63, 39, 37, 21, 19, 13

Pure product of:
1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 14, 16, 17, 19, 23, 29.

Solution code: **16.42.95**

Powers: 2, 4, 6, 8, 10, 12, 14, 16.

Number of terms: **42**
Number of left terms: **21**
Number of right terms: **21**
Left terms:
95, 90, 88, 77, 75, 71, 70, 59, 49, 46, 45, 37, 34, 28, 17, 16, 16, 13, 13, 12, 9
Right terms:
94, 93, 82, 81, 79, 68, 67, 56, 55, 53, 35, 32, 31, 29, 27, 26, 20, 7, 6, 5, 0

Pure product of:
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 16, 17, 19, 22, 23.

Solution code: **17.48.337**

Powers: 1, 3, 5, 7, 9, 11, 13, 15, 17.

Number of terms: **48**
Number of left terms: **24**
Number of right terms: **24**
Left terms:
337, 329, 315, 287, 287, 285, 273, 245, 243, 221, 219, 189, 179, 153, 123, 119, 95, 93, 87, 85, 45, 31, 21, 1
Right terms:
335, 333, 305, 303, 291, 269, 267, 261, 239, 225, 205, 201, 173, 157, 117, 113, 107, 103, 83, 73, 47, 39, 9, 7

Pure product of:
1, 2, 3, 5, 7, 8, 9, 10, 11, 12, 13, 17, 19, 23, 29, 31, 41, 43, 53.

Solution code: **18.58.257**

Powers: 2, 4, 6, 8, 10, 12, 14, 16, 18.

Number of terms: **58**
Number of left terms: **29**
Number of right terms: **29**

Left terms:

257, 247, 243, 219, 215, 213, 211, 191, 185, 177, 155, 147, 143, 141, 127, 119, 103, 97, 77, 73, 71, 69, 69, 45, 45, 21, 13, 11, 3

Right terms:

255, 253, 231, 227, 227, 203, 197, 197, 195, 169, 159, 157, 139, 131, 123, 121, 111, 93, 87, 81, 65, 63, 57, 53, 35, 29, 23, 9, 5

Pure product of:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 15, 17, 19, 21, 23, 25, 27, 31.

Remarks:

My original own search gave 64 terms as the best result.

This solution is taken from *The Prouhet-Tarry-Escott Problem Revisited* by Peter Borwein and Colin Ignalls (1993).

Solution code: **19.65.143**

Powers: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19.

Number of terms: **65**
Number of left terms: **33**
Number of right terms: **32**

Left terms:

143, 138, 136, 124, 122, 121, 120, 110, 103, 101, 99, 88, 87, 86, 85, 84, 83, 66, 65, 64, 53, 51, 51, 50, 49, 37, 25, 18, 16, 14, 13, 9, 4

Right terms:

142, 141, 130, 128, 128, 116, 114, 113, 109, 95, 94, 93, 93, 91, 81, 80, 76, 75, 61, 59, 58, 57, 56, 45, 43, 34, 24, 22, 20, 19, 10, 8

Pure product of:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31.

Remarks:

My original own search gave 74 terms as the best result.

This solution is taken from *The Prouhet-Tarry-Escott Problem Revisited* by Peter Borwein and Colin Ignalls (1993).

Solution code: **20.70.173**

Powers: 2, 4, 6, 8, 10, 12, 14, 16, 18, 20.

Number of terms: **70**
Number of left terms: **35**
Number of right terms: **35**

Left terms:

173, 168, 166, 163, 156, 154, 153, 134, 127, 125, 115, 106, 96, 95, 86, 86, 84, 74, 72, 70, 67, 60, 56, 46, 45, 36, 27, 26, 25, 24, 17, 15, 13, 7, 3

Right terms:

172, 171, 162, 161, 160, 158, 148, 136, 126, 123, 119, 101, 100, 91, 89, 88, 79, 78, 75, 69, 65, 59, 53, 49, 41, 40, 30, 30, 21, 20, 18, 18, 8, 6, 2

Pure product of:

1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 37.

Remarks:

My original own search gave 88 terms as the best result.

This solution is taken from *The Prouhet-Tarry-Escott Problem Revisited* by Peter Borwein and Colin Ignalls (1993).

Pure product polynomials of a special kind.

Here are examples of polynomials of the form

$$\prod_{i=1}^{k}\prod_{p=1}^{a_i}\left(x^p - x^{-p}\right)$$

with

$$n = a_1 + \ldots + a_k$$

and relatively small (the best I could find) half of the sum of absolute values of coefficients (denoted by H).

H seems to fit quite well the estimate

$$E(n) = n^{(\log n)/e}$$

with *log* being base e logarithm.

Although they lead to multigrade equations for exponents up to $n-2$ with total of H terms, multigrade solutions obtained that way aren't very interesting, due to a huge number of terms.

n	H	$a_1,\ldots,a_k$	$H/E(n)$
100	5622	91, 8, 1	2.2993
200	58148	185, 13, 2	1.90327
300	221438	275, 21, 3, 1	1.40442
400	713774	370, 25, 4, 1	1.313
500	1685194	465, 28, 6, 1	1.13813
600	3442128	551, 40, 7, 2	0.997715
700	7730080	639, 50, 8, 3	1.07517
800	11745536	733, 55, 8, 3, 1	0.852695
900	20192374	825, 61, 10, 3, 1	0.817166
1000	35087104	919, 65, 12, 3, 1	0.834611
1100	52615906	1013, 71, 12, 3, 1	0.768469
1200	77839804	1106, 79, 10, 4, 1	0.724091
1300	117903318	1195, 85, 15, 4, 1	0.7207
1400	188230374	1291, 89, 15, 4, 1	0.776696
1500	253188340	1374, 104, 16, 4, 2	0.722005
1600	367054114	1442, 125, 24, 6, 2, 1	0.738491
1700	429572776	1538, 132, 21, 6, 2, 1	0.621076
1800	552094724	1634, 136, 21, 6, 2, 1	0.583101
1900	709247742	1738, 132, 21, 6, 2, 1	0.555347
2000	948582974	1823, 147, 21, 6, 2, 1	0.558063
3000	11686822018	2736, 215, 36, 8, 4, 1	0.670286
4000	51936520048	3654, 275, 49, 15, 4, 2, 1	0.530677
5000	198767088094	4555, 357, 63, 16, 6, 2, 1	0.510929
6000	847295849480	5444, 437, 82, 23, 8, 3, 2, 1	0.686346
7000	2212575241804	6358, 508, 93, 26, 9, 3, 2, 1	0.66237
8000	5482592103696	7230, 617, 106, 30, 10, 4, 2, 1	0.683236