INTRODUCTION TO NONCOMMUTATIVE PROBABILITY
Problem list 3:
CREATION AND ANNIHILATION OPERATOS.

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1. Let $S : l^2 \mapsto l^2$ be the standard unilateral shift on l^2, defined on the standard orthonormal basis $\{e_n : n = 0, 1, 2, \ldots\}$ by the formula: $Se_0 = 0$ and $Se_n = e_{n-1}$ if $n \geq 1$. Consider the state $\varphi(A) = \langle Ae_0, e_0 \rangle$.
   - Compute the formula for the adjoint operator S^*.
   - Compute the operator norm $\|S\|$.
   - Find the moment sequence $m_n := \varphi(G^n)$ for the self-adjoint operator $G := S + S^*$.
   - Find the distribution measure of G with respect to φ, i.e. the measure μ for which m_n is the moment sequence.

- The full Fock space $\mathcal{F}(\mathcal{H})$ on the complex Hilbert space $\mathcal{H}$ is defined as the infinite direct sum

$$\mathcal{F}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}_n, \quad \mathcal{H}_0 := \mathbb{C}\Omega, \quad \mathcal{H}_1 = \mathcal{H}, \quad \mathcal{H}_n := \mathcal{H}^{\otimes n} \quad \text{for} \quad n \geq 2,$$

  where $\Omega \in \mathcal{H}$ is a distinguished norm one *vacuum vector*. Let φ be the vacuum vector state.

2. Let $\{e_j : j = 0, 1, 2, \ldots\}$ be an orthonormal basis of $\mathcal{H}$ with $e_0 = \Omega$. Moreover let $a_j^\dagger$ be the creation operator by the vector e_j for $j \geq 1$, let a_j the corresponding annihilation operator and consider $g_j := a_j^\dagger + a_j$.
   - Show the commutation relations $a_j a_k^\dagger = \delta_{jk}$.
   - Compute the norm $\|a_j^\dagger\|$ of the creation $a_j^\dagger$ and conclude that g_j is bounded and self-adjoint.
   - Compute the distribution of g_j with respect to φ.

- The Muraki's discrete monotonic Fock space $\mathcal{F}_M(\mathcal{H})$ over the complex Hilbert space $\mathcal{H}$ is defined as the infinite direct sum

$$\mathcal{F}_M(\mathcal{H}) := \bigoplus_{r=0}^{\infty} \mathcal{H}_r, \quad \mathcal{H}_r := l^2(T_r)$$

  where

$$T_r := \{\sigma = (j_r, j_{r-1}, \ldots, j_1) \in \mathbb{N}^r \ : \ j_1 < j_2 < \ldots < j_r \geq 1\} \quad \text{for} \quad r \geq 1,$$

  where $T_0 := \{0\}$ is the null sequence.

3. Show that, for a given $r \geq 1$, the set $\{e_\sigma : \sigma \in T_r\}$ defined as

$$e_\sigma(\mu) = \begin{cases} 1 & \text{if } \mu = \sigma \\ 0 & \text{if } \mu \neq \sigma \end{cases}$$

   is an orthonormal basis in $\mathcal{H}_r$.

- The vacuum vector is defined as $\Omega = e_0$
- For given $j \in \mathbb{N}$ the discrete monotonic creation operator $\delta_j^\dagger$ and annihilation δ_j operator are define by

$$\delta_j^\dagger(e_\sigma) = \begin{cases} e_{(j,j_r,\ldots,j_1)} & \text{if} \quad j > j_r > \cdots > j_1 \quad \text{and} \quad \sigma = (j_r, \ldots, j_1) \\ 0 & \text{otherwise} \end{cases}$$

$$\delta_j(e_\sigma) = \begin{cases} e_{(j_{r-1},\ldots,j_1)} & \text{if} \quad j = j_r \quad \text{and} \quad \sigma = (j_r, \ldots, j_1) \\ 0 & \text{otherwise} \end{cases}$$

4. Find the commutation relations for $\delta_j^\dagger \delta_j$, $\delta_j \delta_j^\dagger$ and for $\delta_j^\dagger \delta_k$, $\delta_k \delta_j^\dagger$ with $j \neq k$.
5. Show that the creation and annihilation operators are bounded on $\mathcal{F}_M(\mathcal{H})$.
6. Show that the creation and annihilation operators are adjoint of each other, i.e. $(\delta_j)^* = \delta_j^\dagger$.
7. Find the distribution of $d_j = \delta_j^\dagger + \delta_j$ w.r.t. the vacuum state φ.