INTRODUCTION TO NONCOMMUTATIVE PROBABILITY
Problem list 2: STATES.

- The trace $Tr(M)$ of a matrix $A = (a_{jk})_{j,k=1}^n \in \mathbb{M}_{n\times n}(\mathbb{C})$ is defined as the sum of its diagonal entries: $Tr(A) = \sum_{j=1}^n a_{jj}$.

1. Show the following properties of the trace:
   - Tr is a linear operator on $\mathbb{M}_{n\times n}(\mathbb{C})$,
   - $Tr(A) = Tr(A^T)$ (invariance under transposition),
   - $Tr(AB) = Tr(BA)$ for all $A, B \in \mathbb{M}_{n\times n}(\mathbb{C})$,
   - $Tr(ABC) = Tr(CAB)$ for all $A, B, C \in \mathbb{M}_{n\times n}(\mathbb{C})$.

2. Show that $Tr(AB) = \displaystyle\sum_{j,k=1}^{n} a_{jk}b_{jk}$ if $A = (a_{jk})$ and $B = (b_{jk})$.

3. Show that if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A \in \mathbb{M}_{n\times n}(\mathbb{C})$ then $Tr(A) = \lambda_1 + \ldots + \lambda_n$.
   Hint: use the Jordan form of A.

- A matrix $D \in \mathbb{M}_{n\times n}(\mathbb{C})$ is called *positive definite* if $\langle Dx, x\rangle > 0$ for all nonzero $x \in \mathbb{C}^n$ and *positive semi-definite* if $\langle Dx, x\rangle \geq 0$ for all $x \in \mathbb{C}^n$.

4. Show that a positive (semi-)definite matrix $D \in \mathbb{M}_{n\times n}(\mathbb{C})$ is Hermitian, i.e. show that if $\langle Dx, x\rangle \geq 0$ for all $x \in \mathbb{C}^n$ then $\langle Dx, y\rangle = \langle x, Dy\rangle$ (equivalently $D = D^*$).

5. Show that for a positive (semi-)definite matrix $D \in \mathbb{M}_{n\times n}(\mathbb{C})$ there exists $A \in \mathbb{M}_{n\times n}(\mathbb{C})$ such that $D = AA^*$.

6. Define the following operation $[*, *]$ on matrices:
   $$\mathbb{M}_{n\times n}(\mathbb{C}) \times \mathbb{M}_{n\times n}(\mathbb{C}) \ni (A, B) \longmapsto [A, B] := Tr(AB^*)$$
   Show that this defines a scalar product on $\mathbb{M}_{n\times n}(\mathbb{C})$ by proving that $[*, *]$:
   - is linear on the first argument and antilinear on the second,
   - is conjugate symmetric: $[A, B] = \overline{[B, A]}$,
   - is positive (semi-)definite: $[A, A] \geq 0$ for all $A \in \mathbb{M}_{n\times n}(\mathbb{C})$
   - satisfies $[A, A] = 0$ if and only if $A = 0$.

- The the normalized trace $Tr_n(M)$ on $\mathbb{M}_{n\times n}(\mathbb{C})$ is defined as $Tr_n(A) = \dfrac{1}{n}\displaystyle\sum_{j=1}^{n} a_{jj}$.

7. For a given matrix $D \in \mathbb{M}_{n\times n}(\mathbb{C})$ define a functional $\varphi_D : \mathbb{M}_{n\times n}(\mathbb{C}) \mapsto \mathbb{C}$ by $\varphi_D(A) := Tr_n(AD)$. Show that φ_D is a linear functional and if $D \geq 0$ is positive (semi-)definite, then φ_D is positive. Under what additional assumption φ_D is a state?

8. Characterize all $D \in \mathbb{M}_{n\times n}(\mathbb{C})$ for which φ_D is a vector state.