INTRODUCTION TO NONCOMMUTATIVE PROBABILITY
Problem list 1

- The coin-toss problem is modelled by the classical *Bernoulli* random variable X is defined by
$$\mathbb{P}(X_B = +1) = \mathbb{P}(X_B = -1) = \frac{1}{2}$$

1. Show that the distribution of X is given by the Bernoulli (discrete) probability measure
$$\mu_B := \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1}$$

2. Show that the *moment sequence*
$$M_n(\mu_B) := \int_{-\infty}^{+\infty} t^n \mu(dt)$$
of the Bernoulli measure μ_B is given by
$$M_n(\mu_B) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

- The *operator norm* (of an operator A acting on a Hilbert space $\mathcal{H}$) is defined as $\|A\| := \sup\{\|Ax\|_{\mathcal{H}} : \|x\|_{\mathcal{H}} \leq 1\}$ (where $\|x\|_{\mathcal{H}}$ is the norm of the vector $x \in \mathcal{H}$).
- An operator A, acting on a Hilbert space $\mathcal{H}$, is called *bounded* if the norm $\|A\|$ is finite.
- An operator A, acting on a Hilbert space $\mathcal{H}$, is called self-adjoint if $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all vectors $x, y \in \mathcal{H}$.
- Consider the matrix $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ acting on the two-dimensional Hilbert space $\mathcal{H} := \mathbb{C}^2$ with the orthonormal basis $e_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

3. Show that B is self-adjoint bounded operator and compute the operator norm $\|B\|$ of it.
4. Compute the sequence $m_n := \langle B^n e_0, e_0\rangle$.

- For $\mathcal{H}$ is a Hilbert space let $\mathcal{B}(\mathcal{H})$ be the collection of all bounded operators on $\mathcal{H}$.

5. Show that $\mathcal{B}(\mathcal{H})$ is a unital *-algebra.

- A *state* φ on a unital *-algebra $\mathcal{A}$ is a positive (i.e. $\varphi(A) \geq 0$ for all $A \in \mathcal{A}$) normed (i.e. $\varphi(I) = 1$ for the unit element $I \in \mathcal{A}$) functional $\varphi : \mathcal{A} \mapsto \mathbb{C}$.

6. Let $\mathcal{H}$ be a Hilbert space and $x \in \mathcal{H}$ be a vector. Check if the functional
$$\mathcal{B}(\mathcal{H}) \ni A \mapsto \varphi(A) = \langle Ax, x\rangle$$
is a state on $\mathcal{B}(\mathcal{H})$.