ANALIZA MATEMATYCZNA 3.
WZORY DLA FUNKCJI UWIKŁANYCH.

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1. Dla funkcji $y(x)$ zadanej równaniem uwikłanym $f(x,y)=0$ mamy

$$0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx},$$

Stąd

$$\begin{aligned} 0 &= \frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x\partial y}\cdot\frac{dy}{dx} + \frac{\partial^2 f}{\partial y^2}\cdot\left(\frac{dy}{dx}\right)^2 + \frac{\partial f}{\partial y}\cdot\frac{d^2 y}{dx^2} \\ &= \frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x\partial y}\cdot y'(x) + \frac{\partial^2 f}{\partial y^2}\cdot\left(y'(x)\right)^2 + \frac{\partial f}{\partial y}\cdot y''(x) \end{aligned}$$

Dla punktu stacjonarnego x_0 jest $y'(x_0)=0$ zatem

$$0 = \frac{\partial^2 f}{\partial x^2}(x_0) + \frac{\partial f}{\partial y}(x_0)\cdot y''(x_0), \quad \text{czyli} \quad \texttt{znak}\,[y''(x_0)] = -\texttt{znak}\left[\frac{\partial^2 f}{\partial x^2}(x_0)\cdot\frac{\partial f}{\partial y}(x_0)\right].$$

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2. Dla funkcji $z(x,y)$ zadanej równaniem uwikłanym $F(x,y,z)=0$ mamy

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x} = 0, \qquad \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial y} = 0.$$

Stąd

$$\begin{aligned} 0 &= \frac{\partial F}{\partial z}\cdot\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 F}{\partial x\partial z}\cdot\frac{\partial z}{\partial x} + \frac{\partial^2 F}{\partial z^2}\cdot\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^2 F}{\partial x^2}, \\ 0 &= \frac{\partial F}{\partial z}\cdot\frac{\partial^2 z}{\partial y^2} + 2\frac{\partial^2 F}{\partial y\partial z}\cdot\frac{\partial z}{\partial y} + \frac{\partial^2 F}{\partial y^2}\cdot\left(\frac{\partial z}{\partial y}\right)^2 + \frac{\partial^2 F}{\partial y^2}, \\ 0 &= \frac{\partial F}{\partial z}\cdot\frac{\partial^2 z}{\partial x\partial y} + \frac{\partial^2 F}{\partial x\partial z}\cdot\frac{\partial z}{\partial y} + \frac{\partial^2 F}{\partial y\partial z}\cdot\frac{\partial z}{\partial x} + \frac{\partial^2 F}{\partial z^2}\cdot\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} + \frac{\partial^2 F}{\partial x\partial y}. \end{aligned}$$

W punkcie stacjonarnym $p=(x_0,y_0)$ jest $\frac{\partial F}{\partial x}(p) = 0 = \frac{\partial z}{\partial x}(p)$ i $\frac{\partial F}{\partial y}(p) = 0 = \frac{\partial z}{\partial y}(p)$, więc składniki hesjanu w takim punkcie można uzyskać ze wzorów:

$$\begin{aligned} 0 &= \frac{\partial^2 z}{\partial x^2}(p)\cdot\frac{\partial F}{\partial z}(p) + \frac{\partial^2 F}{\partial x^2}(p) \quad\Longrightarrow\quad \texttt{znak}\left[\frac{\partial^2 z}{\partial x^2}(p)\right] = -\texttt{znak}\left[\frac{\partial F}{\partial z}(p)\cdot\frac{\partial^2 F}{\partial x^2}(p)\right], \\ 0 &= \frac{\partial^2 z}{\partial y^2}(p)\cdot\frac{\partial F}{\partial z}(p) + \frac{\partial^2 F}{\partial y^2}(p) \quad\Longrightarrow\quad \texttt{znak}\left[\frac{\partial^2 z}{\partial y^2}(p)\right] = -\texttt{znak}\left[\frac{\partial F}{\partial z}(p)\cdot\frac{\partial^2 F}{\partial y^2}(p)\right], \\ 0 &= \frac{\partial F}{\partial z}(p)\cdot\frac{\partial^2 z}{\partial x\partial y}(p) + \frac{\partial^2 F}{\partial x\partial y}(p). \end{aligned}$$

$$H_z(p) = \frac{\frac{\partial^2 F}{\partial x^2}(p)\cdot\frac{\partial^2 F}{\partial y^2}(p) - \left[\frac{\partial^2 F}{\partial x\partial y}(p)\right]^2}{\left[\frac{\partial F}{\partial z}(p)\right]^2}$$

Janusz Wysoczański