

**A Collection of
Numerical Solutions of
Multigrade Equations
Related to the
Prouhet-Tarry-Escott
Problem**

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Notes

I am interested only in solutions leading to symmetric PTE solutions, hence I require a multigrade to be satisfied by all exponents of the same parity up to a certain level.

Pure product solution comes from multiplying polynomials of the form

$$x^p - x^{-p}$$

and then extracting solutions in the well known way.

For even powers there is the same number of terms on both sides (I do not discard zero terms).

For odd powers numbers of left and right terms may differ.

Solution code is composed from:

the largest power

total number of terms

the largest term

the second, third ... largest terms of any side (if needed)

The best results known to me at the moment

Max exponent	Number of terms	Ideal	Known/Ideal
8	10	10	1.000
9	12	11	1.091
10	12	12	1.000
11	20	13	1.538
12	26	14	1.857
13	26	15	1.733
14	30	16	1.875
15	34	17	2.000
16	42	18	2.333
17	48	19	2.526
18	58	20	2.900
19	65	21	3.095
20	70	22	3.182

For solutions of higher degree see **Version 5** and/or the last page of this document and/or the following paper:

Mihai Cipu, *Upper bounds for norms of products of binomials*. LMS Journal of Computation and Mathematics, 7 (2004), pp. 37-49

Solution code: **8.10.313**

Powers: 2, 4, 6, 8.

Number of terms: **10**

Number of left terms: **5**

Number of right terms: **5**

Left terms:

313, 301, 188, 100, 99

Right terms:

308, 307, 180, 131, 71

Remarks:

Discovered by Peter Borwein, Petr Lisonek and Colin Percival (2002).

Solution code: **8.10.515**

Powers: 2, 4, 6, 8.

Number of terms: **10**

Number of left terms: **5**

Number of right terms: **5**

Left terms:

515, 452, 366, 189, 103

Right terms:

508, 471, 331, 245, 18

Remarks:

Discovered by Peter Borwein, Petr Lisonek and Colin Percival (2002).

Used by Jarosław Wróblewski (November 27, 2009) to produce solution **9.12.1293**.

Solution code: **8.10.23750**

Powers: 2, 4, 6, 8.

Number of terms: **10**

Number of left terms: **5**

Number of right terms: **5**

Left terms:

23750, 20667, 20449, 11857, 436

Right terms:

23738, 20885, 20231, 11881, 12

Remarks:

The smallest member of family of solutions discovered by A. Letac in 1940's.

Solution code	Right terms	Left terms
8.12.36	36, 31, 30, 17, 7, 1	35, 34, 27, 19, 4, 3
8.12.62	62, 54, 47, 35, 27, 9	61, 57, 42, 37, 30, 1
8.12.66	66, 59, 57, 55, 28, 16	64, 62, 60, 49, 33, 11
8.12.71	71, 63, 50, 23, 18, 9	69, 67, 42, 37, 6, 5
8.12.74	74, 67, 47, 46, 27, 15	73, 69, 45, 41, 38, 2
8.12.109	109, 97, 91, 88, 54, 30	107, 99, 98, 74, 65, 24
8.12.111	111, 86, 81, 28, 25, 22	110, 94, 63, 57, 4, 1
8.12.113	113, 97, 89, 52, 31, 10	109, 107, 74, 67, 20, 13
8.12.114	114, 97, 89, 70, 43, 24	111, 106, 75, 73, 56, 2
8.12.119.115	119, 97, 94, 92, 45, 30	115, 111, 90, 74, 68, 7
8.12.119.118	119, 94, 85, 43, 38, 37	118, 101, 67, 61, 50, 7
8.12.137	137, 120, 110, 73, 57, 3	135, 127, 97, 88, 45, 18
8.12.139	139, 124, 115, 89, 42, 18	135, 133, 106, 93, 46, 4
8.12.151	151, 139, 117, 58, 55, 28	149, 143, 113, 70, 37, 36
8.12.158	158, 137, 126, 125, 76, 55	154, 148, 127, 106, 95, 45
8.12.163	163, 145, 138, 103, 56, 14	161, 152, 131, 105, 58, 2
8.12.167	167, 150, 122, 89, 71, 47	163, 158, 109, 93, 85, 34
8.12.179.172	179, 151, 150, 108, 67, 38	172, 171, 123, 122, 74, 25
8.12.179.173	179, 149, 139, 65, 63, 57	173, 167, 111, 91, 81, 5
8.12.179.178	179, 142, 132, 83, 59, 35	178, 151, 111, 103, 55, 28
8.12.195	195, 169, 148, 98, 71, 42	190, 182, 127, 111, 84, 13
8.12.211	211, 165, 155, 59, 54, 44	209, 180, 121, 111, 31, 10
8.12.212	212, 189, 167, 114, 83, 10	206, 202, 148, 133, 75, 9
8.12.222	222, 182, 164, 59, 55, 41	220, 193, 146, 97, 26, 21
8.12.229	229, 215, 166, 118, 63, 48	224, 222, 162, 113, 89, 5
8.12.237	237, 206, 195, 88, 49, 41	231, 223, 179, 104, 50, 3
8.12.243	243, 219, 178, 112, 71, 50	240, 226, 167, 126, 67, 43
8.12.245	245, 213, 202, 141, 101, 22	239, 231, 178, 158, 97, 15
8.12.265.256	265, 240, 204, 179, 73, 67	256, 255, 197, 172, 111, 5
8.12.265.257	265, 229, 223, 133, 72, 4	257, 252, 200, 149, 43, 41
8.12.265.264	265, 229, 219, 108, 88, 6	264, 236, 211, 122, 57, 45
8.12.267	267, 245, 177, 104, 79, 4	265, 249, 168, 124, 61, 13
8.12.282.277	282, 237, 223, 122, 107, 53	277, 257, 197, 138, 118, 3
8.12.282.278	282, 229, 215, 107, 54, 47	278, 250, 177, 149, 37, 9
8.12.295	295, 216, 203, 106, 91, 78	294, 232, 169, 126, 125, 13

Remarks:

Powers: 2, 4, 6, 8 with $6+6=12$ terms.

Results of a selective search by Jarosław Wróblewski (December 2009).

Solution code	Right terms	Left terms
8.12.303	303, 265, 227, 119, 81, 19	291, 289, 191, 167, 45, 7
8.12.325	325, 277, 271, 201, 131, 57	317, 305, 233, 219, 139, 39
8.12.326	326, 273, 243, 227, 112, 89	317, 301, 208, 207, 186, 7
8.12.347	347, 289, 246, 112, 97, 54	343, 306, 194, 192, 43, 41
8.12.358	358, 329, 323, 281, 75, 12	357, 335, 317, 282, 76, 1
8.12.362	362, 295, 291, 163, 118, 81	353, 333, 233, 205, 134, 6
8.12.365	365, 305, 291, 219, 139, 79	355, 339, 241, 229, 181, 15
8.12.370	370, 311, 252, 201, 101, 61	369, 316, 241, 205, 123, 14
8.12.375	375, 325, 266, 184, 162, 61	371, 338, 230, 215, 171, 24
8.12.389	389, 354, 329, 281, 80, 50	379, 375, 304, 295, 94, 14
8.12.392	392, 328, 308, 241, 125, 109	385, 356, 277, 224, 197, 32
8.12.405	405, 362, 351, 205, 162, 49	393, 390, 331, 210, 167, 29
8.12.407.398.383	407, 365, 284, 242, 163, 79	398, 383, 251, 233, 220, 11
8.12.407.398.385	407, 363, 328, 265, 98, 70	398, 385, 293, 287, 120, 22
8.12.407.405	407, 386, 249, 129, 105, 2	405, 389, 243, 154, 74, 27
8.12.412	412, 358, 337, 161, 154, 27	407, 378, 314, 203, 92, 71
8.12.417	417, 347, 277, 184, 169, 64	416, 353, 248, 233, 139, 69
8.12.418	418, 368, 335, 237, 149, 66	402, 401, 302, 253, 165, 16
8.12.421	421, 357, 343, 274, 76, 62	419, 372, 323, 281, 98, 14
8.12.430	430, 351, 344, 203, 179, 82	424, 386, 283, 259, 162, 65
8.12.438	438, 389, 335, 256, 122, 97	430, 409, 302, 277, 151, 48
8.12.449	449, 397, 381, 374, 300, 28	436, 431, 363, 357, 316, 10
8.12.470	470, 396, 382, 257, 183, 61	465, 423, 349, 268, 194, 22
8.12.471	471, 407, 379, 295, 163, 105	453, 449, 335, 303, 209, 35
8.12.509	509, 429, 420, 248, 172, 111	495, 477, 364, 284, 179, 72
8.12.513	513, 410, 336, 218, 109, 77	512, 418, 315, 241, 123, 14
8.12.531	531, 471, 467, 269, 67, 7	523, 501, 441, 277, 41, 37
8.12.538	538, 472, 415, 321, 137, 114	529, 498, 361, 358, 177, 40

Remarks:

Powers: 2, 4, 6, 8 with $6+6=12$ terms.

Results of a selective search by Jarosław Wróblewski (December 2009).

Solution code: **8.12.541****Powers: 2, 4, 6, 8.**Number of terms: **12**Number of left terms: **6**Number of right terms: **6****Left terms:**

541, 503, 339, 176, 140, 73

Right terms:

532, 517, 305, 251, 96, 31

Remarks:

A member of family of solutions discovered by Jarosław Wróblewski (November 2009).

Left side terms:

$$2 a + 5 b + d$$

$$2 a + 5 b - d$$

$$5 a - 2 b + c$$

$$5 a - 2 b - c$$

$$4 a + 6 b$$

$$6 a - 4 b$$

Right side terms:

$$5 a + 2 b + c$$

$$5 a + 2 b - c$$

$$-2 a + 5 b + d$$

$$-2 a + 5 b - d$$

$$6 a + 4 b$$

$$-4 a + 6 b$$

Assume:

$$c^2 = p * a^2 + q * b^2$$

$$d^2 = p * b^2 + q * a^2$$

$$p = -11/5$$

$$q = 64/5$$

This solution is obtained with

$$a = 118$$

$$b = 89$$

$$c = 266$$

$$d = 401$$

Solution code	Right terms	Left terms
8.12.562	562, 487, 466, 439, 233, 225	549, 523, 470, 373, 326, 163
8.12.575	575, 497, 357, 216, 181, 76	573, 504, 323, 281, 140, 71
8.12.583	583, 501, 427, 205, 113, 109	569, 539, 347, 317, 67, 15
8.12.596	596, 446, 445, 243, 242, 111	594, 485, 354, 353, 149, 148
8.12.797	797, 703, 635, 471, 252, 136	771, 760, 567, 508, 289, 37
8.12.890	890, 732, 653, 470, 233, 213	883, 772, 555, 535, 318, 62

Remarks:

Powers: 2, 4, 6, 8 with $6+6=12$ terms.

Results of a selective search by Jarosław Wróblewski (December 2009).

See the following pages for numerical analysis of **8.12.596**, **8.12.797** and **8.12.890**.

Numerical analysis of **8.12.596**

Performed by Jarosław Wróblewski (December 12, 2009).

Analysis of possible linear constrains in solution **8.12.596** together with the equation for power 2 led me to the following substitutions:

Left side terms:

$$a + 2b + c$$

$$2c + 3x$$

$$c + 3x$$

$$b + c$$

$$b$$

$$a$$

Right side terms:

$$2c + 4x$$

$$2b + c$$

$$a + b + c$$

$$a + b$$

$$c + x$$

$$x$$

with $(a,b,c,x)=(111,242,1,148)$ in case of solution **8.12.596**.

While after this substitution the multigrade equation **is satisfied for power 2**, equation for **power 4** is quadratic in a and can be solved as such.

Equations for **powers 6 and 8** happen to have common quadratic factor and will be both satisfied if

$$3b^2 + 3bc - 2c^2 - 8cx - 8x^2 = 0.$$

This can be solved for x .

Value of x is rational iff

$$6b(b+c) \tag{8.12.596-1}$$

is a square and value of a is rational iff

$$2(17b^2 + 17bc - 4c^2) \tag{8.12.596-2}$$

is a square.

With $b = 2s^2$ and $c = 3t^2 - b$ the expression (8.12.596-1) is a square, while the expression (8.12.596-2), after dividing by 4, takes form

$$-8s^4 + 75s^2t^2 - 18t^4.$$

In the analized numerical solution we have $s = 11$ and $t = 9$.

For $(s,t,a,b,c,x) = (9, 17, -738, 162, 705, -582)$ we get solution **8.12.347**.

For $(s,t,a,b,c,x) = (123, 209, -138504, 30258, 100785, -88953)$ we get
 55358, 46168, 43681, 21763, 10086, 7599
 53767, 51414, 36082, 29651, 3944, 2487

For $(s,t,a,b,c,x) = (113, 137, -91739, 25538, 30769, -38606)$ we get
 92886, 81845, 66201, 38606, 35432, 7837
 91739, 85049, 56307, 54280, 25538, 9894

For $(s,t,a,b,c,x) = (659, 361, -1409216, 868562, -477599, -118049)$ we get
 1427394, 1259525, 1018253, 595648, 540654, 118049
 1409216, 1309345, 868562, 831746, 390963, 149691

For $(s,t,a,b,c,x) = (907, 417, -2175549, 1645298, -1123631, -5513)$ we get
 2269314, 2166965, 1653882, 1129144, 530251, 5513
 2263801, 2175549, 1645298, 1140170, 521667, 8584

For $(s,t,a,b,c,x) = (849, 1697, -5789991, 1441602, 7197825, -5760042)$ we get
 3360767, 2879809, 1929997, 1430346, 961492, 480534
 3360343, 2881506, 1920014, 1449463, 949812, 479261

The above is with the exact notations I have been discovering the family of solutions.
 A nicer form can be obtained by substitutions:

$$b = A - B$$

$$c = 2B$$

$$X^2 = 3(A^2 - B^2)/8 \quad (8.12.596 - 3)$$

$$Y^2 = (17A^2 - 33B^2)/8 \quad (8.12.596 - 4)$$

$$a = -A + Y$$

$$x = -B + X$$

Now with (8.12.596 - 3) and (8.12.596 - 4) satisfied, we get the following solution for
powers 2,4,6,8:

Left side terms:

$$A + Y$$

$$B + 3 X$$

$$-B + 3 X$$

$$A + B$$

$$A - B$$

$$-A + Y$$

Right side terms:

$$4 X$$

$$2 A$$

$$B + Y$$

$$-B + Y$$

$$B + X$$

$$-B + X$$

Numerical analysis of **8.12.797**

Performed by Jarosław Wróblewski (December 13, 2009).

Analysis of possible linear constrains in solution **8.12.797** led me to the following substitutions:

Left terms:

2 a

2 c

2 b

4 n

m - n

m + n

Right terms:

a + b + c

a - b + c

a + b - c

a - b - c

m - 3 n

m + 3 n

This already works for **power 2**.

To get it working for **powers 4,6** we need any 2 of the following 3 equalities satisfied (any equality follows from the remaining two):

$$\begin{aligned} a^4 - 2a^2b^2 + b^4 - 2a^2c^2 - 2b^2c^2 + c^4 &= (-a - b + c)(a - b + c)(-a + b + c)(a + b + c) = \\ &= 8(m - n)n^2(m + n) = 8m^2n^2 - 8n^4 \end{aligned}$$

$$2(a^2 + b^2 + c^2) = m^2 + 11n^2$$

$$8(a^4 + b^4 + c^4) = m^4 + 54m^2n^2 + 89n^4$$

Then **power 8** is for free.

Numerical solutions.

(a, b, c, m, n) = (30, 31, 7, 16, 18) leads to
36, 31, 30, 17, 7, 1
35, 34, 27, 19, 4, 3

(a, b, c, m, n) = (42, 139, 89, 239, 9) leads to
139, 124, 115, 89, 42, 18
135, 133, 106, 93, 46, 4

(a, b, c, m, n) = (56, 163, 103, 283, 7) leads to
163, 145, 138, 103, 56, 14
161, 152, 131, 105, 58, 2

$(a, b, c, m, n) = (98, 195, 71, 317, 21)$ leads to
 195, 169, 148, 98, 71, 42
 190, 182, 127, 111, 84, 13

$(a, b, c, m, n) = (10, 209, 111, 152, 90)$ leads to
 211, 165, 155, 59, 54, 44
 209, 180, 121, 111, 31, 10

$(a, b, c, m, n) = (294, 125, 13, 401, 63)$ leads to
 295, 216, 203, 106, 91, 78
 294, 232, 169, 126, 125, 13

$(a, b, c, m, n) = (192, 343, 43, 235, 153)$ leads to
 347, 289, 246, 112, 97, 54
 343, 306, 194, 192, 43, 41

$(a, b, c, m, n) = (358, 281, 75, 652, 6)$ leads to
 358, 329, 323, 281, 75, 12
 357, 335, 317, 282, 76, 1

$(a, b, c, m, n) = (80, 389, 281, 683, 25)$ leads to
 389, 354, 329, 281, 80, 50
 379, 375, 304, 295, 94, 14

$(a, b, c, m, n) = (98, 407, 265, 691, 35)$ leads to
 407, 363, 328, 265, 98, 70
 398, 385, 293, 287, 120, 22

$(a, b, c, m, n) = (248, 353, 233, 70, 208)$ leads to
 417, 347, 277, 184, 169, 64
 416, 353, 248, 233, 139, 69

$(a, b, c, m, n) = (418, 237, 149, 703, 33)$ leads to
 418, 368, 335, 237, 149, 66
 402, 401, 302, 253, 165, 16

$(a, b, c, m, n) = (14, 419, 281, 695, 49)$ leads to
 421, 357, 343, 274, 76, 62
 419, 372, 323, 281, 98, 14

$(a, b, c, m, n) = (538, 321, 137, 887, 57)$ leads to
 538, 472, 415, 321, 137, 114
 529, 498, 361, 358, 177, 40

$(a, b, c, m, n) = (140, 573, 281, 394, 252)$ leads to
 575, 497, 357, 216, 181, 76
 573, 504, 323, 281, 140, 71

$(a, b, c, m, n) = (436, 575, 93, 1022, 36)$ leads to
 575, 529, 493, 436, 93, 72
 565, 552, 459, 457, 116, 23

$(a, b, c, m, n) = (354, 485, 353, 1, 297)$ leads to
 596, 446, 445, 243, 242, 111
 594, 485, 354, 353, 149, 148

$(a, b, c, m, n) = (358, 615, 53, 964, 90)$ leads to
 617, 513, 460, 347, 155, 102

615, 527, 437, 358, 180, 53

$(a, b, c, m, n) = (82, 749, 347, 1073, 143)$ leads to
751, 589, 507, 322, 242, 160

749, 608, 465, 347, 286, 82

$(a, b, c, m, n) = (252, 797, 471, 1338, 68)$ leads to
797, 703, 635, 471, 252, 136

771, 760, 567, 508, 289, 37

$(a, b, c, m, n) = (664, 865, 41, 502, 440)$ leads to
911, 785, 744, 409, 121, 80

880, 865, 664, 471, 41, 31

$(a, b, c, m, n) = (872, 769, 281, 394, 496)$ leads to
992, 872, 769, 445, 281, 51

961, 941, 680, 547, 192, 89

$(a, b, c, m, n) = (176, 989, 705, 563, 495)$ leads to
1024, 935, 759, 461, 230, 54

990, 989, 705, 529, 176, 34

$(a, b, c, m, n) = (426, 1097, 313, 1592, 198)$ leads to
1097, 895, 697, 426, 396, 313

1093, 918, 605, 499, 492, 179

$(a, b, c, m, n) = (892, 1355, 259, 2281, 133)$ leads to
1355, 1207, 1074, 892, 266, 259

1340, 1253, 994, 941, 361, 102

$(a, b, c, m, n) = (1356, 785, 401, 2243, 135)$ leads to
1356, 1189, 1054, 785, 401, 270

1324, 1271, 919, 870, 486, 85

$(a, b, c, m, n) = (146, 1423, 645, 2017, 279)$ leads to
1427, 1107, 961, 590, 462, 316

1423, 1148, 869, 645, 558, 146

$(a, b, c, m, n) = (794, 1493, 121, 1076, 646)$ leads to
1507, 1204, 1083, 431, 410, 289

1493, 1292, 861, 794, 215, 121

$(a, b, c, m, n) = (14, 1599, 1249, 925, 819)$ leads to
1691, 1431, 1417, 766, 182, 168

1638, 1599, 1249, 872, 53, 14

$(a, b, c, m, n) = (1226, 1823, 125, 1132, 874)$ leads to
1877, 1587, 1462, 745, 361, 236

1823, 1748, 1226, 1003, 129, 125

$(a, b, c, m, n) = (1850, 1803, 291, 1008, 1066)$ leads to
2132, 1850, 1803, 1037, 291, 29

2103, 1972, 1681, 1095, 169, 122

$(a, b, c, m, n) = (2160, 1401, 553, 3678, 176)$ leads to
2160, 1927, 1751, 1401, 553, 352

2103, 2057, 1575, 1504, 656, 103

$(a, b, c, m, n) = (130, 2203, 917, 1864, 850)$ leads to

2207, 1625, 1495, 708, 578, 343

2203, 1700, 1357, 917, 507, 130

$(a, b, c, m, n) = (1172, 2215, 1031, 3829, 47)$ leads to

2215, 1938, 1891, 1172, 1031, 94

2209, 1985, 1844, 1178, 1037, 6

$(a, b, c, m, n) = (1528, 2411, 807, 4177, 117)$ leads to

2411, 2147, 2030, 1528, 807, 234

2373, 2264, 1913, 1566, 845, 38

Solutions added on December 14, 2009

$(a, b, c, m, n) = (2548, 1565, 941, 4423, 91)$ leads to

2548, 2257, 2166, 1565, 941, 182

2527, 2348, 2075, 1586, 962, 21

$(a, b, c, m, n) = (2582, 1903, 3, 1535, 1287)$ leads to

2698, 2244, 2241, 1163, 341, 338

2582, 2574, 1903, 1411, 124, 3

$(a, b, c, m, n) = (1558, 2729, 615, 4384, 342)$ leads to

2729, 2363, 2021, 1558, 684, 615

2705, 2451, 1836, 1679, 893, 278

$(a, b, c, m, n) = (2264, 3169, 631, 5530, 224)$ leads to

3169, 2877, 2653, 2264, 631, 448

3101, 3032, 2429, 2401, 768, 137

$(a, b, c, m, n) = (3126, 2681, 41, 1760, 1674)$ leads to

3391, 2924, 2883, 1631, 243, 202

3348, 3126, 2681, 1717, 43, 41

$(a, b, c, m, n) = (1772, 3745, 1443, 6046, 420)$ leads to

3745, 3233, 2813, 1772, 1443, 840

3653, 3480, 2393, 2037, 1708, 265

$(a, b, c, m, n) = (648, 3871, 1047, 4038, 1232)$ leads to

3871, 2635, 2464, 1403, 1047, 648

3867, 2783, 2135, 1736, 1088, 171

$(a, b, c, m, n) = (4004, 3491, 1137, 1906, 2244)$ leads to

4488, 4004, 3491, 2075, 1137, 169

4319, 4316, 3179, 2413, 825, 312

$(a, b, c, m, n) = (1474, 4535, 1741, 3283, 1925)$ leads to

4535, 3850, 2604, 1741, 1474, 679

4529, 3875, 2401, 2134, 1246, 660

$(a, b, c, m, n) = (2188, 4679, 1813, 7538, 532)$ leads to

4679, 4035, 3503, 2188, 1813, 1064

4567, 4340, 2971, 2527, 2152, 339

$(a, b, c, m, n) = (4900, 3589, 793, 8578, 364)$ leads to

4900, 4471, 4107, 3589, 793, 728

4835, 4641, 3848, 3743, 1052, 259

$(a, b, c, m, n) = (5212, 2253, 2145, 8334, 620)$ leads to

5212, 4477, 3857, 2253, 2145, 1240

5097, 4805, 3237, 2660, 2552, 407

$(a, b, c, m, n) = (5344, 1969, 93, 6781, 1311)$ leads to
5357, 3703, 3610, 1734, 1641, 1424

5344, 4046, 2735, 2622, 1969, 93

$(a, b, c, m, n) = (5410, 2751, 719, 8093, 915)$ leads to
5419, 4440, 3721, 2674, 1689, 970

5410, 4504, 3589, 2751, 1830, 719

$(a, b, c, m, n) = (1378, 5443, 3623, 9355, 403)$ leads to
5443, 4879, 4476, 3623, 1378, 806

5282, 5222, 4073, 3844, 1599, 221

$(a, b, c, m, n) = (514, 5649, 3301, 8875, 819)$ leads to
5666, 4732, 4218, 3209, 1431, 917

5649, 4847, 4028, 3301, 1638, 514

$(a, b, c, m, n) = (5692, 1783, 299, 6142, 1748)$ leads to
5693, 3887, 3588, 2104, 1805, 449

5692, 3945, 3496, 2197, 1783, 299

$(a, b, c, m, n) = (5384, 4285, 581, 3022, 2800)$ leads to
5711, 5125, 4544, 2689, 840, 259

5600, 5384, 4285, 2911, 581, 111

$(a, b, c, m, n) = (5922, 3043, 2395, 9871, 495)$ leads to
5922, 5183, 4688, 3043, 2395, 990

5680, 5678, 4193, 3285, 2637, 242

$(a, b, c, m, n) = (4820, 6211, 949, 11137, 355)$ leads to
6211, 5746, 5391, 4820, 949, 710

6101, 5990, 5041, 5036, 1170, 221

$(a, b, c, m, n) = (6980, 4161, 1651, 11438, 780)$ leads to
6980, 6109, 5329, 4161, 1651, 1560

6889, 6396, 4745, 4549, 2235, 584

$(a, b, c, m, n) = (6332, 4685, 3329, 2002, 3596)$ leads to
7192, 6332, 4685, 3329, 2799, 797

7173, 6395, 4393, 3844, 2488, 841

$(a, b, c, m, n) = (4670, 7447, 2723, 13001, 175)$ leads to
7447, 6588, 6413, 4670, 2723, 350

7420, 6763, 6238, 4697, 2750, 27

$(a, b, c, m, n) = (6606, 5591, 3219, 2100, 3886)$ leads to
7772, 6606, 5591, 3219, 2993, 893

7708, 6879, 4779, 4489, 2117, 1102

$(a, b, c, m, n) = (7826, 7039, 695, 14912, 130)$ leads to
7826, 7521, 7391, 7039, 695, 260

7780, 7651, 7261, 7085, 741, 46

$(a, b, c, m, n) = (7276, 6947, 2585, 3178, 4324)$ leads to
8648, 7276, 6947, 3751, 2585, 573

8404, 8075, 5819, 4897, 1457, 1128

$(a, b, c, m, n) = (7634, 5579, 4673, 1756, 4466)$ leads to

8943, 7577, 5821, 4270, 3364, 1309

8932, 7634, 5579, 4673, 3111, 1355

(a, b, c, m, n) = (3152, 8869, 6731, 4006, 4784) leads to

9568, 8869, 6731, 4395, 3152, 389

9376, 9179, 6224, 5173, 2645, 507

(a, b, c, m, n) = (3320, 9947, 3901, 14878, 1624) leads to

9947, 8251, 6627, 3901, 3320, 3248

9875, 8584, 5264, 5003, 4683, 1363

(a, b, c, m, n) = (6958, 9883, 13, 6025, 4823) leads to

10247, 8427, 8414, 4222, 1469, 1456

9883, 9646, 6958, 5424, 601, 13

(a, b, c, m, n) = (2680, 10569, 4639, 15658, 1800) leads to

10569, 8729, 6929, 4639, 3600, 2680

10529, 8944, 6264, 5129, 4305, 1625

(a, b, c, m, n) = (1532, 11673, 5339, 9058, 4788) leads to

11711, 9272, 7740, 3933, 2653, 2401

11673, 9576, 6923, 5339, 2135, 1532

(a, b, c, m, n) = (4448, 11857, 463, 15230, 2848) leads to

11887, 8384, 7921, 3936, 3473, 3343

11857, 9039, 6191, 5696, 4448, 463

(a, b, c, m, n) = (6756, 9839, 7415, 917, 5985) leads to

12005, 9436, 8519, 5249, 4590, 2166

11970, 9839, 7415, 6756, 3451, 2534

(a, b, c, m, n) = (12190, 12189, 137, 7036, 7038) leads to

14076, 12190, 12189, 7037, 137, 1

14075, 12258, 12121, 7039, 69, 68

Numerical analysis of **8.12.890**

Performed by Jarosław Wróblewski (December 14, 2009).

Analysis of possible linear constrains in solution **8.12.890** led me to the following substitutions:

Left side terms:

$a + 8b + c$
 $-a - 8b + c$
 $2a - b + d$
 $-2a + b + d$
 $3a - 2b + c$
 $-3a + 2b + c$

Right side terms:

$a - 8b + c$
 $-a + 8b + c$
 $2a + b + d$
 $-2a - b + d$
 $3a + 2b + c$
 $-3a - 2b + c$

This solution is obtained with

$a=335/2$
 $b=151/2$
 $c=237/2$
 $d=945/2$

Already works for **power 2**.

To get it working for **power 4** we need

$$-9a^2 + 81b^2 + c^2 - d^2 = 0$$

To get it working for **power 6** we need

$$5a^2 - 32b^2 + 3c^2 = 0$$

Then **power 8** is a free gift from nature.

Equivalently, $96b^2 - 15a^2$ and $825b^2 - 96a^2$ must be squares, equal to $9c^2$ and $9d^2$ respectively.

$(a, b, c, d) = (29, 62, 199, 586)$ gives
 362, 295, 291, 163, 118, 81
 353, 333, 233, 205, 134, 6

$(a, b, c, d) = (202, 83, 74, 443)$ gives
 470, 396, 382, 257, 183, 61
 465, 423, 349, 268, 194, 22

$(a, b, c, d) = (335, 151, 237, 945)$ gives
 890, 732, 653, 470, 233, 213
 883, 772, 555, 535, 318, 62

Solution code: **8.12.55292**

Powers: 2, 4, 6, 8.

Number of terms: **12**

Number of left terms: **6**

Number of right terms: **6**

Left terms:

55292, 50841, 50712, 41050, 23681, 11369

Right terms:

54280, 54151, 45759, 44489, 20902, 14148

Remarks:

A member of family of solutions discovered by Jarosław Wróblewski (November 2009).

Left side terms:

$a + 10 b + d$

$a + 10 b - d$

$10 a - b + c$

$10 a - b - c$

$a + 11 b$

$11 a - b$

Right side terms:

$10 a + b + c$

$10 a + b - c$

$-a + 10 b + d$

$-a + 10 b - d$

$11 a + b$

$-a + 11 b$

Assume:

$$c^2 = p * a^2 + q * b^2$$

$$d^2 = p * b^2 + q * a^2$$

$$p = -27/5$$

$$q = 248/5$$

This solution is obtained with

$a = 9533$

$b = 3439$

$c = 9791$

$d = 66661$

Solution code: **9.12.323**

Powers: 1, 3, 5, 7, 9.

Number of terms: **12**

Number of left terms: **6**

Number of right terms: **6**

Left terms:

323, 289, 269, 173, 91, 7

Right terms:

313, 311, 247, 193, 59, 29

Remarks:

Discovered by Chen Shuwen (2000).

Solution code: **9.12.407**

Powers: 1, 3, 5, 7, 9.

Number of terms: **12**

Number of left terms: **6**

Number of right terms: **6**

Left terms:

407, 347, 341, 181, 163, 23

Right terms:

403, 371, 311, 221, 119, 37

Remarks:

Discovered by Jarosław Wróblewski (November 28, 2009).

Solution code: **9.12.463**

Powers: 1, 3, 5, 7, 9.

Number of terms: **12**

Number of left terms: **6**

Number of right terms: **6**

Left terms:

463, 391, 335, 217, 161, 43

Right terms:

461, 403, 287, 283, 91, 85

Remarks:

Discovered by Jarosław Wróblewski (November 28, 2009).

Solution code: **9.12.1293**

Powers: 1, 3, 5, 7, 9.

Number of terms: **12**

Number of left terms: **6**

Number of right terms: **6**

Left terms:

1293, 1167, 995, 679, 399, 57

Right terms:

1279, 1205, 925, 767, 299, 115

Remarks:

Constructed by Jarosław Wróblewski (November 27, 2009) from solution **8.10.515**.

Solution code: **9.12.1325**

Powers: 1, 3, 5, 7, 9.

Number of terms: **12**

Number of left terms: **5**

Number of right terms: **7**

Left terms:

1325, 1115, 1111, 689, 365

Right terms:

1319, 1177, 1037, 731, 305, 23, 13

Remarks:

Discovered by Jarosław Wróblewski (December 27, 2009).

Solution code: **10.12.151**

Powers: 2, 4, 6, 8, 10.

Number of terms: **12**

Number of left terms: **6**

Number of right terms: **6**

Left terms:

151, 140, 127, 86, 61, 22

Right terms:

148, 146, 121, 94, 47, 35

Remarks:

Discovered by Nuutti Kuosa (1999) using a computer program written by Jean-Charles Meyrignac, as a single-grade solution to power 10. Four days later Chen Shuwen noticed it was in fact a multigrade.

Solution code: **10.12.1511**

Powers: 2, 4, 6, 8, 10.

Number of terms: **12**

Number of left terms: **6**

Number of right terms: **6**

Left terms:

1511, 1138, 1075, 700, 622, 107

Right terms:

1510, 1180, 953, 886, 413, 293

Remarks:

The smallest solution of the infinite family of solutions constructed in:

Ajai Choudhry, Jarosław Wróblewski, *Ideal Solutions of the Tarry-Escott Problem of degree eleven with applications to Sums of Thirteenth Powers*, Hardy-Ramanujan Journal, Vol. 31 (2008) pp. 1-13

The above paper is available at:

<http://www.nias.res.in/hrj/contentsvol31.htm>

Solution code: **10.12.2058**

Powers: 2, 4, 6, 8, 10.

Number of terms: **12**

Number of left terms: **6**

Number of right terms: **6**

Left terms:

2058, 1896, 1618, 1109, 891, 257

Right terms:

2037, 1947, 1514, 1294, 639, 472

Remarks:

Discovered by David Broadhurst (2007):

D. Broadhurst, *A Chinese Prouhet-Tarry-Escott solution*,

<http://physics.open.ac.uk/~dbroadhu/cpte.pdf>

The second known solution.

It was later used by Ajai Choudhry and Jarosław Wróblewski to produce an infinite family of solutions:

Ajai Choudhry, Jarosław Wróblewski, *Ideal Solutions of the Tarry-Escott Problem of degree eleven with applications to Sums of Thirteenth Powers*, Hardy-Ramanujan Journal, Vol. 31 (2008) pp. 1-13

The above paper is available at:

<http://www.nias.res.in/hrj/contentsvol31.htm>

Solution code: **10.12.14770**

Powers: 2, 4, 6, 8, 10.

Number of terms: **12**

Number of left terms: **6**

Number of right terms: **6**

Left terms:

14770, 12638, 11632, 7115, 7043, 929

Right terms:

14693, 13165, 10112, 9718, 4054, 3455

Remarks:

3rd smallest solution of the infinite family of solutions constructed in:

Ajai Choudhry, Jarosław Wróblewski, *Ideal Solutions of the Tarry-Escott Problem of degree eleven with applications to Sums of Thirteenth Powers*, Hardy-Ramanujan Journal, Vol. 31 (2008) pp. 1-13

The above paper is available at:

<http://www.nias.res.in/hrj/contentsvol31.htm>

Solution code: **10.12.23742**

Powers: 2, 4, 6, 8, 10.

Number of terms: **12**

Number of left terms: **6**

Number of right terms: **6**

Left terms:

23742, 18687, 18372, 12734, 9611, 349

Right terms:

23708, 19653, 16426, 14714, 7713, 3309

Remarks:

4th smallest solution of the infinite family of solutions constructed in:

Ajai Choudhry, Jarosław Wróblewski, *Ideal Solutions of the Tarry-Escott Problem of degree eleven with applications to Sums of Thirteenth Powers*, Hardy-Ramanujan Journal, Vol. 31 (2008) pp. 1-13

The above paper is available at:

<http://www.nias.res.in/hrj/contentsvol31.htm>

Solution code: **10.14.68**

Powers: 2, 4, 6, 8, 10.

Number of terms: **14**

Number of left terms: **7**

Number of right terms: **7**

Left terms:

68, 61, 55, 32, 31, 28, 1

Right terms:

67, 64, 49, 44, 23, 20, 17

Remarks:

Found by direct search (Jarosław Wróblewski, November 2009) - see solution **10.14.400** for details of a family it belong to.

Solution code: **10.14.400**

Powers: 2, 4, 6, 8, 10.

Number of terms: **14**

Number of left terms: **7**

Number of right terms: **7**

Left terms:

400, 365, 359, 254, 242, 89, 35

Right terms:

395, 383, 341, 271, 230, 70, 64

Remarks:

Constructed by Tito Piezas and Jarosław Wróblewski (November 2009):

Left terms:

-15 a - 4 e

15 a - 4 e

8 e

-3 a - 6 y

3 a - 6 y

-12 e + 6 z

12 e + 6 z

Right terms:

-16 e

-12 a + 8 e

12 a + 8 e

-9 a - 6 y

9 a - 6 y

-3 a + 6 z

3 a + 6 z

Constrains:

$$y^2 = a^2 + 52e^2/9$$

$$z^2 = 4a^2 + 4e^2/9$$

The above is as it has appeared in the original construction - it can be simplified by dividing corresponding coefficients by a common factor.

Solutions:

(a,e,y,z) = (3,12,29,10) leads to solution **10.14.68**

(a,e,y,z) = (612,105,662,1226) leads to this solution

(a,e,y,z) = (783,168,881,1570) leads to **10.14.4139**

(a,e,y,z) = (989,1980,4861,2378) leads to **10.14.12676**

(a,e,y,z) = (4876,2079,6982,9850) leads to

3502, 3394, 2701, 2355, 1423, 1136, 693

3574, 3131, 3072, 1853, 1745, 1386, 83

$(a,e,y,z) = (13231,10440,28369,27362)$ leads to
96484, 80075, 69969, 52235, 43507, 27840, 12964
96431, 80764, 67955, 55680, 41493, 25084, 17045

$(a,e,y,z) = (2941,44460,106909,30218)$ leads to
238276, 216759, 210877, 118560, 117404, 73985, 44575
237120, 222641, 204995, 130324, 106796, 63377, 57495

Solution code: **10.14.4139**

Powers: 2, 4, 6, 8, 10.

Number of terms: 14

Number of left terms: 7

Number of right terms: 7

Left terms:

4139, 3812, 3691, 2545, 2468, 979, 448

Right terms:

4111, 3923, 3580, 2684, 2357, 896, 587

Remarks:

Constructed by Tito Piezas and Jarosław Wróblewski (November 2009) - see solution 10.14.400 for details.

Solution code: **10.14.12689**

Powers: 2, 4, 6, 8, 10.

Number of terms: 14

Number of left terms: 7

Number of right terms: 7

Left terms:

12689, 10560, 9236, 6755, 5745, 3767, 1324

Right terms:

12676, 10711, 8733, 7585, 5280, 3164, 2305

Remarks:

Constructed by Tito Piezas and Jarosław Wróblewski (November 2009) - see solution 10.14.400 for details.

Solution code: **10.16.93**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**

Number of left terms: **8**

Number of right terms: **8**

Left terms:

93, 87, 72, 52, 44, 41, 29, 1

Right terms:

92, 89, 67, 61, 39, 36, 33, 8

Solution code: **10.16.113**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**

Number of left terms: **8**

Number of right terms: **8**

Left terms:

113, 103, 91, 64, 59, 34, 10, 2

Right terms:

112, 106, 85, 74, 53, 26, 23, 1

Solution code: **10.16.132**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**

Number of left terms: **8**

Number of right terms: **8**

Left terms:

132, 117, 103, 71, 61, 59, 56, 12

Right terms:

131, 121, 92, 84, 72, 43, 39, 37

Solution code: **10.16.155**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**

Number of left terms: **8**

Number of right terms: **8**

Left terms:

155, 135, 133, 97, 93, 93, 55, 17

Right terms:

153, 145, 115, 107, 105, 83, 43, 33

Solution code: **10.16.172**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**

Number of left terms: **8**

Number of right terms: **8**

Left terms:

172, 159, 149, 135, 106, 78, 43, 14

Right terms:

169, 166, 140, 138, 111, 74, 37, 27

Solution code: **10.16.173**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**

Number of left terms: **8**

Number of right terms: **8**

Left terms:

173, 156, 143, 104, 92, 91, 39, 13

Right terms:

168, 167, 123, 116, 113, 61, 44, 29

Solution code: **10.16.188**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**

Number of left terms: **8**

Number of right terms: **8**

Left terms:

188, 168, 166, 125, 114, 83, 39, 25

Right terms:

183, 182, 151, 131, 120, 62, 60, 19

Solution code: **10.16.193**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**

Number of left terms: **8**

Number of right terms: **8**

Left terms:

193, 179, 164, 100, 75, 72, 51, 31

Right terms:

191, 184, 159, 109, 68, 60, 53, 45

Remarks:

By a proper sign changes we can make the solution work for powers 1 and 3:

Left terms:

193, -179, 164, -100, 75, 72, 51, -31

Right terms:

191, -184, 159, 109, -68, -60, 53, 45

Solution code: **10.16.275**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**

Number of left terms: **8**

Number of right terms: **8**

Left terms:

275, 239, 211, 158, 100, 98, 2, 1

Right terms:

274, 245, 188, 185, 89, 86, 37, 22

Remarks:

Derived from solution **9.12.463**.

Solution code: **10.16.2567**

Powers: 2, 4, 6, 8, 10.

Number of terms: **16**

Number of left terms: **8**

Number of right terms: **8**

Left terms:

2567, 2339, 2283, 1544, 1426, 710, 479, 237

Right terms:

2536, 2449, 2173, 1654, 1347, 631, 510, 347

Remarks:

Constructed by Tito Piezas and Jarosław Wróblewski (November 2009).

Solution code: **11.20.107**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**

Number of left terms: **10**

Number of right terms: **10**

Left terms:

107, 101, 86, 78, 66, 55, 43, 25, 19, 13

Right terms:

106, 103, 81, 79, 73, 50, 38, 30, 27, 6

Remarks:

Derived from solution **10.16.93**.

Solution code: **11.20.139**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**

Number of left terms: **11**

Number of right terms: **9**

Left terms:

139, 125, 125, 113, 95, 85, 67, 65, 31, 5, 1

Right terms:

137, 133, 119, 107, 101, 91, 61, 53, 49

Remarks:

Derived from solution **10.14.68**.

Solution code: **11.20.178**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**

Number of left terms: **11**

Number of right terms: **9**

Left terms:

178, 167, 154, 119, 94, 88, 67, 49, 20, 8, 5

Right terms:

175, 173, 148, 124, 100, 74, 62, 59, 34

Remarks:

Derived from solution **10.12.151**.

Solution code: **11.20.199**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**

Number of left terms: **11**

Number of right terms: **9**

Left terms:

199, 182, 169, 141, 118, 97, 90, 39, 35, 18, 3

Right terms:

194, 193, 149, 147, 139, 78, 70, 66, 55

Remarks:

Derived from solution **10.16.173**.

Solution code: **11.20.327**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**

Number of left terms: **10**

Number of right terms: **10**

Left terms:

327, 305, 279, 271, 217, 197, 163, 69, 69, 45

Right terms:

321, 317, 277, 255, 229, 213, 119, 97, 95, 19

Remarks:

Derived from solution **10.12.151**.

Solution code: **11.20.329**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**

Number of left terms: **10**

Number of right terms: **10**

Left terms:

329, 307, 281, 265, 199, 161, 149, 71, 67, 43

Right terms:

323, 319, 275, 253, 227, 145, 121, 97, 95, 17

Remarks:

Derived from solution **10.12.151**.

Solution code: **11.20.431**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**

Number of left terms: **10**

Number of right terms: **10**

Left terms:

431, 409, 383, 301, 251, 167, 163, 113, 85, 7

Right terms:

425, 421, 371, 317, 223, 199, 151, 125, 43, 35

Remarks:

Derived from solution **10.12.151**.

Solution code: **11.20.569**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**

Number of left terms: **10**

Number of right terms: **10**

Left terms:

569, 547, 521, 439, 389, 311, 223, 145, 95, 29

Right terms:

563, 559, 509, 455, 361, 337, 197, 173, 79, 35

Remarks:

Derived from solution **10.12.151**.

Solution code: **11.20.3615**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**

Number of left terms: **10**

Number of right terms: **10**

Left terms:

3615, 2869, 2743, 2427, 1993, 1837, 1767, 1313, 379, 233

Right terms:

3613, 2953, 2499, 2429, 2365, 1683, 1557, 1419, 651, 7

Remarks:

Derived from solution **10.12.1511**.

Solution code: **11.20.5155**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**

Number of left terms: **9**

Number of right terms: **11**

Left terms:

5155, 4409, 4283, 3533, 3377, 2347, 1919, 887, 733

Right terms:

5153, 4493, 4039, 3905, 2959, 2719, 1547, 1307, 361, 143, 17

Remarks:

Derived from solution **10.12.1511**.

Solution code: **11.20.6269**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**

Number of left terms: **10**

Number of right terms: **10**

Left terms:

6269, 5945, 5389, 4371, 3935, 2667, 1921, 1741, 435, 371

Right terms:

6227, 6047, 5181, 4741, 3431, 3097, 1963, 1209, 1083, 65

Remarks:

Derived from solution **10.12.2058**.

Solution code: **11.20.6625**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**

Number of left terms: **11**

Number of right terms: **9**

Left terms:

6625, 6301, 5745, 4727, 4291, 3023, 1995, 1385, 519, 291, 79

Right terms:

6583, 6403, 5537, 5097, 3787, 3453, 1607, 1283, 1231

Remarks:

Derived from solution **10.12.2058**.

Solution code: **11.20.43107**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**

Number of left terms: **9**

Number of right terms: **11**

Left terms:

43107, 38843, 36831, 27797, 27653, 15819, 15425, 12763, 5869

Right terms:

42953, 39897, 33791, 33003, 21675, 20477, 15973, 9697, 5459, 663, 519

Remarks:

Derived from solution **10.12.14770**.

Solution code: **11.20.48287**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**

Number of left terms: **11**

Number of right terms: **9**

Left terms:

48287, 44023, 42011, 32977, 32833, 20605, 16889, 7583, 4661, 1477, 689

Right terms:

48133, 45077, 38971, 38183, 26855, 25657, 11837, 10793, 6529

Remarks:

Derived from solution **10.12.14770**.

Solution code: **11.20.65507**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**

Number of left terms: **9**

Number of right terms: **11**

Left terms:

65507, 55397, 54767, 43491, 37245, 29393, 21283, 17325, 14829

Right terms:

65439, 57329, 50875, 47451, 33449, 29461, 24641, 19351, 7445, 2597, 1199

Remarks:

Derived from solution **10.12.23742**.

Solution code: **11.20.78905**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **20**

Number of left terms: **10**

Number of right terms: **10**

Left terms:

78905, 68795, 68165, 56889, 50643, 32119, 30723, 12199, 7885, 1431

Right terms:

78837, 70727, 64273, 60849, 46847, 38039, 24803, 16063, 5323, 1993

Remarks:

Derived from solution **10.12.23742**.

Solution code: **11.22.65**

Powers: 1, 3, 5, 7, 9, 11.

Number of terms: **22**

Number of left terms: **11**

Number of right terms: **11**

Left terms:

65, 60, 59, 47, 45, 40, 30, 18, 16, 10, 4

Right terms:

64, 63, 56, 49, 43, 42, 27, 21, 14, 13, 2

Pure product of:

1, 2, 3, 4, 5, 7, 9, 11, 13, 16, 17, 19, 23.

Solution code: **12.26.79**

Powers: 2, 4, 6, 8, 10, 12.

Number of terms: **26**

Number of left terms: **13**

Number of right terms: **13**

Left terms:

79, 76, 72, 69, 58, 53, 48, 44, 26, 21, 17, 14, 11

Right terms:

78, 77, 74, 64, 61, 54, 51, 33, 31, 28, 19, 8, 4

Pure product of:

1, 2, 5, 6, 7, 8, 9, 11, 13, 15, 17, 19, 20, 25.

Remarks:

This solution is taken from the following paper:

Mihai Cipu, *Upper bounds for norms of products of binomials*. LMS Journal of Computation and Mathematics, 7 (2004), pp. 37-49

Solution code: **12.28.81**

Powers: 2, 4, 6, 8, 10, 12.

Number of terms: **28**

Number of left terms: **14**

Number of right terms: **14**

Left terms:

81, 74, 73, 72, 59, 52, 50, 48, 44, 30, 23, 15, 14, 1

Right terms:

80, 78, 71, 69, 62, 54, 49, 47, 40, 33, 25, 18, 4, 4

Pure product of:

1, 3, 4, 5, 6, 7, 10, 11, 13, 16, 17, 19, 23, 27.

Solution code: **12.28.82**

Powers: 2, 4, 6, 8, 10, 12.

Number of terms: **28**

Number of left terms: **14**

Number of right terms: **14**

Left terms:

82, 75, 74, 73, 60, 51, 49, 47, 33, 24, 21, 15, 12, 6

Right terms:

81, 79, 72, 70, 63, 54, 45, 44, 36, 27, 18, 17, 9, 5

Pure product of:

1, 3, 4, 5, 6, 7, 10, 11, 13, 16, 17, 19, 23, 29.

Solution code: **12.28.119**

Powers: 2, 4, 6, 8, 10, 12.

Number of terms: **28**

Number of left terms: **14**

Number of right terms: **14**

Left terms:

119, 109, 105, 103, 81, 81, 67, 65, 57, 29, 27, 19, 19, 11

Right terms:

117, 115, 101, 99, 89, 79, 73, 51, 49, 45, 41, 7, 3, 1

Solution code: **12.28.169**

Powers: 2, 4, 6, 8, 10, 12.

Number of terms: **28**

Number of left terms: **14**

Number of right terms: **14**

Left terms:

169, 151, 149, 145, 105, 101, 99, 95, 71, 63, 51, 47, 43, 23

Right terms:

167, 161, 139, 135, 133, 89, 85, 83, 81, 79, 61, 39, 33, 25

Pure product of:

1, 4, 5, 6, 7, 8, 9, 11, 13, 15, 17, 19, 23, 31.

Solution code: **13.26.173**

Powers: 1, 3, 5, 7, 9, 11, 13.

Number of terms: **26**

Number of left terms: **13**

Number of right terms: **13**

Left terms:

173, 159, 157, 131, 129, 107, 103, 79, 75, 51, 25, 9, 1

Right terms:

171, 167, 141, 139, 137, 97, 93, 89, 85, 43, 19, 15, 3

Pure product of:

1, 3, 4, 5, 6, 7, 9, 10, 11, 13, 16, 17, 19, 23, 29.

Remarks:

Published by L.J. Lander (1973), *Mathematics of Computation* 27 (122), 1973, p. 397

Solution code: **13.28.191**

Powers: 1, 3, 5, 7, 9, 11, 13.

Number of terms: **28**

Number of left terms: **14**

Number of right terms: **14**

Left terms:

191, 177, 175, 173, 147, 131, 129, 125, 113, 69, 57, 51, 27, 7

Right terms:

189, 185, 171, 167, 153, 137, 123, 119, 115, 71, 67, 37, 21, 17

Pure product of:

1, 3, 4, 5, 6, 7, 10, 11, 13, 16, 17, 19, 23, 27, 29.

Solution code: **13.30.69**

Powers: 1, 3, 5, 7, 9, 11, 13.

Number of terms: **30**

Number of left terms: **14**

Number of right terms: **16**

Left terms:

69, 64, 62, 61, 50, 48, 41, 40, 38, 27, 19, 16, 15, 15

Right terms:

68, 67, 60, 59, 54, 46, 45, 34, 32, 31, 31, 10, 9, 8, 6, 5

Pure product of:

1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 15, 16, 17, 19.

Solution code: **13.30.129**

Powers: 1, 3, 5, 7, 9, 11, 13.

Number of terms: **30**

Number of left terms: **13**

Number of right terms: **17**

Left terms:

129, 119, 115, 89, 87, 85, 83, 47, 43, 41, 37, 25, 23

Right terms:

127, 125, 103, 101, 95, 81, 63, 59, 51, 49, 19, 17, 9, 9, 7, 5, 3

Pure product of:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 17, 19.

Solution code: **14.30.153**

Powers: 2, 4, 6, 8, 10, 12, 14.

Number of terms: **30**

Number of left terms: **15**

Number of right terms: **15**

Left terms:

153, 143, 139, 115, 111, 111, 103, 89, 61, 59, 57, 53, 47, 5, 3

Right terms:

151, 149, 127, 123, 123, 99, 95, 93, 73, 65, 51, 43, 41, 23, 1

Pure product of:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 15, 17, 19, 23.

Solution code: **14.32.187**

Powers: 2, 4, 6, 8, 10, 12, 14.

Number of terms: **32**

Number of left terms: **16**

Number of right terms: **16**

Left terms:

187, 173, 171, 157, 127, 125, 123, 121, 83, 79, 75, 71, 39, 23, 15, 1

Right terms:

185, 181, 159, 155, 151, 115, 111, 107, 103, 99, 61, 57, 37, 33, 17, 13

Pure product of:

1, 3, 4, 5, 6, 7, 9, 10, 11, 13, 14, 16, 17, 19, 23, 29.

Solution code: **15.34.189**

Powers: 1, 3, 5, 7, 9, 11, 13, 15.

Number of terms: **34**

Number of left terms: **18**

Number of right terms: **16**

Left terms:

189, 179, 175, 173, 159, 149, 129, 127, 97, 85, 55, 41, 31, 25, 17, 11, 3, 1

Right terms:

187, 185, 171, 169, 161, 155, 121, 119, 117, 69, 63, 39, 37, 21, 19, 13

Pure product of:

1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 14, 16, 17, 19, 23, 29.

Solution code: **16.42.95**

Powers: 2, 4, 6, 8, 10, 12, 14, 16.

Number of terms: **42**

Number of left terms: **21**

Number of right terms: **21**

Left terms:

95, 90, 88, 77, 75, 71, 70, 59, 49, 46, 45, 37, 34, 28, 17, 16, 16, 13, 13, 12, 9

Right terms:

94, 93, 82, 81, 79, 68, 67, 56, 55, 53, 35, 32, 31, 29, 27, 26, 20, 7, 6, 5, 0

Pure product of:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 16, 17, 19, 22, 23.

Solution code: **17.48.337**

Powers: 1, 3, 5, 7, 9, 11, 13, 15, 17.

Number of terms: **48**

Number of left terms: **24**

Number of right terms: **24**

Left terms:

337, 329, 315, 287, 287, 285, 273, 245, 243, 221, 219, 189, 179, 153, 123, 119, 95, 93, 87, 85, 45, 31, 21, 1

Right terms:

335, 333, 305, 303, 291, 269, 267, 261, 239, 225, 205, 201, 173, 157, 117, 113, 107, 103, 83, 73, 47, 39, 9, 7

Pure product of:

1, 2, 3, 5, 7, 8, 9, 10, 11, 12, 13, 17, 19, 23, 29, 31, 41, 43, 53.

Solution code: **18.58.257**

Powers: 2, 4, 6, 8, 10, 12, 14, 16, 18.

Number of terms: **58**

Number of left terms: **29**

Number of right terms: **29**

Left terms:

257, 247, 243, 219, 215, 213, 211, 191, 185, 177, 155, 147, 143, 141, 127, 119, 103, 97, 77, 73, 71, 69, 69, 45, 45, 21, 13, 11, 3

Right terms:

255, 253, 231, 227, 227, 203, 197, 197, 195, 169, 159, 157, 139, 131, 123, 121, 111, 93, 87, 81, 65, 63, 57, 53, 35, 29, 23, 9, 5

Pure product of:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 15, 17, 19, 21, 23, 25, 27, 31.

Remarks:

My original own search gave 64 terms as the best result.

This solution is taken from *The Prouhet-Tarry-Escott Problem Revisited* by Peter Borwein and Colin Ignalls (1993).

Solution code: **19.65.143**

Powers: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19.

Number of terms: **65**

Number of left terms: **33**

Number of right terms: **32**

Left terms:

143, 138, 136, 124, 122, 121, 120, 110, 103, 101, 99, 88, 87, 86, 85, 84, 83, 66, 65, 64, 53, 51, 51, 50, 49, 37, 25, 18, 16, 14, 13, 9, 4

Right terms:

142, 141, 130, 128, 128, 116, 114, 113, 109, 95, 94, 93, 93, 91, 81, 80, 76, 75, 61, 59, 58, 57, 56, 45, 43, 34, 24, 22, 20, 19, 10, 8

Pure product of:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31.

Remarks:

My original own search gave 74 terms as the best result.

This solution is taken from *The Prouhet-Tarry-Escott Problem Revisited* by Peter Borwein and Colin Ignalls (1993).

Solution code: **20.70.173**

Powers: 2, 4, 6, 8, 10, 12, 14, 16, 18, 20.

Number of terms: **70**

Number of left terms: **35**

Number of right terms: **35**

Left terms:

173, 168, 166, 163, 156, 154, 153, 134, 127, 125, 115, 106, 96, 95, 86, 86, 84, 74, 72, 70, 67, 60, 56, 46, 45, 36, 27, 26, 25, 24, 17, 15, 13, 7, 3

Right terms:

172, 171, 162, 161, 160, 158, 148, 136, 126, 123, 119, 101, 100, 91, 89, 88, 79, 78, 75, 69, 65, 59, 53, 49, 41, 40, 30, 30, 21, 20, 18, 18, 8, 6, 2

Pure product of:

1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 37.

Remarks:

My original own search gave 88 terms as the best result.

This solution is taken from *The Prouhet-Tarry-Escott Problem Revisited* by Peter Borwein and Colin Ignalls (1993).

Pure product polynomials of a special kind.

Here are examples of polynomials of the form

$$\prod_{i=1}^k \prod_{p=1}^{a_i} (x^p - x^{-p})$$

with

$$n = a_1 + \dots + a_k$$

and relatively small (the best I could find) half of the sum of absolute values of coefficients (denoted by H).

H seems to fit quite well the estimate

$$E(n) = n^{(\log n)/e}$$

with \log being base e logarithm.

Although they lead to multigrade equations for exponents up to $n-2$ with total of H terms, multigrade solutions obtained that way aren't very interesting, due to a huge number of terms.

n	H	a_1, \dots, a_k	$H/E(n)$
100	5622	91, 8, 1	2.2993
200	58148	185, 13, 2	1.90327
300	221438	275, 21, 3, 1	1.40442
400	713774	370, 25, 4, 1	1.313
500	1685194	465, 28, 6, 1	1.13813
600	3442128	551, 40, 7, 2	0.997715
700	7730080	639, 50, 8, 3	1.07517
800	11745536	733, 55, 8, 3, 1	0.852695
900	20192374	825, 61, 10, 3, 1	0.817166
1000	35087104	919, 65, 12, 3, 1	0.834611
1100	52615906	1013, 71, 12, 3, 1	0.768469
1200	77839804	1106, 79, 10, 4, 1	0.724091
1300	117903318	1195, 85, 15, 4, 1	0.7207
1400	188230374	1291, 89, 15, 4, 1	0.776696
1500	253188340	1374, 104, 16, 4, 2	0.722005
1600	367054114	1442, 125, 24, 6, 2, 1	0.738491
1700	429572776	1538, 132, 21, 6, 2, 1	0.621076
1800	552094724	1634, 136, 21, 6, 2, 1	0.583101
1900	709247742	1738, 132, 21, 6, 2, 1	0.555347
2000	948582974	1823, 147, 21, 6, 2, 1	0.558063
3000	11686822018	2736, 215, 36, 8, 4, 1	0.670286
4000	51936520048	3654, 275, 49, 15, 4, 2, 1	0.530677
5000	198767088094	4555, 357, 63, 16, 6, 2, 1	0.510929
6000	847295849480	5444, 437, 82, 23, 8, 3, 2, 1	0.686346
7000	2212575241804	6358, 508, 93, 26, 9, 3, 2, 1	0.66237
8000	5482592103696	7230, 617, 106, 30, 10, 4, 2, 1	0.683236