

ON TAME AMENABLE FLOWS AND GENERALIZED NEWELSKI'S CONJECTURE

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ABSTRACT. We are developing topological dynamics “below” a given ambit; we define and study Bohr compactifications below ambits and other related objects. We describe a natural epimorphism from the Ellis group of an ambit to the Bohr compactification below its Ellis semigroup ambit. The first main result says that, assuming tameness and hereditary amenability, this natural map is an isomorphism of topological groups. This is an abstract dynamical generalization of Newelski's conjecture, or rather a theorem of Chernikov and Simon that a natural map from the Ellis group of the flow $(S_G(M), G)$ to the definable Bohr compactification of G is an isomorphism when G is a definably amenable group definable in a NIP structure M . We give two proofs of our theorem: a shorter dynamical proof and an analytic proof. In the course of the dynamical proof, we prove some results on the universal compactification flow and on the generalized Bohr compactification below a given ambit which admits a left continuous semigroup operation extending the action of G . The considerations around the analytic proof yield a description of the measured structure of amenable tame minimal flows, which is the second main result of this paper. As a corollary of the first main result mentioned above, we obtain another structural result: each amenable tame minimal flow is a proximal extensions of its maximal equicontinuous factor. Our results do not require any metrizability assumption.

We also study specializations of the above abstract topological dynamics context to model theory contexts to recover some known and to obtain some new notions and results. The main new results in the model theory contexts are: (i) a description of the Bohr compactification of the group of automorphisms of a sufficiently saturated model $\mathfrak{C}$ of a first order theory T below a certain fundamental ambit associated with $\mathfrak{C}$; (ii) transfer of amenability of a NIP theory T (i.e., amenability of the aforementioned ambit) to amenability of the Ellis semigroup ambit; (iii) a version of Newelski's conjecture for spaces of local types with a description of Bohr compactifications below such spaces.

1. INTRODUCTION

Recall that a G -flow is a pair (X, G) consisting of a compact (Hausdorff) space X and a topological group G together with a continuous action of G on X . In this paper, the topology on G is irrelevant, so G is assumed to be a discrete group. A G -ambit is a flow (X, x_0, G) with a distinguished point x_0 with dense orbit. The *Ellis semigroup* of a flow (X, G) is the closure of $\{g_X : g \in G\}$ in the space X^X equipped with the pointwise convergence topology, where $g_X(x) := gx$, with composition of

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functions as the semigroup operation. This is also naturally a G -ambit (which we call the *Ellis flow* or *Ellis ambit* of (X, G)) with id_X as the distinguished element with dense orbit.

1.1. Motivation. The motivation and general goals for the research in this paper come from several directions.

- (1) Understand the structure of amenable tame minimal flows *without* the metrizable assumption. Recall that metrizability was an essential assumption in many topological dynamics papers, in particular in the very important paper [Gla18] describing the structure of *metric* tame minimal flows and *metric* amenable tame minimal flows.
- (2) Generalize to the abstract context of topological dynamics the various important notions, results, and phenomena known in topological dynamics in model theory (i.e., results about the flows of the form $(S_G(M), G)$ where G is a group definable in a structure M and $S_G(M)$ is the space of complete types over M concentrated on G , and also some other natural flows considered in model theory).
- (3) Develop topological dynamics “below” a given ambit. For a given group G the usual topological dynamics automatically takes place below the universal G -ambit. A motivation is that if we replace the universal G -ambit by other G -ambits, we can impose interesting dynamical properties (e.g., tameness) which are almost never satisfied by the universal G -ambit. Also, the “definable topological dynamics” is essentially topological dynamics below certain natural ambits (type spaces) considered in model theory.

The above three items are strongly related. The general goal in item (2) was successfully realized in [HK26] under the tameness assumption (which corresponds to the model-theoretic NIP) for results concerning the transfer of amenability of a flow to its Ellis flow, unique ergodicity, and a description of ergodic measures. In this paper, we generalize the so-called Newelski’s conjecture to the abstract context and prove it for hereditarily amenable tame flows. As a corollary, we obtain a structural result saying that each amenable tame minimal flow is a proximal extensions of its maximal equicontinuous factor, which is a contribution belonging to item (1). Our analytic proof of the Generalized Newelski’s Conjecture led us to a description of the measured structure of amenable tame minimal flows which is the main result belonging to (1). Considerations around generalized Newelski’s conjecture naturally belong to (3).

Going beyond the metrizable context is a highly non-trivial task. For example, originally we had some results from [HK26] only in the metrizable case, as we were using [Gla18] which requires this assumption, but then we developed techniques which allowed us to drop the metrizability assumption. Similarly in the case of the so-called Revised Newelski Conjecture from [KP26], which was first deduced in [CGK26] from [Gla18] in a metrizable context, and then generalized to arbitrary tame minimal flows in [BZ24] by using set-theoretic methods related to forcing. In this paper, we observe that the Generalized Newelski’s Conjecture follows from [Gla18] for metrizable minimal flows, and then we give two different proofs which work in general (i.e., without the metrizability and minimality assumption). The two structural results mentioned in the previous paragraph are, of course, also general (i.e., without metrizability).

1.2. Main results in abstract topological dynamics. For precise definitions and more references see further sections.

Recall that the *Bohr compactification* of a discrete group G is a unique (up to isomorphism) universal group homomorphism from G to a compact group with dense image. We define the Bohr compactification of G below a given G -ambit (X, x_0, G) as a unique (up to isomorphism) universal homomorphism as above which additionally factors through the ambit (X, x_0, G) (see Definition 3.1). In Subsections 3.1 and 3.2, we study such Bohr compactifications and compare them to the maximal equicontinuous factors (MEF's). We observe that in general the MEF of (X, x_0, G) may be strictly bigger than the Bohr compactification of G below (X, x_0, G) , but these notions coincide when there is a left continuous semigroup operation on X extending the action of G (equivalently, when (X, x_0, G) is isomorphic to an Ellis flow $(E(Y, G), \text{id}_Y, G)$ for some G -flow (Y, G)).

To state the Generalized Newelski's Conjecture, we need to recall a fundamental theorem of Ellis theory. As already mentioned, with a flow (X, G) one associates its Ellis semigroup $E(X, G)$. This is a left-topological semigroup (i.e., the semigroup operation is continuous in the left coordinate). Ellis theorem tells us that there exists a minimal left ideal $\mathcal{M}$ in $E(X, G)$ and each such ideal $\mathcal{M}$ can be written as the disjoint union of all sets $u\mathcal{M}$, where u ranges over the idempotents in $\mathcal{M}$. Moreover, each $u\mathcal{M}$ is a group, and all these groups are isomorphic for varying minimal left ideal $\mathcal{M}$ and idempotent $u \in \mathcal{M}$. Each group $u\mathcal{M}$ is equipped with the so-called τ -topology which makes it a quasi-compact semitopological group (i.e., the group operation is separately continuous). The groups $u\mathcal{M}$ remain isomorphic also as semitopological groups. The isomorphism type of these groups (or any of these group separately) will be called the *Ellis group* of (X, G) .

Let us fix a flow (X, G) . Set $E := E(X, G)$, choose a minimal left ideal $\mathcal{M}$ in E and an idempotent $u \in \mathcal{M}$. Let $\mathfrak{b}: G \rightarrow B$ be the Bohr compactification of G below (E, id_X, G) . There is a unique left continuous action of E on B extending the natural action of G . We define an abstract counterpart of Newelski's map (whose definition is recalled in Subsection 6.1):

$$\widehat{\theta}: u\mathcal{M} \rightarrow B, \quad \widehat{\theta}(\eta) := \eta e_B,$$

where e_B denotes the neutral element of B . In Subsection 3.3, we explain that $\widehat{\theta}$ is a group epimorphism and a quotient topological map which factors through the largest Hausdorff quotient of $u\mathcal{M}$ (even through a certain Galois group).

Conjecture 1.1 (Generalized Newelski's Conjecture (GNC)). Under suitable assumptions on (X, G) , the map $\widehat{\theta}$ is a topological group isomorphism.

The flow (X, G) is said to be *hereditarily amenable* if every subflow admits an invariant Borel probability measure. Below is our first main result (see Section 2 for the definition of tameness). This theorem is an abstract generalization of Theorem 1.5 in [CS18] being one of the main results of that paper.

Theorem 1.2. *The GNC holds when (X, G) is tame and hereditarily amenable.*

For an equivalent form of the above conjecture and theorem and for other comments see Subsection 4.1.

We provide two different proofs of the above theorem: a dynamical proof in Subsection 4.3 and an analytic one in Subsection 5.1.

A deep theorem which is behind both proofs is a weaker version of GNC which one could call the Generalized Revised Newelski's Conjecture (GRNC); see the discussion below on the model-theoretic origins. This result was proved in [CGK26, Corollary 5.14] in the metrizable context and in [BZ24, Theorem 11.7] in full generality. For a short deduction of the statement below from [BZ24, Theorem 11.7] see [HK26, Lemma 3.3].

Fact 1.3 (Generalized Revised Newelski's Conjecture (GRNC)). *If (X, G) is tame, then $u\mathcal{M}$ is Hausdorff in the τ -topology, and so a compact topological group.*

Another non-trivial ingredient of the dynamical proof is the following result contained in [HK26, Lemma 5.7] (using also [HK26, Corollary 5.6] which says that if (X, G) is tame and hereditarily amenable, then $E(X, G)$ is hereditarily amenable, and so $\mathcal{M}$ is amenable.)

Fact 1.4. *If (X, G) is tame and hereditarily amenable, then $u\mathcal{M}$ is dense in $\mathcal{M}$.*

Having the above two results and [HK26, Corollary 5.6], the dynamical proof is short and uses arguments as in [Gla76]. We take the opportunity and also study a bit topological dynamics below a given ambit, yielding some information on the universal compactification flow and on the generalized Bohr compactification in this context (see Subsections 3.3 and 4.3).

The main engine of the analytic proof is Lemma 5.1, which for a tame hereditarily amenable flow (X, G) identifies the L^2 -spaces $L^2(B, \mathfrak{h}_B)$ and $L^2(\mathcal{M}, \mu)$, where $\mathfrak{h}_B$ is the unique normalized Haar measure on B and μ is the unique G -invariant probability measure on $\mathcal{M}$. A final deduction of GNC is then contained in the proof of Theorem 5.3 and afterwards, which also uses the uniqueness of a G -invariant probability measure on $\mathcal{M}$. This last fact is the non-trivial Theorem 4.2 in [HK26] (which relies on Fact 1.3).

In passing from the study of metrizable minimal tame amenable flows (e.g. as in [Gla18]) to the context of general minimal tame amenable flows, we use measure-theoretic substitutes for the algebra $C(X)$ and for techniques based on tameness. The spaces $L^2(X, \mu)$ and the von Neumann algebras $L^\infty(X, \mu)$ provide such a framework, allowing us to study the measured structure without metrizability assumptions. In Subsection 5.2, Lemma 5.1 is used to prove the second main result of this paper, Theorem 5.7, i.e. a characterization of the measured structure of the minimal, tame and amenable flows. We establish there

Theorem 1.5 (Theorem 5.7). *If (X, G) is an amenable tame minimal flow, then*

$$L^\infty(X, \mu) \cong_G L^\infty(B / \text{Gal}_B, \mathfrak{h}_{B / \text{Gal}_B}),$$

where μ and $\mathfrak{h}_{B / \text{Gal}_B}$ are the unique G -invariant Borel probability measures on X and B / Gal_B , respectively (for the definition of Gal_B , see Subsection 5.2), and $\cong_G$ denotes a normal trace-preserving G -equivariant $$ -isomorphism of von Neumann algebras.*

This can be expressed also as follows. For a probability space (Y, ν) , let $\text{MAlg}(Y, \nu)$ denote its complete Boolean algebra of measurable sets modulo null sets.

Corollary 1.6 (Corollary 5.8). *If (X, G) is an amenable tame minimal flow, then*

$$\text{MAlg}(X, \mu) \cong_G \text{MAlg}(B / \text{Gal}_B, \mathfrak{h}_{B / \text{Gal}_B}).$$

Here $\cong_G$ denotes a measure-preserving G -equivariant isomorphism of complete Boolean algebras.

In Subsection 5.3, we use Theorem 5.7 to get a similar identification of $\text{MAlg}(X, \mu)$ and $\text{MAlg}(Z, \lambda_Z)^1$, where $\pi: X \rightarrow Z$ is the MEF of (X, G) and λ_Z is the unique G -invariant Borel probability measure on Z . Using GNC (i.e., Theorem 1.2), we also show that π is a proximal map, and this is already used in an ongoing project of the first author with Andy Zucker on extending the main results of [Gla18] to non-metrizable flows.

Example 1.7. The fully-split circle example $X := \text{Split}(\mathbb{T}; \mathbb{T})$ from [HK26, Appendix A.2], i.e. the circularly ordered space obtained by replacing each $t \in \mathbb{T}$ by two adjacent points $t^- < t^+$ and with the action of $G := D_\infty$ defined in [HK26], illustrates the situation nicely. This flow (X, G) is minimal, tame, amenable, but not metrizable. The map $\pi: X \rightarrow \mathbb{T}$, $\pi(t^\pm) := t$, is its maximal equicontinuous factor. Since every fiber of π has exactly two points, π is not almost one-to-one as would be the case in the metrizable set-up [Gla18]. Nevertheless, by our results, π is proximal and induces a measure-algebra isomorphism.

A related example, which has the above properties, is $(X, \mathbb{T})$ with the natural action of the circle group $\mathbb{T}$ on X . This example appears naturally in model theory as a unique minimal subflow of the flow $(S_G(M), G)$ for $G := \mathbb{T}$ treated as a group definable in $M := (\mathbb{R}, +, \cdot)$.

All these results form a contribution to understanding of the structure of amenable tame minimal flows without the metrizability assumption.

1.3. Original Newelski’s Conjecture and new results in model-theoretic contexts. First, we recall some history related to Newelski’s conjecture and its variants. For more details (including definitions) see Subsection 6.1.

Let $G = G(M)$ be a group definable in a structure M . We consider the ambit $(S_{G, \text{ext}}(M), \text{tp}_{\text{ext}}(e/M), G)$ of all complete external types over M concentrated on G (i.e., the space of ultrafilters in the Boolean algebra of externally definable subsets of G). It turns out that there is a left continuous semigroup operation $*$ on $S_{G, \text{ext}}(M)$ extending the action of G , and so $S_{G, \text{ext}}(M)$ can be identified with its Ellis semigroup (as a semigroup and G -flow). Let $\mathcal{M}$ be a minimal left ideal in $S_{G, \text{ext}}(M)$ and $u \in \mathcal{M}$ an idempotent. Let $\bar{G}_M^{00}$ be the smallest M -type-definable subgroup of bounded index in $\bar{G}$, where $\bar{G}$ is the interpretation of G in a sufficiently saturated model (so-called monster model). Then $\bar{G}/\bar{G}_M^{00}$ equipped with the *logic topology* (a subset is closed if its preimage in $\bar{G}$ is type-definable) is a compact (Hausdorff) topological group. The component $\bar{G}_M^{00}$ and the quotient $\bar{G}/\bar{G}_M^{00}$ play a very important role in model theory (especially in theories with NIP).

In [New09], Newelski introduced a natural epimorphism $\rho: u\mathcal{M} \rightarrow \bar{G}/\bar{G}_M^{00}$ (see Definition 6.1). *Newelski’s conjecture* predicted that it is an isomorphism under suitable model-theoretic assumptions. This was an interesting connection between the topological dynamics object $u\mathcal{M}$ and the model-theoretic object $\bar{G}/\bar{G}_M^{00}$. Later, the various refinements and variants of Newelski’s map [KP17], [KPR18], [KR20], [KP26] led to important contributions in model theory (e.g., to understand the complexity of strong types) and to additive combinatorics (to construct generalized locally compact models for arbitrary approximate subgroups in a simpler and “more classical” way than originally done by Hrushovski).

Newelski’s conjecture was refuted in [GPP15] for groups definable in NIP theories, concretely for $G := \text{SL}_2(\mathbb{R})$ treated as a group definable in $M := (\mathbb{R}, +, \cdot)$. The

¹Alternatively, the corresponding von Neumann algebras.

main positive result is the following, where definable amenability of G means that the flow $(S_G(M), G)$ is amenable (i.e., $S_G(M)$ admits an invariant Borel probability measure).

Fact 1.8 (Theorem 1.5 in [CS18]). *Newelski's conjecture holds for definably amenable groups definable in NIP theories.*

A very important refinement of Newelski's map was obtained in [KP17]:

$$u\mathcal{M} \rightarrow u\mathcal{M}/H(u\mathcal{M}) \rightarrow \bar{G}/\bar{G}_M^{000} \rightarrow \bar{G}/\bar{G}_M^{00}$$

with all the maps being epimorphisms of groups and topological quotient maps, where $u\mathcal{M}/H(u\mathcal{M})$ is the largest Hausdorff quotient of $u\mathcal{M}$ and $\bar{G}_M^{000}$ is the smallest M -invariant bounded index subgroup of $\bar{G}$. This yields many counter-examples to Newelski's conjecture. The variant of $u\mathcal{M}/H(u\mathcal{M})$ for definable approximate subgroups in place of definable groups led to the aforementioned dynamical construction of generalized locally compact models for arbitrary approximate subgroups in [KP26]. The above refinement also led to a weakening of Newelski's conjecture stated as [KP26, Conjecture 5.3] and called *Revised Newelski's conjecture* (RNC): If G is definable in a NIP structure M , then $u\mathcal{M}$ is Hausdorff in the τ -topology (equivalently, $H(u\mathcal{M}) = \{u\}$). It was first confirmed in [CGK26] for countable structures. The additional countability assumption was needed, as the proof relied on a structure theorem by Glasner from [Gla18] stated for metric tame minimal flows. Then [BZ24] used forcing techniques to remove this countability assumption. In fact, Theorem 11.7 from [BZ24] yields a generalization of RNC to the abstract context of tame flows which we stated in Fact 1.3 and called GRNC. The GRNC was fundamental for the results in [HK26] and is also fundamental in this paper. RNC also played a fundamental role in the recent beautiful paper [Sto26] by A. Stonestrom to describe the structure of Ellis groups and of quotients of $\bar{G}$ by certain local components in the NIP context (under some additional assumption).

There is also a variant of the above theory for first order theories in place of definable groups (see Subsection 6.2 for definitions, more details, and references). Let T be a first order theory and $\mathfrak{C} \models T$ a sufficiently saturated model. Let $\bar{c}$ be an enumeration of $\mathfrak{C}$ and $S_{\bar{c}}(\mathfrak{C})$ the space of complete types over $\mathfrak{C}$ which contain $\text{tp}(\bar{c}/\emptyset)$. Then $(S_{\bar{c}}(\mathfrak{C}), \text{tp}(\bar{c}/\mathfrak{C}), \text{Aut}(\mathfrak{C}))$ is an ambit with respect to the obvious action. This ambit plays a central role in the topological dynamics of first order theories, as it captures a lot of information about the underlying theory T (e.g., see [KPR18; KNS19; Hru19; KR20; HKP26; GHK25]).

Let $E := E(S_{\bar{c}}(\mathfrak{C}), \text{Aut}(\mathfrak{C}))$, $\mathcal{M}$ be a minimal left ideal in E , and $u \in \mathcal{M}$ an idempotent. A counterpart of Newelski's map was introduced in [KPR18]; it is an epimorphism $\rho: u\mathcal{M} \rightarrow \text{Gal}_{KP}(T)$ from the Ellis group to the *Kim-Pillay Galois group* of T . An analogous refinement of ρ to the one described above for definable groups was also obtained in the context of theories, which led in [KPR18; KR20] to important applications to the study of complexity of strong types which were not accessible by the previously known methods.

We then have an obvious counterpart of Newelski's Conjecture, and Theorem 0.7 in [KNS19] is a counterpart of Fact 1.8.

Fact 1.9 (Theorem 0.7 in [KNS19]). *If T is NIP and amenable (meaning that the flow $(S_{\bar{c}}(\mathfrak{C}), \text{Aut}(\mathfrak{C}))$ is amenable), then ρ is an isomorphism.*

Coming back to definable groups, in Subsection 6.1 we explain that Fact 1.8 follows from Theorem 1.2. It is not completely direct, as the quotient map $G \rightarrow \bar{G}/\bar{G}_M^{00}$ is the Bohr compactification of G below the ambit $(S_G(M), \text{tp}(e/M), G)$ (see [GPP14]) which is a factor of the Bohr compactification of G below the ambit $(S_{G,\text{ext}}(M), \text{tp}_{\text{ext}}(e/M), G)$, and we observe that the map ρ is the composition of the map $\hat{\theta}: u\mathcal{M} \rightarrow B$ with this factor map. Moreover, in Theorem 1.2, we assume hereditary amenability (equivalently amenability) of $(S_{G,\text{ext}}(M), G)$ while in Fact 1.8 only amenability of $(S_G(M), G)$ is assumed. However, by the results from [CPS14], assuming NIP, the aforementioned factor map is an isomorphism and amenability transfers from $S_G(M)$ to $S_{G,\text{ext}}(M)$, so all the obstacles vanish.

In the context of first order theories, we prove a counterpart of the aforementioned result from [GPP14]: the natural map $\text{Aut}(\mathfrak{C}) \rightarrow \text{Gal}_{KP}(T)$ is the Bohr compactification of $\text{Aut}(\mathfrak{C})$ below the ambit $(S_{\bar{\mathfrak{e}}}(\mathfrak{C}), \text{tp}(\bar{\mathfrak{e}}/\mathfrak{C}), \text{Aut}(\mathfrak{C}))$. We also prove that under NIP amenability of T implies amenability of the Ellis flow $E := E(S_{\bar{\mathfrak{e}}}(\mathfrak{C}), \text{Aut}(\mathfrak{C}))$, which is a counterpart of the aforementioned result from [CPS14]. We conjecture that NIP implies that the Bohr compactifications below the ambits $(S_{\bar{\mathfrak{e}}}(\mathfrak{C}), \text{tp}(\bar{\mathfrak{e}}/\mathfrak{C}), \text{Aut}(\mathfrak{C}))$ and $(E, \text{id}_{S_{\bar{\mathfrak{e}}}(\mathfrak{C})}, \text{Aut}(\mathfrak{C}))$ coincide (as for definable groups). Because we have not proved this last conjecture, Theorem 1.2 specializes to a slightly weaker statement than Fact 1.9: the map ρ is the composition of $\hat{\theta}$ with an epimorphism from the Bohr compactification below $(E, \text{id}_{S_{\bar{\mathfrak{e}}}(\mathfrak{C})}, \text{Aut}(\mathfrak{C}))$ to the one below $(S_{\bar{\mathfrak{e}}}(\mathfrak{C}), \text{tp}(\bar{\mathfrak{e}}/\mathfrak{C}), \text{Aut}(\mathfrak{C}))$ (which is $\text{Gal}_{KP}(T)$), so the fact that $\hat{\theta}$ is an isomorphism is weaker than the fact that ρ is an isomorphism.

Example 4.10 shows that in the general context one cannot improve Theorem 1.2 by saying that the composition of $\hat{\theta}$ with the canonical epimorphism from the Bohr compactification below $(E(X, G), \text{id}_X, G)$ to the one below (X, x_0, G) is an isomorphism. This leads to an interesting general problem on extracting a dynamical reason why something like that is true in the above two model-theoretic contexts. This and another problem in this spirit is stated in Subsection 6.2.4.

Besides recovering Fact 1.8 and a weaker form of Fact 1.9, Theorem 1.2 may lead to variants of Newelski's conjecture for some more specific model-theoretic ambits (e.g. the ones considered in [GHK25]). In Subsection 6.3, we obtain a local variant of Newelski's conjecture for definable groups, where the ambit in question is a hereditarily amenable ambit of the form $(S_{G,\Delta}(M), \text{tp}_{\Delta}(e/M), G)$ for a definable group G and a set of formulas Δ each of which has NIP and the set of the instances of the formulas from Δ is closed under the left translations (where $S_{G,\Delta}(M)$ denotes the space of complete Δ -types over M concentrated on G). Under the additional assumption that the set of the instances of the formulas from Δ is closed under group inversion, we give a description of the Bohr compactification of G below $(S_{G,\Delta}(M), \text{tp}_{\Delta}(e/M), G)$.

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2. PRELIMINARIES

Throughout, G denotes a discrete group. All compact spaces are assumed to be Hausdorff, all probability measures are regular Borel measures. We are following

the notation and conventions from our previous work, [HK26], some of them shortly recalled below.

2.1. Flows and invariant measures. A *subflow* of a flow is a nonempty closed G -invariant subset, and a *factor map* is a continuous surjective G -map between G -flows.

For a compact space X , let $C(X) := C(X, \mathbb{C})$, endowed with the uniform norm and the involution $f^*(x) := \overline{f(x)}$.

We write $\text{Prob}(X)$ for the compact convex space of regular Borel probability measures on X , equipped with the weak-* topology inherited from $C(X)^*$. If $\pi : X \rightarrow Y$ is Borel and $\mu \in \text{Prob}(X)$, then $\pi_*\mu$ denotes the pushforward of μ . For $g \in G$, we write $g_*\mu$ for the pushforward under $x \mapsto gx$, and put

$$\text{Prob}_G(X) := \{\mu \in \text{Prob}(X) : g_*\mu = \mu \text{ for every } g \in G\}.$$

An invariant measure is *ergodic* if it is an extreme point of $\text{Prob}_G(X)$. The flow (X, G) is *amenable* if $\text{Prob}_G(X) \neq \emptyset$, and it is *uniquely ergodic* if $\text{Prob}_G(X)$ is a singleton. It is *hereditarily amenable* if every subflow of X is amenable. The action of G on $C(X)$ is given by $(g \cdot f)(x) := f(g^{-1}x)$.

Fact 2.1 (Theorem 4.2 in [HK26]). *Every minimal tame amenable flow is uniquely ergodic.*

Recall also the definition of a group extension of a flow.

Definition 2.2. Let $\chi : (Y, G) \rightarrow (X, G)$ be a homomorphism of minimal flows. We say that χ is a *group extension* if there is a compact Hausdorff group K acting faithfully and continuously on the right on Y in such a way that:

- (1) for every $y \in Y$, $\chi^{-1}(\chi(y)) = y \cdot K$, and
- (2) for all $y \in Y$, $g \in G$ and $k \in K$, $(gy) \cdot k = g(y \cdot k)$.

In the above situation, we also say that (Y, G, K) is a *group extension* of (X, G) .

2.2. Ellis semigroups and tameness. The *enveloping semigroup* of a flow (X, G) is

$$E(X, G) := \overline{\{g_X : g \in G\}}^{X^X}, \quad g_X(x) := gx,$$

where X^X carries the product topology. Multiplication is composition: $(pq)(x) := p(q(x))$. For every fixed $q \in E(X, G)$, the map $p \mapsto pq$ is continuous. For $x \in X$, the evaluation at x is the continuous map

$$\text{ev}_x : E(X, G) \longrightarrow X, \quad \text{ev}_x(p) := p(x),$$

and $\text{ev}_x[E(X, G)] = \overline{Gx}$. A factor map $\pi : X \rightarrow Y$ induces a continuous surjective G -equivariant semigroup homomorphism

$$\pi_E : E(X, G) \longrightarrow E(Y, G)$$

satisfying $\pi \circ p = \pi_E(p) \circ \pi$ for every $p \in E(X, G)$.

Put $E := E(X, G)$. We write $\mathcal{M} \leq_m E$ for a minimal left ideal. If $u^2 = u \in \mathcal{M}$, then $K := u\mathcal{M}$ is the corresponding *Ellis group*. For $p \in E$ and $A \subseteq E$, define the Ellis circle operation by

$$p \circ A := \left\{ q \in E : \text{there are nets } (g_i)_i \subseteq G \text{ and } (a_i)_i \subseteq A \right. \\ \left. \text{such that } g_i \rightarrow p \text{ and } g_i a_i \rightarrow q \right\}.$$

For $A \subseteq K$, put

$$\text{cl}_\tau(A) := K \cap (u \circ A) = u(u \circ A).$$

The operator cl_τ is a closure operator on K . The topology determined by this closure operator is called the τ -topology. The τ -topology is coarser than the topology on K inherited from E . Moreover, (K, τ) is always a quasi-compact T_1 semitopological group, that is, multiplication is separately continuous. No Hausdorffness is asserted here. See [Gla76] for the classical construction in the βG setting, and [Rze18, Appendix A] for the general formulation used here.

Whenever (K, τ) is a compact Hausdorff topological group, $\mathfrak{h}_K$ denotes its normalized Haar measure.

The next fact follows from Facts 2.3 and 2.5 in [KLM22].

Fact 2.3. *Let (X, G) and (Y, G) be flows, and let $\Phi: E(X, G) \rightarrow E(Y, G)$ be a continuous surjective G -equivariant semigroup homomorphism. Let $\mathcal{M} \trianglelefteq_m E(X, G)$, $u^2 = u \in \mathcal{M}$, and put $\mathcal{N} := \Phi[\mathcal{M}]$, $v := \Phi(u)$, $K := u\mathcal{M}$, $L := v\mathcal{N} = \Phi[K]$. Then $\mathcal{N} \trianglelefteq_m E(Y, G)$, $v^2 = v \in \mathcal{N}$, and the restriction*

$$\Phi|_K: (K, \tau) \rightarrow (L, \tau)$$

is a continuous group epimorphism and a quotient map for the τ -topologies. (In particular, the conclusion applies when $\Phi = \pi_E$ is induced by a factor map $\pi: X \rightarrow Y$, and when $Y \subseteq X$ is a subflow and $\Phi: E(X, G) \rightarrow E(Y, G)$ is the restriction map.)

There are many equivalent definitions of tame flows (see Theorems 2.4, 3.2 and Definition 3.1 in [GM18]). We give the one which immediately points to a strong connection with the NIP property in model theory.

A sequence $(f_n)_{n < \omega}$ of real valued functions on a set X is said to be *independent* if there exist real numbers $a < b$ such that

$$\bigcap_{n \in P} f_n^{-1}((-\infty, a)) \cap \bigcap_{n \in Q} f_n^{-1}((b, \infty)) \neq \emptyset$$

for all finite disjoint subsets P, Q of ω .

Definition 2.4. Let (X, G) be a flow. A function $f \in C(X, \mathbb{R})$ (i.e. a continuous real valued function on X) is *tame* if the family of translates $\{gf : g \in G\}$ does not contain an infinite independent sequence. The flow (X, G) is *tame* if all functions in $C(X, \mathbb{R})$ are tame.

Let (X, G) be a flow. Put $E := E(X, G)$, choose $\mathcal{M} \trianglelefteq_m E$ and $u^2 = u \in \mathcal{M}$. Choose $x \in uX$, so $ux = x$. We set

$$\text{Gal}(X, x) := \{\eta \in u\mathcal{M} : \eta(x) = x\}.$$
²

Remark 2.5. $\text{Gal}(X, x)$ is closed in the τ -topology. This is standard, for example, we can re-use the short argument of the first paragraph of [Gla76, Chapter IX.2].

Fact 2.6 (Theorem 8.2 in [GM13]). *If (Y, G) is tame, then the canonical restriction homomorphism*

$$\text{res}: E(\text{Prob}(Y), G) \rightarrow E(Y, G)$$

is a topological semigroup isomorphism.

²We will be also using the same notation for Galois groups computed in Ellis semigroups of some ambits which project onto X .

Fact 2.7 ([HK26, Corollary 5.6]). *For a tame flow (X, G) , hereditary amenability of (X, G) is equivalent to amenability (and also to hereditary amenability) of $(E(X, G), G)$.*

Fact 2.8. *Let (X, G) be a tame hereditarily amenable flow. Put $E := E(X, G)$, choose $\mathcal{M} \leq_m E$ and $u^2 = u \in \mathcal{M}$. Then the following statements hold:*

- (1) *$(\mathcal{M}, G)$ is amenable.*
- (2) *The Ellis group $u\mathcal{M}$ is dense in $\mathcal{M}$.*

Proof. By Fact 2.7, the Ellis flow (E, G) is hereditarily amenable, and therefore $\mathcal{M}$ is amenable. The second point follows from [HK26, Lemma 5.7]. $\square$

Recall that a flow (X, G) is *proximal* if for every $x_1, x_2 \in X$ there exists $\eta \in E(X, G)$ such that $\eta(x_1) = \eta(x_2)$.

The following easy fact is folklore.

Fact 2.9. *Let $(\mathcal{A}, a_0, G)$ be an ambit. Then the following conditions are equivalent.*

- (1) *There exists a left continuous semigroup operation $*$ extending the action of G (i.e., $(ga_0) * q = gq$).*
- (2) *There exists a left continuous semigroup operation $*$ on $\mathcal{A}$ such that the function $\Phi: \mathcal{A} \rightarrow E(\mathcal{A}, G)$ defined by $\Phi(p) := l_p$ (where $l_p: \mathcal{A} \rightarrow \mathcal{A}$ is given by $l_p(q) := p * q$) is a semigroup epimorphism such that $\Phi(ga_0) = g_{\mathcal{A}}$ (where $g_{\mathcal{A}}(a) := ga$); then Φ is also an isomorphism of G -flows.*
- (3) *$(\mathcal{A}, a_0, G)$ is isomorphic to $(E(X, G), \text{id}, G)$ for some flow (X, G) .*

For further background on tame flows and Ellis semigroups, see [Aus88; Gla76; Gla06; GM18; CH23].

2.3. Operator theory. Let (Y, G) be a flow and let $\mu \in \text{Prob}_G(Y)$. For $1 \leq p \leq \infty$, we write $L^p(Y, \mu) = L^p(Y, \mu; \mathbb{C})$. The inner product on $L^2(Y, \mu)$ is

$$\langle F_1, F_2 \rangle_{L^2(Y, \mu)} = \int_Y F_1(y) \overline{F_2(y)} d\mu(y),$$

where $F_1, F_2 \in L^2(Y, \mu)$. For $g \in G$ and $F = [f] \in L^2(Y, \mu)$, define $U_g F = [f \circ g^{-1}]$. This is well-defined because μ is G -invariant. As usual, after choosing representatives, we write $(U_g F)(y) = F(g^{-1}y)$ with equality understood μ -almost everywhere.

The G -invariance of μ implies that

$$\langle U_g F_1, U_g F_2 \rangle_{L^2(Y, \mu)} = \langle F_1, F_2 \rangle_{L^2(Y, \mu)}$$

for all $F_1, F_2 \in L^2(Y, \mu)$. Since $U_g^{-1} = U_{g^{-1}}$, each U_g is unitary. Moreover, $U_g U_h = U_{gh}$ for every $g, h \in G$, so $g \mapsto U_g$ is a unitary representation of G on $L^2(Y, \mu)$.

For a surjective continuous map $\varphi: Y \rightarrow Z$, we denote by $\varphi^*: C(Z) \rightarrow C(Y)$, $\varphi^*(F) := F \circ \varphi$, the corresponding pullback map. If $\mu \in \text{Prob}_G(Y)$ and $\nu = \varphi_* \mu$, then

$$\|\varphi^* F\|_{L^2(Y, \mu)}^2 = \int_Y |F \circ \varphi|^2 d\mu = \int_Z |F|^2 d\nu = \|F\|_{L^2(Z, \nu)}^2$$

for every $F \in C(Z)$. Since $C(Z)$ is dense in $L^2(Z, \nu)$, the pullback extends uniquely to a linear isometry

$$\varphi^*: L^2(Z, \nu) \rightarrow L^2(Y, \mu).$$

Its range satisfies $\text{ran}(\varphi^*) = \overline{\varphi^*[C(Z)]}$. Indeed, this follows from the density of $C(Z)$ in $L^2(Z, \nu)$ and the fact that the range of an isometry is closed. Therefore, the isometry φ^* is a unitary isomorphism if and only if $\overline{\varphi^*[C(Z)]} = L^2(Y, \mu)$.

Let $\mathcal{H}$ be a complex Hilbert space and let $B(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$. The *strong operator topology* (SOT) on $B(\mathcal{H})$ is the topology generated by the seminorms

$$p_\xi(T) := \|T\xi\|_{\mathcal{H}} \quad (\xi \in \mathcal{H}).$$

Equivalently, a net $(T_i)_i$ in $B(\mathcal{H})$ converges to $T \in B(\mathcal{H})$ in SOT if and only if

$$\|T_i\xi - T\xi\|_{\mathcal{H}} \longrightarrow 0 \quad \text{for every } \xi \in \mathcal{H}.$$

We equip the unitary group $U(\mathcal{H})$ and each of its subgroups with the corresponding subspace topology.

Definition 2.10 (See Section 3.1 in [Fol16]). Let B be a topological group and let $\mathcal{H}$ be a complex Hilbert space. A *unitary representation* of B on $\mathcal{H}$ is a group homomorphism

$$\rho : B \longrightarrow U(\mathcal{H})$$

which is continuous when $U(\mathcal{H})$ is equipped with the strong operator topology. Equivalently, for every $\xi \in \mathcal{H}$, the orbit map

$$B \longrightarrow \mathcal{H}, \quad b \longmapsto \rho(b)\xi,$$

is norm-continuous.

In the proof of Lemma 5.1, we define a unitary representation in the following way: let B be a compact Hausdorff group, and let $K \leq U(\mathcal{H})$ carry the subspace topology induced by SOT. If $\rho_0 : B \longrightarrow K$ is a continuous group homomorphism, then the composition $B \xrightarrow{\rho_0} K \hookrightarrow U(\mathcal{H})$ is a unitary representation of B on $\mathcal{H}$.

Fact 2.11 (Theorem 5.2 in [Fol16]). *Let B be a compact Hausdorff group.*

- (1) *Every irreducible unitary representation of B is finite-dimensional.*
- (2) *Every unitary representation of B is a Hilbert direct sum of irreducible unitary representations. More precisely, if $\mathcal{H}$ is a complex Hilbert space and*

$$\rho : B \longrightarrow U(\mathcal{H})$$

is a continuous unitary representation, then there is a family $(\mathcal{H}_i)_{i \in I}$ of pairwise orthogonal closed $\rho(B)$ -invariant subspaces such that

$$\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i$$

and each restricted representation $\rho|_{\mathcal{H}_i} : B \longrightarrow U(\mathcal{H}_i)$ is irreducible.

For a compact group B , we write $\mathfrak{h}_B$ for normalized Haar probability. If $H \leq B$ is closed, then $\mathfrak{h}_{B/H}$ is the pushforward of $\mathfrak{h}_B$ under the quotient map $B \rightarrow B/H$. In the fact below, the left-translation condition in (i) is the function-algebra form of the condition called right invariance in the terminology of [TT71].

Fact 2.12 (Theorem 2 in [TT71]). *Let B be a compact Hausdorff group, and let $\mathcal{N}$ be a weak-* closed unital *-subalgebra of $L^\infty(B, \mathfrak{h}_B)$. For $b, h \in B$, define*

$$(L_b F)(c) := F(b^{-1}c), \quad (R_h F)(c) := F(ch).$$

Then the following are equivalent:

(i) $\mathcal{N}$ is invariant under every left translation, that is,

$$L_b[\mathcal{N}] = \mathcal{N} \quad \text{for every } b \in B;$$

(ii) there is a unique closed subgroup $H \leq B$ such that

$$\mathcal{N} = L^\infty(B, \mathfrak{h}_B)^H := \{F \in L^\infty(B, \mathfrak{h}_B) : R_h F = F \text{ for every } h \in H\}.$$

In this correspondence, the subgroup H is recovered from $\mathcal{N}$ by

$$H = \{h \in B : R_h F = F \text{ for every } F \in \mathcal{N}\}.$$

Equivalently, if $p_H : B \rightarrow B/H$ is the quotient map, then

$$\mathcal{N} = p_H^*[L^\infty(B/H, \mathfrak{h}_{B/H})].$$

3. BOHR COMPACTIFICATIONS, EQUICONTINUOUS FACTORS AND CATEGORY OF AMBITS

Let (X, x_0, G) be an ambit.

3.1. Bohr compactifications below ambits. The following definition is a natural extension of the notion of the Bohr compactification of the group G . It recovers the usual definition when applied to the universal G -ambit.

Definition 3.1 (Definition (A)). By a *compactification of G below (X, x_0, G)* we mean a group homomorphism $c : G \rightarrow K$ with dense image for some compact group K such that $c = f \circ \text{ev}_{x_0}$, where $\text{ev}_{x_0} : G \rightarrow X$ is given by $\text{ev}_{x_0}(g) := gx_0$ and $f : X \rightarrow K$ is an epimorphism of G -ambits (mapping x_0 to the neutral element e_K of K) with the G -action on K given by $g \cdot k := c(g)k$.

A *Bohr compactification below (X, x_0, G)* is a universal compactification $\mathfrak{b}$ of G below (X, x_0, G) in the sense that for any compactification $c : G \rightarrow K$ of G below (X, x_0, G) there is a (unique) continuous homomorphism $h : B \rightarrow K$ such that $c = h \circ \mathfrak{b}$. Abusing terminology, the target group B of $\mathfrak{b}$ may be occasionally also called the Bohr compactification below (X, x_0, G) .

Remark 3.2. For any ambit (X, x_0, G) a Bohr compactification of G below (X, x_0, G) exists and is unique up to isomorphism, so will be called *the* Bohr compactification of G below (X, x_0, G) .

Proof. Uniqueness follows from the density of the image. For the existence, take a set $\{c_i : G \rightarrow K_i\}_{i \in I}$ of all (up to isomorphism) compactifications of G below (X, x_0, G) . Put $K := \overline{\{(c_i(g))_{i \in I} : g \in G\}} \subseteq \prod_{i \in I} K_i$. It is clear that the map $\mathfrak{b} : G \rightarrow K$ given by $\mathfrak{b}(g) := (c_i(g))_{i \in I}$ is a compactification below (X, x_0, G) which is universal. $\square$

Notice that the epimorphism f in Definition 3.1 is unique (by the density of Gx_0 in X and continuity of f). So with each compactification c of G below (X, x_0, G) we have a uniquely associated triple (K, c, f) . This leads to the following equivalent definition of the Bohr compactification of G below (X, x_0, G) which will be particularly useful in this paper.

Definition 3.3 (Definition (B)). We say that a triple (K, c, f) is a *compactification of G below (X, x_0, G)* if the following diagram commutes

$$\begin{array}{ccc} G & \xrightarrow{c} & K \\ \text{ev}_{x_0}: g \mapsto g(x_0) \searrow & & \nearrow f \\ & X & \end{array}$$

where K is a compact group, c is a group homomorphism, and f is an epimorphism of G -ambits (mapping x_0 to e_K) where the G -action on K is given by $g \cdot k := c(g)k$.

The *Bohr compactification of G below (X, x_0, G)* is a triple $(B, \mathfrak{b}, w)$ as above which is universal in the sense that for any other triple $(B', \mathfrak{b}', w')$ as above there exists a (unique) continuous group epimorphism $h: B \rightarrow B'$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{w} & B \\ & \searrow w' & \nearrow h \\ & B' & \end{array}$$

Bearing in mind the paragraph preceding Definition 3.3, we leave as an easy exercise to check the equivalence of both definitions. Note also that the map h in the above definition is an epimorphism of G -ambits.

Moreover, we may describe the Bohr compactification of G below (X, x_0, G) by mimicking some techniques from model theory (cf. [GPP14]). Let us sketch this approach shortly.

Definition 3.4. (1) Let Y be a compact Hausdorff space. We call a map $f: G \rightarrow Y$ *X -definable* if for every disjoint compact sets $C_0, C_1 \subseteq Y$ there is a continuous function $\varphi: X \rightarrow [0, 1]$ such that

$$f^{-1}[C_0] \subseteq \text{ev}_{x_0}^{-1}[\varphi^{-1}[\{0\}]], \quad f^{-1}[C_1] \subseteq \text{ev}_{x_0}^{-1}[\varphi^{-1}[\{1\}]].$$

(2) An X -definable homomorphism $f: G \rightarrow H$, where H is a compact Hausdorff group, with $f[G]$ being a dense subset of H , is called an *X -definable compactification of G* .

The universal X -definable compactification of G is precisely the Bohr compactification of G below (X, x_0, G) . This might be proved by adapting arguments from [GPP14], but as we do not refer to this approach in what follows, we skip the proof.

Working with a G -ambit (X, x_0, G) and $E := E(X, G)$, we have the Bohr compactification of G below (X, x_0, G) :

$$\begin{array}{ccc} G & \xrightarrow{\mathfrak{b}_1} & B_1 \\ g \mapsto g(x_0) \searrow & & \nearrow w_1 \\ & X & \end{array}$$

which will be called the *X-Bohr compactification* of G . Then we also have the Bohr compactification of G below (E, id_X, G) :

$$\begin{array}{ccc} G & \xrightarrow{\mathfrak{b}_2} & B_2 \\ & \searrow g \mapsto g \text{id}_X & \nearrow w_2 \\ & E & \end{array}$$

which will be called the *E-Bohr compactification* of G . We keep the above notation $(\mathfrak{b}_1, \mathfrak{b}_2, w_1, w_2, B_1 \text{ and } B_2)$ for the rest of the paper.

Both (B_1, e_{B_1}, G) and (B_2, e_{B_2}, G) are G -ambits with the G -action induced by $\mathfrak{b}_1$ and $\mathfrak{b}_2$, respectively. These flows are minimal equicontinuous (also called almost periodic) and satisfy $E(B_1, G) \cong B_1$ and $E(B_2, G) \cong B_2$. In particular, they are WAP.

Remark 3.5. $w_2: E \rightarrow B_2$ is a continuous semigroup epimorphism.

Proof. Consider $\eta, \xi \in E$ and a net $(g_i)_{i \in I} \subseteq G$ with $\lim_{i \in I} g_i = \eta$ (which formally means that $\lim_{i \in I} g_i \text{id}_X = \eta$) and simply compute, using left-continuity of the product in E :

$$\begin{aligned} w_2(\eta\xi) &= w_2((\lim_i g_i)\xi) = w_2(\lim_i (g_i\xi)) \\ &= \lim_i w_2(g_i\xi) = \lim_i \mathfrak{b}_2(g_i)w_2(\xi) \\ &= (\lim_i \mathfrak{b}_2(g_i))w_2(\xi) = (\lim_i w_2(g_i \text{id}_X))w_2(\xi) \\ &= w_2(\lim_i (g_i \text{id}_X))w_2(\xi) = w_2(\eta)w_2(\xi) \end{aligned}$$

□

Remark 3.6. Let $\pi_0: E \rightarrow X$ be given by $\pi_0(\eta) := \eta(x_0)$. Then, by universality of B_2 , there is a unique continuous group epimorphism $\mathfrak{f}: B_2 \rightarrow B_1$ completing the following diagram:

$$\begin{array}{ccccc} & & G & & \\ & \swarrow \mathfrak{b}_2 & & \searrow \mathfrak{b}_1 & \\ & E & \xrightarrow{\pi_0} & X & \\ & \swarrow w_2 & & \searrow w_1 & \\ B_2 & \text{-----} & & & B_1 \\ & & \mathfrak{f} & & \end{array}$$

Remark 3.7. If G is a topological group and $\text{Sam}(G)$ is its Samuel compactification (*greatest G -ambit* in terminology of [Usp02]), then the Bohr compactification B of G is the maximal compact Hausdorff group factor of $\text{Sam}(G)$. Note also, that B is in fact the $\text{Sam}(G)$ -Bohr compactification of G and also the $E(\text{Sam}(G), G)$ -Bohr compactification of G . Moreover, for any ambit (X, x_0, G) we have the following commuting diagram of epimorphisms of G -flows (and [semi]groups where it makes sense):

$$\begin{array}{ccccc} \text{Sam}(G) & \twoheadrightarrow & E & \twoheadrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ B & \twoheadrightarrow & B_2 & \twoheadrightarrow & B_1 \end{array}$$

3.2. Maximal Equicontinuous Factors. Recall that (X, x_0, G) is a G -ambit. Let

$$(X, G) \xrightarrow{\pi} (Z, G)$$

be the Maximal Equicontinuous Factor (MEF) of (X, G) , which will be called the X -MEF of (X, G) [Aus88, Chapter 9]. Set $K_1 := E(Z, G) \leq \text{Homeo}(Z)$, which is a compact group [Gla76, Theorem I.3.3(4)]. Note that π lifts up to the Ellis semigroups as

$$\pi_*: E \rightarrow K_1$$

We need a slight generalization of a classical theorem. The original theorem is stated for minimal flows, but being an ambit is sufficient to repeat the same proof:

Fact 3.8 (Theorem I.3.3(5) in [Gla76]). *Let (Y, y_0, G) be an equicontinuous ambit with $K := E(Y, G)$. Then*

$$(Y, G) \cong (K/F, G)$$

for $F := \{k \in K : ky_0 = y_0\} \leq K$.

By definition, $Z = X/Q^*$, where Q^* denotes the smallest closed G -invariant equivalence relation on X such that the flow $(X/Q^*, G)$ is equicontinuous. Recall that Q^* contains the relation Q of regional proximality (cf. Theorem 3 in Chapter 9 of [Aus88]). By the above fact, $(Z, G) \cong (K_1/F, G)$ for the closed subgroup $F := \{k \in K_1 : kz_0 = z_0\}$, where $z_0 := \pi(x_0)$.

Remark 3.9. By universality of the X -MEF, there is a unique epimorphisms of flows $(X/Q^*, G) \rightarrow (B_1, G)$, which lifts to the Ellis semigroups as a continuous epimorphism of groups $\mathfrak{t}: K_1 \rightarrow B_1$ ³.

Example 3.10. [X -MEF $\neq X$ -Bohr comp.] Let X be the 2-dimensional sphere $S^2 \cong \text{SO}(3)/\text{SO}(2)$. Set $G = \text{SO}(3)$. Let $x_0 \in X$ be the point $(0, 0, 1)$. We have that (X, x_0, G) is equicontinuous, because G is collection of isometries. Hence, (X, x_0, G) is already its own X -MEF. Note that $E(X, G) = \text{SO}(3)$, and if we apply Fact 3.8, we indeed obtain that $S^2 = X = \text{SO}(3)/\text{SO}(2)$ as G -flows. Now, let B_1 be the Bohr compactification of G below X . We have that (B_1, G) is an equicontinuous flow and $w_1: X \rightarrow B_1$ is an epimorphism of G -flows. Therefore, we obtain the following diagram of G -epimorphisms:

$$\begin{array}{ccc} \text{SO}(3) & & \\ \downarrow \iota & \searrow f := w_1 \circ \iota & \\ \text{SO}(3)/\text{SO}(2) & \xrightarrow{w_1} & B_1 \end{array}$$

Note that f is a group epimorphism and $\text{SO}(2) \leq \ker f$. Because $\text{SO}(3)$ is simple, we have $\ker f = \text{SO}(3)$, and so $B_1 = \{e_{B_1}\}$.

By Q_E^* we denote the smallest closed G -invariant equivalence relation on $E = E(X, G)$ such that the flow $(E/Q_E^*, G)$ is equicontinuous. (it contains the relation Q_E of regional proximity on E). Then $(E/Q_E^*, G)$ is the MEF of (E, G) which will be called the E -MEF of (X, G) . Set $K_2 := E(E/Q_E^*, G) \leq \text{Homeo}(E/Q_E^*)$, which is a compact Hausdorff group. We fix $u^2 = u \in \mathcal{M} \leq_m E$ (an idempotent in a left minimal ideal of E).

³We could try to reverse the last arrow by universality of the Bohr compactification $\mathfrak{b}_1$, but for that we would need a map $X \rightarrow K_1$, whereas we only have a map $X \rightarrow K_1/F$.

Proposition 3.11. *$(E/Q_E^*, G) \cong (K_2, G)$, there exists a unique topological group structure on E/Q_E^* extending the (left) action of G on E/Q_E^* , and there exists an epimorphism $u\mathcal{M}/H(u\mathcal{M}) \rightarrow K_2$ of compact topological groups.*

Proof. We have $\sigma: (E, G) \rightarrow (E/Q_E^*, G)$, which lifts up to

$$\sigma_*: E(E, G) \rightarrow K_2.$$

On the other hand, $E \cong E(E, G)$, and the composition of these two maps is a semigroup and G -flow epimorphism $\sigma_E: E \rightarrow K_2$ such that $\sigma_E|_{u\mathcal{M}}: u\mathcal{M} \rightarrow K_2$ is a group epimorphism and topological quotient map with respect to the τ -topologies (using Fact 2.3).

By Fact 3.8, we have an isomorphism $f: (E/Q_E^*, G) \rightarrow (K_2/F_E, G)$ given by

$$f(\xi(\text{id}/Q_E^*)) := \xi/F_E,$$

for some closed subgroup $F_E \leq K_2$. So $(E, G) \rightarrow (K_2/F_E, G)$ is the maximal equicontinuous factor. Note that $E(K_2, G) \cong K_2$ and so $\sigma_E: (E, G) \rightarrow (K_2, G)$ is also an equicontinuous factor. The following diagram of ambits commutes:

$$\begin{array}{ccccc} (E, G) & \xrightarrow{\sigma} & (E/Q_E^*, G) & \xrightarrow{f, \cong} & (K_2/F_E, G) \\ \eta(\text{id}_X) \mapsto \eta \downarrow \cong & \searrow \sigma_E & \uparrow \xi \mapsto \xi(\sigma(\text{id}_X)) & \nearrow \xi \mapsto \xi/F_E & \\ E(E, G) & \xrightarrow{\sigma_*} & (E(E/Q_E^*, G), G) = (K_2, G) & & \end{array}$$

Hence, by maximality of $(E/Q_E^*, G) \cong (K_2/F_E, G)$, the quotient map $K_2 \rightarrow K_2/F_E$ is an isomorphism, so F_E is trivial and $(E/Q_E^*, G) \cong (K_2, G)$.

So a posteriori there is a unique topological group structure on E/Q_E^* extending the left action of G on E/Q_E^* .

By Corollary A.15 from [Rze18], we have the following diagram of continuous group epimorphisms:

$$\begin{array}{ccc} u\mathcal{M} & \xrightarrow{\sigma_E|_{u\mathcal{M}}} & K_2 \\ & \searrow & \nearrow \\ & u\mathcal{M}/H(u\mathcal{M}) & \end{array}$$

□

Proposition 3.12. *There exists a semigroup and G -flow isomorphism $B_2 \cong K_2$.*

Proof. We work with the continuous semigroup epimorphism σ_E from the proof of Proposition 3.11, which is also a G -ambit epimorphism.

The G -ambit epimorphism $w_2: (E, G) \rightarrow (B_2, G)$ is in fact an equicontinuous factor. By the proof of Proposition 3.11, σ_E is the MEF of (E, G) . Thus, there is a unique epimorphism of G -ambits $f_1: (K_2, G) \rightarrow (B_2, G)$ such that the following diagram commutes:

$$\begin{array}{ccc} (E, G) & & \\ \sigma_E \downarrow & \searrow w_2 & \\ (K_2, G) & \xrightarrow{f_1} & (B_2, G). \end{array}$$

Since σ_E and w_2 are semigroup epimorphisms (by Remark 3.5), so is f_1 .

On the other hand, the existence of σ_E allows us to use the universality of the Bohr compactification below E and find a unique continuous epimorphism $\mathfrak{f}_2: B_2 \rightarrow K_2$ making the following diagram commutative (where $\iota: G \rightarrow E$ is the usual map given by $\iota(g) := g \text{id}_X$):

$$\begin{array}{ccc}
 & G & \\
 \mathfrak{b}_2 \swarrow & \downarrow \iota & \searrow \sigma_E \circ \iota \\
 & E & \\
 w_2 \swarrow & & \searrow \sigma_E \\
 B_2 & \xrightarrow{\mathfrak{f}_2} & K_2
 \end{array}$$

Note that $\mathfrak{f}_2$ is also an epimorphism of G -ambits, $(B_2, G) \rightarrow (K_2, G)$.

Combining the universality properties of the E -MEF and the Bohr compactification below E , gives us

$$\begin{array}{ccccc}
 & (E, G) & & E & \\
 \sigma_E \swarrow & \downarrow w_2 & \searrow \sigma_E & \swarrow w_2 & \searrow w_2 \\
 (K_2, G) & \xrightarrow[\mathfrak{f}_1]{\exists!} & (B_2, G) & \xrightarrow[\mathfrak{f}_2]{\exists!} & (K_2, G) \\
 & \searrow \text{id}_{K_2} & & \swarrow \text{id}_{B_2} & \\
 & & & &
 \end{array}$$

Therefore $\mathfrak{f}_2 \mathfrak{f}_1 = \text{id}_{K_2}$ and $\mathfrak{f}_1 \mathfrak{f}_2 = \text{id}_{B_2}$. $\square$

Let us comment and provide some intuition. On the level of (X, G) , by Remark 3.9, we have the epimorphism $\mathfrak{t}: K_1 \rightarrow B_1$, but we lack a flow epimorphism from (X, G) to (K_1, G) to make $\mathfrak{t}$ an isomorphism, and Example 3.10 shows that $\mathfrak{t}$ need not be an isomorphism. However, on the level of (E, G) , we actually pass over this issue and obtain an isomorphism $K_2 \cong B_2$. Let $\mathfrak{f}: B_2 \rightarrow B_1$ be the continuous group epimorphism from Remark 3.6. Recall the Ellis semigroup and G -flow epimorphism $\pi_*: E \rightarrow K_1$ from the beginning of Subsection 3.2. By universality of $(B_2, \mathfrak{b}_2, w_2)$, there is a unique continuous group epimorphism $h: B_2 \rightarrow K_1$ such that $\pi_* = h \circ w_2$. The following diagram of continuous group epimorphism commutes:

$$\begin{array}{ccc}
 B_1 & \xleftarrow{\mathfrak{f}} & B_2 \\
 \uparrow \mathfrak{t} & \nearrow h & \uparrow \mathfrak{f}_1, \cong \\
 K_1 & & K_2 \longleftarrow u\mathcal{M}/H(u\mathcal{M})
 \end{array}$$

Note that there is no need to introduce higher Bohr compactifications or higher MEFs of (X, G) , simply because $E \cong E(E, G)$.

Corollary 3.13. *If there exists a left-continuous semigroup operation on X extending the action of G on X , then $B_2 \cong B_1 \cong K_1 \cong K_2$. Thus, if $K_1 \not\cong B_1$, then such a semigroup operation does not exist.*

Proof. By assumption and Fact 2.9, the ambits (X, x_0, G) and $(E(X, G), \text{id}_X, G)$ are isomorphic, so $B_2 \cong B_1$ and $K_2 \cong K_1$. The fact that $B_2 \cong K_2$ follows from Proposition 3.12. $\square$

3.3. Category of ambits below an Ellis flow. Our goal is to develop some topological dynamics “below” any given G -ambit $(\mathcal{A}, a_0, G)$ on which there is a left continuous semigroup operation extending the action of G . This generalizes the classical topological dynamics which takes place below the universal G -ambit. Some basic issues are similar, but there are also essential new obstacles. An important motivation is that one can impose interesting dynamical conditions (e.g., tameness) on $(\mathcal{A}, a_0, G)$ which are almost never satisfied by the universal G -ambit.

Let $(\mathcal{A}, a_0, G)$ be an ambit on which there exists a left continuous semigroup operation $*$ extending the action of G (i.e. $(ga_0) * q = gq$). Recall that, by Fact 2.9, the existence of a left continuous semigroup operation $*$ on $\mathcal{A}$ extending the action of G is equivalent to saying that $(\mathcal{A}, a_0, G)$ is isomorphic to $(E(X, G), \text{id}_X, G)$ for some flow (X, G) .

We will be interested in the category $\mathcal{C}$ of ambits which are G -factors of $(\mathcal{A}, a_0, G)$. Note that the semigroup $(\mathcal{A}, *)$ acts naturally on the left on every ambit in $\mathcal{C}$. Indeed, take an ambit (X, x_0, G) from $\mathcal{C}$. By Remark 3.14, for any $x \in X$ there exists a flow homomorphism $\pi_x: \mathcal{A} \rightarrow X$ with $\pi_x(a_0) = x$. We define the action of $(\mathcal{A}, *)$ on X by $a \bullet x := \pi_x(a)$ (associativity is left as an exercise). This action is left continuous and extends the action of G (i.e. $(ga_0) \bullet x = gx$), because π_x is a flow homomorphism. From now on, we will skip $\bullet$.

Remark 3.14. If (X, x_0, G) belongs to $\mathcal{C}$, then for every $x \in X$ the ambit $(\overline{Gx}, x, G)$ also belongs to $\mathcal{C}$.

Proof. Take an epimorphism $f: (\mathcal{A}, a_0) \rightarrow (X, x_0)$ of G -ambits. There is $a_x \in \mathcal{A}$ which is mapped to x . We can find a homomorphism (of pointed G -flows) $h: (\mathcal{A}, a_0) \rightarrow (\mathcal{A}, a_x)$ (just $z \mapsto z * a_x$). Then the composition $f \circ h: (\mathcal{A}, a_0) \rightarrow (\overline{Gx}, x)$ witnesses that $(\overline{Gx}, x, G)$ belongs to $\mathcal{C}$. $\square$

By the above remark, abusing terminology, we can say that a minimal G -flow M belongs to $\mathcal{C}$ if (M, m, G) belongs to $\mathcal{C}$ for some (equivalently, every) $m \in M$. It is very easy to see that any minimal subflow of a product of minimal G -flows in the category $\mathcal{C}$ still belongs to $\mathcal{C}$. Thus, standard proofs (cf. [Gla76, Proposition II.4.2] and [KP17, Remark 1.22]) yield the following fact (where uniqueness is up to isomorphism). Item (3) is a particular case of Remark 3.2.

- Remark 3.15.**
- (1) There exists a unique universal minimal proximal G -flow $(\pi_{\mathcal{A}}, G)$ in $\mathcal{C}$.
 - (2) For a given minimal G -flow (X, G) in $\mathcal{C}$ there exists a unique universal group extension of X in the category $\mathcal{C}$. Denote the universal group extension of $(\pi_{\mathcal{A}}, G)$ in $\mathcal{C}$ by $\psi_{\mathcal{A}}: (\pi_{\mathcal{A}}^{\#}, G, \Sigma(G)_{\mathcal{A}}) \rightarrow (\pi_{\mathcal{A}}, G)$.
 - (3) There exists a unique Bohr compactification $\mathfrak{b}_{\mathcal{A}}: G \rightarrow G_{\mathcal{A}}^{\text{Bohr}}$ of G with the induced minimal flow $(G_{\mathcal{A}}^{\text{Bohr}}, G)$ belonging to $\mathcal{C}$.
 - (4) Let (X, x_0, G) be a G -ambit and set $(\mathcal{A}, a_0, G) = (E(X, G), \text{id}_X, G)$. We have that $\mathfrak{b}_{\mathcal{A}}: G \rightarrow G_{\mathcal{A}}^{\text{Bohr}}$ coincides with the Bohr compactification of G below $(E(X, G), \text{id}_X, G)$, i.e. with $\mathfrak{b}_2: G \rightarrow B_2$.

A *compactification flow* is a group extension (X, G, K) of a minimal proximal flow. Then the compact group K witnessing the group extension is precisely

$\text{Aut}(X, G)$ with the pointwise convergence topology and with the opposite composition. (See the paragraph after Def. 1.23 in [KP17] and the references in there.) The flow $(\pi_{\mathcal{A}}^{\#}, G)$ is the *universal compactification flow of G in $\mathcal{C}$* (see Proposition 3.18), and $\Sigma(G)_{\mathcal{A}}$ will be called the *generalized Bohr compactification of G in $\mathcal{C}$* (it is uniquely determined as $\text{Aut}(\pi_{\mathcal{A}}^{\#}, G)$ with the opposite composition).

Let us fix a minimal left ideal $\mathcal{M}$ in $\mathcal{A}$ and an idempotent $u \in \mathcal{M}$. Recall that $\mathcal{A}$ acts transitively and left-continuously on any minimal flow from $\mathcal{C}$ by an action which extends the action of G (i.e., $(ga_0)x = gx$). Choose $b_0 \in \pi_{\mathcal{A}}^{\#}$ with $ub_0 = b_0$ ($b_0 := ub$ for any choice of $b \in \pi_{\mathcal{A}}^{\#}$ has this property). Let

$$\text{Gal}_{\mathcal{A}} = \text{Gal}(\pi_{\mathcal{A}}^{\#}, b_0) := \{\eta \in u\mathcal{M} : \eta b_0 = b_0\}.$$

In the classical case when $\mathcal{A} = \beta G$ (i.e., the universal G -ambit), Theorem IX.4.2 of [Gla76] describes $\pi_{\mathcal{A}}^{\#}$ as a quotient of $\mathcal{M}$ by a certain closed G -invariant equivalence relation $\sim$, and $\Sigma(G)_{\mathcal{A}}$ as $u\mathcal{M}/H(u\mathcal{M})$. However, in general, this approach fails, as the relation $\sim$ need not be closed. It is an interesting general question how to describe both $\pi_{\mathcal{A}}^{\#}$ and $\Sigma(G)_{\mathcal{A}}$. In a model-theoretic category of externally definable flows there was an attempt to do that in Theorem 2.5 of [KP17], but recently Andy Zucker found a gap in the proof (there is no reason for the relation $\mathcal{M}$ defined there to be an equivalence relation). Using the ideas around Theorem 2.5 of [KP17], the second author has found some description of the universal compactification flow of G in $\mathcal{C}$, which shows that under some special assumptions the generalized Bohr compactification in $\mathcal{C}$ is $u\mathcal{M}/H(u\mathcal{M})$, but it does not prove it in general; we do not present in this paper those partial results. We expect that it may be false in general. However, in the situation in which we are interested in this paper (i.e. for a tame and amenable ambit $\mathcal{A}$), it will be true.

By the proof of [Gla76, Theorem IX.2.1] or just using this theorem and passing to the suitable quotients of Ellis groups (the difference is that in [Gla76, Theorem IX.2.1] we compute all the Ellis groups and Galois groups in the universal G -ambit βG , whereas here in the ambit $\mathcal{A}$ which is a quotient of βG), we get the following fact.

Fact 3.16. *There is a topological group isomorphism $\kappa: u\mathcal{M}/\text{Gal}_{\mathcal{A}} \rightarrow \Sigma(G)_{\mathcal{A}}$ determined by $(pb_0) \cdot \kappa(\eta/\text{Gal}_{\mathcal{A}}) = (p\eta)b_0$ for $p \in \mathcal{M}$. Moreover, $H(u\mathcal{M}) \leq \text{Gal}_{\mathcal{A}}$.*

For the next fact see [Gla76, Lemma VIII.1.1].

Fact 3.17. *Let $\chi: (Z, G, K) \rightarrow (Y, G)$ and $\phi: (X, G, L) \rightarrow (Y, G)$ be group extensions and assume that there is a homomorphism $\psi: Z \rightarrow X$ such that $\chi = \phi \circ \psi$. Then there is a topological group epimorphism $\theta: K \rightarrow L$ determined by $\psi(z) \cdot \theta(k) = \psi(z \cdot k)$.*

Proposition 3.18. (1) $(\pi_{\mathcal{A}}^{\#}, G)$ is the unique universal compactification G -flow in $\mathcal{C}$.

- (2) If (X, G) is a compactification flow in $\mathcal{C}$, then there is a continuous epimorphism θ from the compact group $\Sigma(G)_{\mathcal{A}}$ to the compact group $\text{Aut}(X, G)$. More precisely, bearing in mind Fact 3.16, if we choose $x_0 \in X$ such that $ux_0 = x_0$, then we can find the above θ determined by $(px_0) \cdot \theta(\eta/\text{Gal}_{\mathcal{A}}) = (p\eta)x_0$ for $p \in \mathcal{M}$.

Proof. (1) Let (X, G) be a compactification flow in $\mathcal{C}$. So there is a minimal proximal flow (Y, G) and a group extension $\phi: (X, G, K) \rightarrow (Y, G)$. Since (Y, G) belongs

to $\mathcal{C}$, by universality of $\pi_{\mathcal{A}}$, we get an epimorphism $\chi: \pi_{\mathcal{A}} \rightarrow Y$. Recall also that we have the universal group extension $\psi_{\mathcal{A}}: (\pi_{\mathcal{A}}^{\#}, G, \Sigma(G)_{\mathcal{A}}) \rightarrow (\pi_{\mathcal{A}}, G)$.

Set $Z' := \{(x, p) \in X \times \pi_{\mathcal{A}} : \phi(x) = \chi(p)\}$. Pick $x_0 \in X$ and $p_0 \in \pi_{\mathcal{A}}$ so that $ux_0 = x_0$ and $up_0 = p_0$. Then $u(x_0, p_0) = (x_0, p_0)$, so $Z := \overline{G(x_0, p_0)}$ is a minimal subflow of Z' (to see that Z is a subflow of Z' , note that $(x_0, p_0) \in Z'$; indeed, the elements $\phi(x_0), \chi(p_0) \in uY$ are distal by [Gla76, Proposition I.3.2(3)], and so they are equal by proximality of (Y, G)). Since both (X, G) and $(\pi_{\mathcal{A}}, G)$ belong to $\mathcal{C}$, so does (Z, G) . We have a natural faithful right action of K on Z given by $(x, p) \cdot k := (x \cdot k, p)$. This action is jointly continuous, and each $k \in K$ acts by a G -flow automorphism. Let $\pi_1: Z \rightarrow X$ and $\pi_2: Z \rightarrow \pi_{\mathcal{A}}$ be the projections.

Claim 1. π_2 is a group extension with the above action of K as a witness.

Proof. Consider any $(x, p) \in Z$. Then $\pi_2^{-1}[\pi_2(x, p)] = \pi_2^{-1}(p)$, so the goal is to show that $\pi_2^{-1}(p) = (x, p) \cdot K$.

($\subseteq$) Take any $(x', p) \in \pi_2^{-1}(p)$. Since $(x, p), (x', p) \in Z$, we have $\phi(x') = \chi(p) = \phi(x)$. Thus, as ϕ is a group extension witnessed by K , there exists $k \in K$ such that $x' = x \cdot k$. Then $(x', p) = (x, p) \cdot k \in (x, p) \cdot K$.

($\supseteq$) Take any $(x', p) \in (x, p) \cdot K$, so $x' \in x \cdot K = \phi^{-1}[\phi(x)]$ as ϕ is a group extension witnessed by K . Choose an idempotent $v \in \mathcal{M}$ such that $v(x, p) = (x, p)$ (such a v exists by minimality of Z). Since (Y, G) is proximal, Proposition I.4.1(2) from [Gla76] yields $v\mathcal{M}\phi(x) = \{\phi(x)\}$. Hence, by [Gla76, Proposition I.4.1(3)], we conclude that $\phi^{-1}[\phi(x)] = v\mathcal{M}x$ (we use here that group extensions are distal homomorphisms). All of this implies that $x' \in v\mathcal{M}x$. On the other hand, since $(\pi_{\mathcal{A}}, G)$ is proximal, again by [Gla76, Proposition I.4.1(2)], we get that $v\mathcal{M}p = \{p\}$. Therefore, $(x', p) \in (v\mathcal{M}x, p) = v\mathcal{M}(x, p) \subseteq Z$ and clearly $\pi_2(x', p) = p$, so $(x', p) \in \pi_2^{-1}(p)$. $\square$ (claim)

By the claim, the fact that (Z, G) is in $\mathcal{C}$, and by universality of $\psi_{\mathcal{A}}$, there exists an epimorphism $\Theta: \pi_{\mathcal{A}}^{\#} \rightarrow Z$ with $\pi_2 \circ \Theta = \psi_{\mathcal{A}}$. Then $\pi_1 \circ \Theta: \pi_{\mathcal{A}}^{\#} \rightarrow X$ is an epimorphism, which witnesses universality of $(\pi_{\mathcal{A}}^{\#}, G)$.

Uniqueness follows from an easy observation that every endomorphism of a compactification flow is an automorphism.

(2) Consider any $x_0 \in X$ with $ux_0 = x_0$. Let $p_0 = \psi_{\mathcal{A}}(b_0) \in \pi_{\mathcal{A}}$; then $up_0 = p_0$. The construction in the proof of (1) yields an epimorphism $\Theta: \pi_{\mathcal{A}}^{\#} \rightarrow Z$ with $\pi_2 \circ \Theta = \psi_{\mathcal{A}}$. We can find $b'_0 \in \pi_{\mathcal{A}}^{\#}$ satisfying $ub'_0 = b'_0$ and $\Theta(b'_0) = (x_0, p_0)$. Since $\psi_{\mathcal{A}}(b'_0) = \pi_2(\Theta(b'_0)) = \pi_2(x_0, p_0) = p_0 = \psi_{\mathcal{A}}(b_0)$ and $\psi_{\mathcal{A}}$ is a group extension, there exists an automorphism σ of the flow $(\pi_{\mathcal{A}}^{\#}, G)$ mapping b_0 to b'_0 . Then $\Theta' := \Theta \circ \sigma: \pi_{\mathcal{A}}^{\#} \rightarrow Z$ is an epimorphism satisfying $\Theta'(b_0) = (x_0, p_0)$. This implies that $\pi_2(\Theta'(b_0)) = p_0 = \psi_{\mathcal{A}}(b_0)$, and so $\pi_2 \circ \Theta' = \psi_{\mathcal{A}}$.

By Fact 3.16, we can identify $\Sigma(G)_{\mathcal{A}}$ with $u\mathcal{M}/\text{Gal}_{\mathcal{A}}$ and then the right action on $\pi_{\mathcal{A}}^{\#}$ is given by $(pb_0) \cdot (\eta/\text{Gal}_{\mathcal{A}}) = (p\eta)b_0$ for $p \in \mathcal{M}$.

Since π_2 and $\psi_{\mathcal{A}}$ are group extensions, by Fact 3.17, there is a continuous epimorphism $\theta: \Sigma(G)_{\mathcal{A}} \rightarrow K$ given by $\Theta'(pb_0) \cdot \theta(\eta/\text{Gal}_{\mathcal{A}}) = \Theta'(pb_0 \cdot (\eta/\text{Gal}_{\mathcal{A}}))$ which equals $\Theta'((p\eta)b_0)$ by the previous paragraph. But $\Theta'(pb_0) = p\Theta'(b_0) = p(x_0, p_0) = (px_0, pp_0)$ and $\Theta'((p\eta)b_0) = ((p\eta)x_0, (p\eta)p_0)$. So

$$\begin{aligned} ((p\eta)x_0, (p\eta)p_0) &= \Theta'(pb_0) \cdot \theta(\eta/\text{Gal}_{\mathcal{A}}) = (px_0, pp_0) \cdot \theta(\eta/\text{Gal}_{\mathcal{A}}) \\ &= ((px_0) \cdot \theta(\eta/\text{Gal}_{\mathcal{A}}), pp_0) \end{aligned}$$

which yields $(px_0) \cdot \theta(\eta/\text{Gal}_A) = (p\eta)x_0$, as required. $\square$

Corollary 3.19. *Let e_B be the neutral element in G_A^{Bohr} . The map $\theta: u\mathcal{M}/\text{Gal}_A \rightarrow G_A^{\text{Bohr}}$ given by $\theta(\eta/\text{Gal}_A) := \eta e_B$ is a well-defined continuous epimorphism, which satisfies $\theta(ugu/\text{Gal}_A) = \mathbf{b}_A(g)$ for all $g \in G$. Since $H(u\mathcal{M})$ is contained in Gal_A , this yields a continuous epimorphism $u\mathcal{M}/H(u\mathcal{M}) \rightarrow G_A^{\text{Bohr}}$. This also yields an epimorphism and topological quotient map $\hat{\theta}: u\mathcal{M} \rightarrow G_A^{\text{Bohr}}$ defined as the composition of θ with the quotient map $u\mathcal{M} \rightarrow u\mathcal{M}/\text{Gal}_A$.*

Proof. Let $(X, G) := (G_A^{\text{Bohr}}, G)$, where the action is given by $gx := \mathbf{b}_A(g)x$. Note that X acts transitively, jointly continuously and by automorphisms on the flow (X, G) by multiplying on the right (i.e., for every $h \in X$ the assignment $x \mapsto xh$ is an automorphism of (X, G)). By minimality of (X, G) , this yields an identification of X with $\text{Aut}(X, G)$. In particular, (X, G) is a group extension of the trivial flow, so it is a compactification flow.

We check that $uh = h$ for every $h \in X$. There exists $h_0 \in X$ with $uh_0 = h_0$. Then $uh = u(h_0(h_0^{-1}h)) = (uh_0)(h_0^{-1}h) = h_0h_0^{-1}h = h$, where the second equality follows from the fact that the assignment $x \mapsto x(h_0^{-1}h)$ is an automorphism of (X, G) .

By the above two paragraphs, we can apply Proposition 3.18(2) to the flow (X, G) and $x_0 := e_B$. Bearing in mind the above identification of $\text{Aut}(X, G)$ with X , the map $\theta: u\mathcal{M}/\text{Gal}_A \rightarrow \text{Aut}(X, G)$ from Proposition 3.18(2) is in fact a continuous epimorphism onto X . By the last formula in Proposition 3.18(2) and the identification of $\text{Aut}(X, G)$ with X , we see that $\theta(\eta/\text{Gal}_A) = x_0 \cdot \theta(\eta/\text{Gal}_A) = \eta x_0$. Finally, by the second paragraph, $\theta(ugu/\text{Gal}_A) = ugu x_0 = ugx_0 = gx_0 = \mathbf{b}_A(g)x_0 = \mathbf{b}_A(g)$.

Since θ is continuous and $u\mathcal{M}/\text{Gal}_A$ and G_A^{Bohr} are compact Hausdorff, we have that θ is a topological quotient map, and thus so is $\hat{\theta}$. $\square$

Question 3.20. Is the set $\{ugu/\text{Gal}_A : g \in G\}$ dense in $u\mathcal{M}/\text{Gal}_A$? Whenever this is the case, the epimorphism θ from Corollary 3.19 is the unique continuous epimorphism $u\mathcal{M}/\text{Gal}_A \rightarrow G_A^{\text{Bohr}}$ satisfying $\theta(ugu/\text{Gal}_A) = \mathbf{b}_A(g)$ for all $g \in G$.

By minimality of $\mathcal{M}$, we get that Gu is dense in $\mathcal{M}$. Thus, in the case when $\overline{u\mathcal{M}} = \mathcal{M}$, a positive answer to the last question follows from [KPR18, Lemma 3.1]. However, this argument does not work if $\overline{u\mathcal{M}} \subsetneq \mathcal{M}$.

Note that if the map $\theta: u\mathcal{M}/\text{Gal}_A \rightarrow G_A^{\text{Bohr}}$ from Corollary 3.19 is an isomorphism, then the map $\zeta: G \rightarrow u\mathcal{M}/\text{Gal}_A$ given by $g \mapsto ugu/\text{Gal}_A$ is the Bohr compactification of G in $\mathcal{C}$ (in particular, ζ has dense image).

Now, we present a generalization to $\mathcal{C}$ of a more precise version of [Gla76, Corollary IX.4.3] (note that [Gla76, Corollary IX.4.3] was left without a proof in [Gla76]).

Proposition 3.21. *Assume that the ambit $\mathcal{A}$ is strongly amenable in the sense that it has no non-trivial minimal proximal factors. Then the map $\theta: u\mathcal{M}/\text{Gal}_A \rightarrow G_A^{\text{Bohr}}$ from Corollary 3.19 is an isomorphism.*

Proof. By strong amenability, π_A (i.e., the universal minimal proximal ambit in $\mathcal{C}$) is trivial. So $(Y, G, K) := (\pi_A^\#, G, \Sigma(G)_A)$ is the universal group extension in $\mathcal{C}$ of the trivial flow. So $Y = b_0 \cdot K$ and the map $\Phi: K \rightarrow Y$ given by $\Phi(k) := b_0 \cdot k$ is a homeomorphism (where recall that $b_0 \in Y$ was chosen so that $ub_0 = b_0$). This induces a compact group structure on Y , that is $(b_0 \cdot k_1) \odot (b_0 \cdot k_2) := b_0 \cdot (k_1 k_2)$. For every $g \in G$ there exists $k_g \in K$ such that $gb_0 = b_0 \cdot k_g$. A direct computation shows

that for any $g_1, g_2 \in G$, we have $(g_1 b_0) \odot (g_2 b_0) = (g_1 g_2) b_0$, and so $k_{g_1} k_{g_2} = k_{g_1 g_2}$. Thus, $k: G \rightarrow K$ given by $k(g) := k_g$ is a homomorphism with dense image (as $\Phi[\text{im}(k)] = b_0 \cdot \text{im}(k) = G b_0$ and $G b_0$ is dense in Y), i.e. k is a compactification of G . The G -flow structure on K induced by the homeomorphism Φ is $ga := \Phi^{-1}(g(b_0 \cdot a)) = \Phi^{-1}((g b_0) \cdot a) = \Phi^{-1}(b_0 \cdot (k_g a)) = k_g a$, and so we see that it coincides with the G -flow structure induced by the compactification k . Since $(K, G) \cong (Y, G)$ we have that (K, G) is in $\mathcal{C}$, so K is a compactification of G in $\mathcal{C}$. By universality of the Bohr compactification, there is a unique topological group epimorphism $h: G_{\mathcal{A}}^{\text{Bohr}} \rightarrow K$ such that $h \circ \mathbf{b}_{\mathcal{A}} = k$.

On the other hand, by Fact 3.16, we have a topological group isomorphism $\kappa: u\mathcal{M}/\text{Gal}_{\mathcal{A}} \rightarrow K$ determined by $(pb_0) \cdot \kappa(\eta/\text{Gal}_{\mathcal{A}}) = (p\eta)b_0$ for $p \in \mathcal{M}$. For $g \in G$ we compute:

$$\begin{aligned} b_0 \cdot \kappa(ugu/\text{Gal}_{\mathcal{A}}) &= (ub_0) \cdot \kappa(ugu/\text{Gal}_{\mathcal{A}}) = ugub_0 = ugb_0 \\ &= u(b_0 \cdot k_g) = (ub_0) \cdot k_g = b_0 \cdot k_g, \end{aligned}$$

and so $\kappa(ugu/\text{Gal}_{\mathcal{A}}) = k_g = k(g)$.

We conclude that for any $g \in G$, $(\kappa^{-1} \circ h \circ \mathbf{b}_{\mathcal{A}})(g) = ugu/\text{Gal}_{\mathcal{A}}$. Hence, by Corollary 3.19, we get $\theta \circ \kappa^{-1} \circ h \circ \mathbf{b}_{\mathcal{A}} = \mathbf{b}_{\mathcal{A}}$, so $\theta \circ \kappa^{-1} \circ h = \text{id}_{G_{\mathcal{A}}^{\text{Bohr}}}$, so θ is injective by surjectivity of $\kappa^{-1} \circ h$. $\square$

Now, we explain that the map $\hat{\theta}$ from Corollary 3.19 coincides with w_2 from Subsection 3.1.

Remark 3.22. Let $(\mathcal{A}, a_0, G) = (E(X, G), \text{id}_X, G)$ for an ambit (X, x_0, G) . Then for the map θ from Corollary 3.19 we have $\theta(\eta/\text{Gal}_{\mathcal{A}}) = w_2(\eta)$, i.e. $\eta e_B = w_2(\eta)$. Hence, the following diagram commutes

$$\begin{array}{ccc} u\mathcal{M} & \xrightarrow{w_2|_{u\mathcal{M}}} & B_2 \\ \downarrow & \nearrow \theta & \\ u\mathcal{M}/\text{Gal}_{\mathcal{A}} & & \end{array}$$

and $\hat{\theta} = w_2|_{u\mathcal{M}}$. Thus, $w_2|_{u\mathcal{M}}$ is a topological quotient map.

Proof. For every $g \in G$ we have $w_2(g \text{id}_X) = \mathbf{b}_2(g) = \mathbf{b}_2(g)e_B = ge_B$, and then we use standard continuity and density arguments. $\square$

In Proposition 3.21, we obtained that the map θ is an isomorphism under suitable assumptions on the ambit $\mathcal{A}$. In the next section, we give assumptions on the ambit $\mathcal{A}$ guaranteeing that the map $\hat{\theta} = w_2|_{u\mathcal{M}}$ is an isomorphism.

4. THE GENERALIZED NEWELSKI'S CONJECTURE

4.1. Formulation. We keep the notation from the previous section. In particular, for a flow (X, G) and $u^2 = u \in \mathcal{M} \trianglelefteq_m E(X, G)$, we have the Bohr compactification $\mathbf{b}_2: G \rightarrow B_2$ and the group epimorphism $\hat{\theta} = w_2|_{u\mathcal{M}}: u\mathcal{M} \rightarrow B_2$ which is also a topological quotient map (see Remark 3.22).

Conjecture 4.1 (GNC - Generalized Newelski's Conjecture). Let (X, G) be a flow. Then, under suitable assumptions on the flow (X, G) , the map $w_2|_{u\mathcal{M}}: u\mathcal{M} \rightarrow B_2$ is a topological group isomorphism.

As in model theory, we have

Remark 4.2. The conclusion of Conjecture 4.1 (i.e., $w_2|_{u\mathcal{M}}: u\mathcal{M} \rightarrow B_2$ is a topological group isomorphism) is equivalent to injectivity of $w_2|_{u\mathcal{M}}: u\mathcal{M} \rightarrow B_2$, and trivially implies the conclusion of the GRNC (i.e., $u\mathcal{M}$ is compact Hausdorff in the τ topology).

Proof. It follows from Remark 3.22. $\square$

A slightly different form of GNC is:

Conjecture 4.3. Let $(\mathcal{A}, a_0, G)$ be an ambit on which there exists a left continuous semigroup operation $*$ extending the action of G . Let $u^2 = u \in \mathcal{M} \trianglelefteq_m \mathcal{A}$. Then, under suitable assumptions on the ambit $\mathcal{A}$, the map $\hat{\theta}: u\mathcal{M} \rightarrow G_{\mathcal{A}}^{\text{Bohr}}$ (from Corollary 3.19) is a topological group isomorphism.

Of course, the question “What are the suitable assumptions?” is the main point. Our first, motivated by [CS18] and [KNS19], guess was: if (X, G) is tame and amenable, then $w_2|_{u\mathcal{M}}: u\mathcal{M} \rightarrow B_2$ is an isomorphism of topological groups. However, it might be easily refuted — by adding a single point to a flow we obtain an amenable flow, then we do it for a tame flow for which $w_2|_{u\mathcal{M}}: u\mathcal{M} \rightarrow B_2$ is not injective (e.g., the one from [GPP15]). On the other hand, we prove Conjecture 4.3 for tame and amenable $(\mathcal{A}, a_0, G)$:

Theorem 4.4. *Let $(\mathcal{A}, a_0, G)$ be a tame amenable ambit on which there exists a left continuous semigroup operation $*$ extending the action of G . Then $\hat{\theta}: u\mathcal{M} \rightarrow G_{\mathcal{A}}^{\text{Bohr}}$ is a topological group isomorphism.*

By the fact that tameness of a flow (X, G) transfers to $(E(X, G), G)$ (see [GM12, Lemma 5.4] and the sentence immediately following its proof) and by Fact 2.7 and Remark 3.22, Theorem 4.4 is equivalent to the following theorem which says that Conjecture 4.1 holds for tame hereditarily amenable flows and which is the first main result of this paper.

Theorem 4.5. *If (X, G) is tame and hereditarily amenable, then $w_2|_{u\mathcal{M}}: u\mathcal{M} \rightarrow B_2$ is an isomorphism of topological groups.*

A topological dynamics proof will be given in Subsection 4.3; an analytic proof will appear in Subsection 5.1.

In Corollary 6.5, we will see that Theorem 4.4 implies one of the main results of [CS18] saying that the original Newelski’s Conjecture holds for definably amenable groups definable in NIP theories (i.e., Fact 1.8). In Subsection 6.3, we will use Theorem 4.4 to get a local variant of Newelski’s Conjecture.

4.2. GNC under special assumptions. We say that GNC holds for a flow (X, G) if the conclusion of Conjecture 4.1 holds, equivalently if $w_2|_{u\mathcal{M}}: u\mathcal{M} \rightarrow B_2$ is injective.

Note that GNC trivially holds for flows with $u\mathcal{M} = \{u\}$ (e.g., proximal flows). However, it is not enough to assume only finiteness of $u\mathcal{M}$, as we have a counterexample for GNC where $|u\mathcal{M}| = 2$ and $|B_1| = |B_2| = 1$ (see [GPP15] and e.g. Lemma 3.10 in [GPP14] to note that $B_1 = B_2$). Working in the model-theoretic context of a group G definable in a p -adically closed field M acting on the type space $S_G(M)$, Theorem 2 in [YZ26] yields the equivalence between the conclusion of Newelski’s conjecture and amenability of the flow $(S_G(M), G)$. In Subsection 4.1, we gave a

simple reason why the right-to-left implication does not hold in general for tame flows (and Theorem 4.5 shows that it holds with hereditary amenability in place of amenability). Below we give an example showing that in general the left-to-right implication does not hold either.

Example 4.6. Consider an $\aleph_0$ -saturated and strongly $\aleph_0$ -homogeneous model M of the theory T of the circular order (cf. [Kim96]). Let $\bar{m}$ be an enumeration of the model M , and let $S_{\bar{m}}(M)$ be the space of complete types over M extending $\text{tp}(\bar{m}/\emptyset)$. Since T has NIP, the flow $(X, G) := (S_{\bar{m}}(M), \text{Aut}(M))$ is tame by [KR20, Corollary 5.8]. The theory T is a classical example for which $\emptyset$ is not an extension base for forking. Then Corollaries 2.7 and 2.24 from [HKP26] show that (X, G) is not amenable. On the other hand, by [KR20, Proposition A.1], the Ellis groups are trivial (and so the compactifications B_1 and B_2 are trivial as well) and thus GNC holds. Note also that Claim 1 in the proof of [KR20, Proposition A.1] tells us that (X, G) is proximal.

Remark 4.7. GNC holds for WAP flows.

Proof. If (X, G) is WAP, then there exists a unique minimal left ideal $\mathcal{M}$ in $E := E(X, G)$, which also coincides with its Ellis group $u\mathcal{M}$, and (E, G) is an amenable flow (cf. [EN89] or Theorem 2.5 in [Con25] for short summary). Let $\rho: E \rightarrow \mathcal{M}$ be given by $\rho(\eta) := \eta u$ and $\iota: G \rightarrow E$ by $\iota(g) := g \text{id}_X$. Then ρ is clearly an epimorphism of ambits and $\rho \circ \iota: G \rightarrow \mathcal{M} = u\mathcal{M}$ is a group homomorphism, as $(\rho \circ \iota)(g_1 g_2) = g_1 g_2 u = g_1 u g_2 u = (\rho \circ \iota)(g_1)(\rho \circ \iota)(g_2)$ (we used here that since $\mathcal{M} = u\mathcal{M}$, for every $\eta \in \mathcal{M}$ we have $\eta = u\eta$). We use the universal property of $(B_2, \mathfrak{b}_2, w_2)$ to find the dashed arrow in the following commutative diagram

$$\begin{array}{ccc}
 & G & \\
 \mathfrak{b}_2 \swarrow & \downarrow \iota & \searrow \rho \circ \iota \\
 & E & \\
 \swarrow w_2 & & \searrow \rho \\
 B_2 & \text{---} \text{---} \text{---} f \text{---} & \mathcal{M}
 \end{array}$$

Then for $\eta \in E$,

$$(fw_2)(u\eta u) = \rho(u\eta u) = u\eta u^2 = u\eta u,$$

hence $(fw_2)|_{u\mathcal{M}} = \text{id}_{u\mathcal{M}}$ and so $w_2|_{u\mathcal{M}}$ is injective. Therefore, GNC holds for WAP flows. $\square$

Recall that a factor $\pi: (X, G) \rightarrow (Z, G)$ is said to be *almost 1-1*, if $\{z \in Z : |\pi^{-1}(z)| = 1\}$ is a dense subset of Z .

Lemma 4.8. *If (X, G) is minimal and the MEF factor $\pi: (X, G) \rightarrow (Z, G)$ is almost 1-1⁴, then GNC holds and $u\mathcal{M} \cong K_2 \cong B_2 \cong K_1$.*

Proof. By Lemma 5.7 from [CGK26], $\pi_*|_{u\mathcal{M}}: u\mathcal{M} \rightarrow \pi_*(u)\pi_*[\mathcal{M}] = K_1$ is an isomorphism of semitopological groups considered with the τ -topologies (and because K_1 is a compact group, we have that also $u\mathcal{M}$ is a compact group in the τ -topology).

⁴In this case (Z, G) is the maximal Kronecker factor, i.e. a MEF which is a minimal flow. Alternatively, in the lemma we may assume that π is almost 1-1 and (Z, G) is minimal, as this implies that (X, G) is minimal by Remark 5.6 from [CGK26].

By universality of $(B_2, \mathfrak{b}_2, w_2)$ for the triple $(K_1, \pi_* \circ (g \mapsto g \text{id}_X), \pi_*)$, there exists a unique continuous group epimorphism $h: B_2 \rightarrow K_1$ such that

$$\begin{array}{ccc} G & \xrightarrow{\mathfrak{b}_2} & B_2 \\ g \mapsto g \text{id}_X \searrow & & \nearrow w_2 \\ E = E(X, G) & \xrightarrow{\pi_*} & E(Z, G) \cong K_1 \end{array} \quad \begin{array}{c} \\ \\ h \end{array}$$

Since $(h \circ w_2)|_{u\mathcal{M}} = \pi_*|_{u\mathcal{M}}$, we have that $w_2|_{u\mathcal{M}}$ is injective, so GNC holds.

Then, h is injective on the image of $w_2|_{u\mathcal{M}}$, which is equal to the entire group B_2 , so h is a continuous isomorphism of groups. Both are compact groups, hence h is an isomorphism of topological groups.

Since $w_2|_{u\mathcal{M}}$ is an isomorphism, so is $\mathfrak{f}_1: K_2 \rightarrow B_2$ in the first diagram in the proof of Proposition 3.12. As $\mathfrak{f}_1$ is continuous, we get that $K_2 \cong B_2$ as topological groups. $\square$

The notation in the next result is taken from Remarks 3.6 and 3.9.

Remark 4.9. Assume that GNC holds, i.e. $w_2|_{u\mathcal{M}}$ is an isomorphism. Then $u\mathcal{M} \cong K_2 \cong B_2$. If additionally the unique map $\mathfrak{f}: B_2 \rightarrow B_1$ from Remark 3.6 is an isomorphism, then $u\mathcal{M} \cong K_2 \cong B_2 \cong K_1 \cong B_1$.

Proof. The first conclusion follows as in the last paragraph of the proof of Lemma 4.8.

Now, assume that $w_2|_{u\mathcal{M}}$ is an isomorphism of groups (i.e. GNC holds), and $\mathfrak{f}$ is an isomorphism of groups as well. By an easy diagram chase (recall that the map $\mathfrak{t}: K_1 \rightarrow B_1$ was given in Remark 3.9) on the following commutative diagram

$$\begin{array}{ccccc} & & \exists! & & \\ & \curvearrowright & & \curvearrowleft & \\ B_2 & \xrightarrow{\cong} & B_1 & \xleftarrow{\mathfrak{t}} & K_1 \\ & \nwarrow \mathfrak{f} & \uparrow w_1 \pi_0 & \nearrow \mathfrak{t} & \\ & & E & & \end{array} \quad \begin{array}{c} \\ \\ \pi_* \end{array}$$

we conclude that $\mathfrak{t}$ is an isomorphism as well. $\square$

One could ask whether $B_1 \cong B_2$ under the assumptions of Lemma 4.8. Example 3.10 gives us a negative answer. Indeed, the flow in Example 3.10 satisfies the assumptions of Lemma 4.8, hence $B_2 \cong K_1 \cong \text{SO}(3)$, but $B_1 = \{e_{B_1}\}$ in that case.

We now give a general source for examples where the compactifications $\mathfrak{b}_1$ and $\mathfrak{b}_2$ are not isomorphic despite of the very strong properties of the underlying flow. This will reveal an intriguing difference between the abstract and the model theory situations discussed in Section 6.

Example 4.10. If (X, G) is an equicontinuous flow, then the Ellis semigroup $E(X)$ is a compact group and the map $\iota: G \rightarrow E(X, G)$, $g \mapsto g \text{id}_X$, is precisely the Bohr compactification of G below $E(X)$, so we can assume that $\mathfrak{b}_2 = \iota$. On the other hand, if (X, G) is additionally minimal, then, after distinguishing any point $x_0 \in X$, we have that $(X, G) \cong (E(X, G)/H, G)$ for the compact subgroup

$$H := \ker(\text{ev}_{x_0}) := \{\eta \in E(X, G) : \eta(x_0) = x_0\}$$

of $E(X)$. Thus, whenever H is a non-trivial subgroup of $E(X)$, then the Bohr compactification below $E(X)$ (i.e., $\mathfrak{b}_2$) is not isomorphic to $\mathfrak{b}_1$. Moreover, each minimal equicontinuous flow is WAP (hence tame) and hereditarily amenable.

Example 3.10 is an instance in which even the target groups of the above two compactifications are not isomorphic. For a simpler “finite” example consider $G := S_3$, $H := \langle (1, 2) \rangle \leq G$, and the action of G on G/H by left translations. Then $(G/H, G)$ is equicontinuous, $E(G/H, G) = G$, but $|G/H| = 3 < 6 = |G|$, so the Bohr compactification below G/H is of smaller cardinality (in fact, it is trivial) than the one below $E(G/H, G)$.

Proof. For the fact that equicontinuity of (X, G) implies that $E(X, G)$ is a compact group, and equicontinuity together with minimality imply that $(X, G) \cong (E(X, G)/H, G)$ for $H = \ker(\text{ev}_{x_0})$ see [Gla76, Theorem I.3.3]. This yields amenability of (X, G) (and so hereditary amenability by minimality of (X, G)) with a witnessing G -invariant measure being the pushforward of the Haar measure on $E(X, G)$ by the quotient map $E(X, G) \rightarrow E(X, G)/H$ (in fact, this is a unique G -invariant measure on (X, G) by [EN89, Proposition II.10]). By [Gla76, Theorem I.3.3], if (X, G) is equicontinuous, then each member of $E(X, G)$ is a continuous function, so (X, G) is WAP (hence tame) by [EN89, Proposition II.2].

The fact that $\iota: G \rightarrow E(X, G)$ is the Bohr compactification of G below $E(X, G)$ follows from the fact that ι is a group homomorphism with dense image and w_2 is a continuous group epimorphism by Remark 3.5.

To see that if H is nontrivial then $\mathfrak{b}_1$ is not isomorphic to $\mathfrak{b}_2$, note that the map f in Remark 3.6 is a unique continuous group homomorphism such that $f \circ \mathfrak{b}_2 = \mathfrak{b}_1$ and it is not an isomorphism because $f = f \circ w_2 = w_1 \circ \text{ev}_{x_0}$ and $\ker(\text{ev}_{x_0}) = H$ is nontrivial and contained in $\ker(f)$. $\square$

Corollary 4.11. *If (X, G) is tame, amenable, minimal and metric, then GNC holds.*

Proof. By Corollary 5.4.2 from [Gla18], we have that (X, G) is almost automorphic, i.e. the X -MEF π is almost 1-1. Then we apply Lemma 4.8. $\square$

In Theorem 4.5 (which will be proven in the next subsection) we drop the minimality and metrizability assumption. This requires a different method of a proof.

4.3. Dynamical proof of GNC. In this subsection, we give a dynamical proof of Theorem 4.4, which is equivalent to Theorem 4.5 as explained in Subsection 4.1.

From now on, let $(\mathcal{A}, a_0, G)$ be an ambit with a left continuous semigroup operation extending the action of G , $\mathcal{M}$ a minimal left ideal, and $u \in \mathcal{M}$ an idempotent.

A very important fact for the proof of Theorem 4.4 is Lemma 5.7 in [HK26], i.e. if $\mathcal{A}$ is tame and amenable, then $\overline{u\mathcal{M}} = \mathcal{M}$ (see Fact 2.8(2)).

Lemma 4.12. *Assume that $\overline{u\mathcal{M}} = \mathcal{M}$. Then $\mathcal{A}$ is strongly amenable (i.e., has no non-trivial minimal proximal factors).*

Proof. Let Y be a minimal proximal factor of $\mathcal{A}$. Pick $y_0 \in Y$ such that $uy_0 = y_0$. By minimality of Y , we have $\mathcal{M}y_0 = Y$. Since Y is proximal, we have that $u\mathcal{M}y_0 = \{y_0\}$ (by Prop. I.3.2(3) or I.4.1(2) in [Gla76]). As by assumption $\overline{u\mathcal{M}} = \mathcal{M}$ and the evaluation map $\text{ev}_{y_0}: \mathcal{M} \rightarrow Y$ is continuous, we conclude that $Y = \{y_0\}$. $\square$

By virtue of Proposition 3.21, Fact 2.8(2), and Lemma 4.12, in order to prove Theorem 4.4, it is enough to show that $\text{Gal}_{\mathcal{A}} = \{u\}$. The final highly non-trivial

reduction is the Generalized Revised Newelski's Conjecture for tame flows, i.e. Fact 1.3: the Ellis group $u\mathcal{M}$ of a tame flow is always Hausdorff, equivalently, $H(u\mathcal{M}) = \{u\}$. Therefore, in order to prove Theorem 4.4, it is enough to show that $\text{Gal}_{\mathcal{A}} = H(u\mathcal{M})$.

The desired equality $\text{Gal}_{\mathcal{A}} = H(u\mathcal{M})$ follows by an obvious adaptation (and simplification due to the fact that the universal minimal proximal factor of $\mathcal{A}$ is trivial here) of either the proofs of Propositions IX.4.1 and IX.4.2 from [Gla76] or Theorem 2.5 from [KP17], because under the assumption $\overline{u\mathcal{M}} = \mathcal{M}$ the relations on $\mathcal{M}$ in both these sources are indeed closed, invariant, equivalence relations. Just for the reader's convenience we briefly repeat the argument from [Gla76], which completes the proof of Theorems 4.4. As in Subsection 3.3, $\mathcal{C}$ denotes the category of minimal flows which are factors of $\mathcal{A}$.

Lemma 4.13. *Assume that $\overline{u\mathcal{M}} = \mathcal{M}$. Then the universal minimal proximal factor of $\mathcal{A}$ is trivial.*

Let $\sim$ be a relation on $\mathcal{M}$ defined by

$$p \sim q \iff (up)^{-1}q \in H(u\mathcal{M}).$$

Then $\sim$ is a closed, G -invariant, equivalence relation on $\mathcal{M}$. Moreover, $Z := \mathcal{M}/\sim$ is the universal group extension in $\mathcal{C}$ of the trivial flow and

$$\text{Gal}(Z, u/\sim) = \{\eta \in u\mathcal{M} : \eta(u/\sim) = u/\sim\} = H(u\mathcal{M})$$

so $u\mathcal{M}/H(u\mathcal{M})$ is the generalized Bohr compactification of G in $\mathcal{C}$.

Before the proof let us comment that although (in contrast to the case of $\mathcal{A} = \beta G$) in general we do not have an action of $\mathcal{A}$ on the flow $2^{\mathcal{A}}$ (because the subambits of this flow need not be in the category $\mathcal{C}$), using the definition of the circle operation $\circ$ for the τ -topology the proofs of 1.2-1.12 (except 1.12(2)) from [Gla76, Chapter IX] go through (with some slight modifications) in our context (see [Rze18, Appendix A] for the proofs). Observe that Lemma 3.4 from [Gla76, Chapter IX] also goes through in our context.

Note also that the additional condition $p \circ u\mathcal{M} = q \circ u\mathcal{M}$ which was needed in the definition of $\sim$ in [Gla76, Chapter IX.4] is automatically satisfied under the assumption $\overline{u\mathcal{M}} = u\mathcal{M}$. Indeed, then $p \circ u\mathcal{M} = p \circ \overline{u\mathcal{M}} = p \circ \mathcal{M} = \mathcal{M}$ for every $p \in \mathcal{M}$.

Proof of Lemma 4.13. Triviality of the universal minimal proximal factor of $\mathcal{A}$ is Lemma 4.12.

It is clear that $p \sim q$ iff $upH(u\mathcal{M}) = uqH(u\mathcal{M})$, so $\sim$ is an equivalence relation on $\mathcal{M}$. To see invariance, consider any $p \sim q$ and $g \in G$. Since $q \in \mathcal{M} = p \circ u\mathcal{M}$, by [Gla76, Lemma 3.4], $q \in p \circ N$ for any τ -neighborhood N of $(up)^{-1}q$ in $u\mathcal{M}$. Hence, $gq \in (gp) \circ N$, and so $(u(gp))^{-1}(gq) \in u \circ N$ by [Gla76, Lemma 3.4]. Thus, $(u(gp))^{-1}(gq) \in \text{cl}_{\tau}(N)$. Since by assumption $(up)^{-1}q \in H(u\mathcal{M})$, the above conclusion holds for all τ -neighborhoods N of $H(u\mathcal{M})$. As $u\mathcal{M}/H(u\mathcal{M})$ is Hausdorff, the intersection of all those $\text{cl}_{\tau}(N)$ equals $H(u\mathcal{M})$, so we get $(u(gp))^{-1}(gq) \in H(u\mathcal{M})$, i.e. $gp \sim gq$.

Finally, we show that $\sim$ is closed. So take any net $(p_i, q_i) \in \mathcal{M} \times \mathcal{M}$ with $p_i \sim q_i$, which converges to (p, q) . Let N be any τ -neighborhood of $H(u\mathcal{M})$ so also of each $(up_i)^{-1}q_i$. Since $q_i \in \mathcal{M} = p_i \circ u\mathcal{M}$, by [Gla76, Lemma 3.4], $q_i \in p_i \circ N$. Using limits, we conclude that $q \in p \circ N$ and so $(up)^{-1}q \in \text{cl}_{\tau}(N)$ by [Gla76, Lemma 3.4]. Thus, as in the previous paragraph, we get $(up)^{-1}q \in H(u\mathcal{M})$, i.e. $p \sim q$.

In this way, Z becomes a minimal G -flow which is a factor of $(\mathcal{M}, G)$ so also of $\mathcal{A}$. It is clear that $u(u/\sim) = u/\sim$. By definition,

$$\begin{aligned} \text{Gal}(Z, u/\sim) &= \{\eta \in u\mathcal{M} : \eta(u/\sim) = u/\sim\} = \{\eta \in u\mathcal{M} : \eta \sim u\} \\ &= [u]_\sim \cap u\mathcal{M} = H(u\mathcal{M}). \end{aligned}$$

Applying the obvious counterpart of Theorem IX.2.1 from [Gla76] (which can be shown either by the same proof or by an application of that theorem in the original context of βG to the quotient objects obtained from the epimorphism $\beta G \rightarrow \mathcal{A}$), in order to show that $(Z, G, u\mathcal{M}/H(u\mathcal{M}))$ is the universal group extension of the trivial flow (and, in consequence, $u\mathcal{M}/H(u\mathcal{M})$ is the generalized Bohr compactification of G in $\mathcal{C}$), it remains to prove that Z is distal. By [Gla76, Proposition I.3.2(3)], it is enough to show that $uz = z$ for every $z = p/\sim \in Z$; equivalently, $p \sim up$. The last thing holds as $(up)^{-1}up = u \in H(u\mathcal{M})$. $\square$

Although Theorem 4.4 follows by the sequence of reductions described in this subsection, we summarize it below.

Proof of Theorem 4.4. By assumption, $\mathcal{A}$ is tame and amenable.

By Fact 2.8(2) and Lemma 4.13, working with the notation from Lemma 4.13 and from Subsection 3.3, we have $\mathcal{M}/\sim = \pi_{\mathcal{A}}^{\#}$, and we may choose $b_0 := u/\sim$ in the definition of $\text{Gal}_{\mathcal{A}}$ (see Subsection 3.3); then $\text{Gal}_{\mathcal{A}} = \text{Gal}(Z, u/\sim)$. So, by Lemma 4.13, $\text{Gal}_{\mathcal{A}} = H(u\mathcal{M})$. This together with Fact 1.3 implies that $\text{Gal}_{\mathcal{A}} = \{u\}$.

By Fact 2.8(2) and Lemma 4.12, $\mathcal{A}$ is strongly amenable, so the conclusion follows from Proposition 3.21. $\square$

In fact, one can prove Theorem 4.4 a bit faster, i.e. without most of the general considerations in Subsection 3.3 involved above, which we now describe. But first of all, if we want to avoid using Proposition 3.18 and Corollary 3.19, we have to explain what the map $\hat{\theta}$ in the statement of Theorem 4.4 is. Bearing in mind the natural identification of $\mathcal{A}$ with $E(\mathcal{A}, G)$ the map w_2 from Subsection 3.1 considered for the ambit $\mathcal{A}$ becomes a semigroup epimorphism from $\mathcal{A}$ to $G_{\mathcal{A}}^{\text{Bohr}}$, and we take $\hat{\theta} := w_2|_{u\mathcal{M}} : u\mathcal{M} \rightarrow G_{\mathcal{A}}^{\text{Bohr}}$ which is a continuous group epimorphism by Remark 3.5. As in the short proof of Remark 3.22, we get that $\hat{\theta}(\eta) = \eta e_B$.

Faster proof of Theorem 4.4. Let $\mathcal{A}$ be a G -ambit with a left continuous semigroup operation extending the action of G . Define $\hat{\theta} : u\mathcal{M} \rightarrow G_{\mathcal{A}}^{\text{Bohr}}$ as in the above paragraph. By the second paragraph of the proof of Corollary 3.19, we know that $uh = h$ for all $h \in G_{\mathcal{A}}^{\text{Bohr}}$. Thus, for every $g \in G$, $\hat{\theta}(ugu) = ugue_B = ge_B = \mathfrak{b}_{\mathcal{A}}(g)e_B = \mathfrak{b}_{\mathcal{A}}(g)$.

From now on, assume that $\mathcal{A}$ is additionally tame and amenable. By Fact 2.8(2) and Lemma 4.12, $\mathcal{A}$ is strongly amenable. By Facts 2.8(2) and 1.3 and Lemma 4.13, the universal compactification flow below $\mathcal{A}$ is of the form $(Z, G, u\mathcal{M})$, $b_0 := u/\sim$ satisfies $ub_0 = b_0$ (actually any point in Z has this property), and $\text{Gal}(Z, b_0) = \{u\}$. As in Fact 3.16, the proof of Lemma 4.13 (more precisely, the application of the explicit action in the proof of Theorem IX.2.1 from [Gla76]) shows that the witnessing right action of $u\mathcal{M}$ on Z is given by $(pb_0) \cdot \eta := (p\eta)b_0$ for every $p \in \mathcal{M}$.

Having the above two paragraphs, we apply the proof of Proposition 3.21 to $u\mathcal{M}$ in place of $\text{Gal}_{\mathcal{A}}$ (now $\text{Gal}_{\mathcal{A}} = u\mathcal{M}$) and to $\hat{\theta}$ defined above in place of θ defined in Corollary 3.19 (in fact, these functions coincide, but we do not use it) and we get that $\hat{\theta}$ is a topological group isomorphism (continuity of $\hat{\theta}$ follows from

the proof of Proposition 3.21, namely $\hat{\theta} \circ \kappa^{-1} \circ h = \text{id}_{G_A^{\text{Bohr}}}$ where $\kappa^{-1} \circ h$ is a continuous epimorphism between compact Hausdorff groups), which finishes the proof of Theorem 4.4. $\square$

5. MORE ON THE STRUCTURE OF AMENABLE AND TAME FLOWS

5.1. Analytic proof of GNC. This part is dedicated to the proof of GNC via methods from operator theory. The main engine of the analytic proof of GNC is the following lemma which is interesting by its own rights and useful for the structural results in Subsections 5.2 and 5.3.

Here, $E = E(X, G)$, $\mathcal{M} \trianglelefteq_m E$, $u^2 = u \in \mathcal{M}$, and $(B_2, \mathfrak{b}_2, w_2)$ is the Bohr compactification of G below (E, id_X, G) .

Lemma 5.1. *Let (X, G) be tame, $\mathcal{M} \trianglelefteq_m E(X, G)$, and let $\mu \in \text{Prob}_G(\mathcal{M})$. Then:*

- (1) μ is the unique G -invariant probability measure on $\mathcal{M}$;
- (2) $(w_2|_{\mathcal{M}})_*\mu = \mathfrak{h}_{B_2}$;
- (3) the pullback along $w_2|_{\mathcal{M}}$ extends uniquely to a unitary isomorphism

$$(w_2|_{\mathcal{M}})^* : L^2(B_2, \mathfrak{h}_{B_2}) \longrightarrow L^2(\mathcal{M}, \mu).$$

Proof. By [HK26, Lemma 2.2(3)], the Ellis flow $(E(X, G), G)$ is tame; hence its minimal subflow $(\mathcal{M}, G)$ is minimal and tame. Fact 2.1 makes μ unique, and therefore ergodic. Since $\mathcal{M}$ is minimal, $\text{supp}(\mu) = \mathcal{M}$.

The measure $(w_2|_{\mathcal{M}})_*\mu$ is invariant under the dense subgroup $\mathfrak{b}_2[G]$, and therefore $(w_2|_{\mathcal{M}})_*\mu = \mathfrak{h}_{B_2}$ (e.g. by Fact 2.1). Thus pullback initially gives an isometry

$$(w_2|_{\mathcal{M}})^* : L^2(B_2, \mathfrak{h}_{B_2}) \longrightarrow L^2(\mathcal{M}, \mu), \quad (w_2|_{\mathcal{M}})^*(F) = F \circ (w_2|_{\mathcal{M}}). \quad (1)$$

It remains to prove that $(w_2|_{\mathcal{M}})^*$ is onto, i.e. that $(w_2|_{\mathcal{M}})^*[C(B_2)]$ is dense. For this task, we will find an auxiliary unitary representation $B_2 \rightarrow U(L^2(\mathcal{M}, \mu))$. Let us start with “target” subgroup of $U(L^2(\mathcal{M}, \mu))$ and, working with the strong operator topology, set

$$K := \overline{\{U_g : g \in G\}} \subseteq B(L^2(\mathcal{M}, \mu)).$$

Claim 1. K is a compact Hausdorff group.

Proof. Let $\Gamma := \{U_g : g \in G\}$. If $F \in C(\mathcal{M})$ and $(g_n)_{n < \omega}$ is a sequence in G , then tameness and [GM18, Definition 3.1(1), Theorem 2.4], applied successively to the real and imaginary parts of F , yield a subsequence $(g_{n_k})_k$ for which $(U_{g_{n_k}}F)_k$ converges pointwise on $\mathcal{M}$, say, to a function $F_\infty : \mathcal{M} \rightarrow \mathbb{C}$. Since $|U_{g_{n_k}}F| \leq \|F\|_\infty$ and $|F_\infty| \leq \|F\|_\infty$ pointwise on $\mathcal{M}$, the dominated convergence theorem, applied to the sequence of functions $(|U_{g_{n_k}}F - F_\infty|^2)_k$ dominated by the integrable constant function $4\|F\|_\infty^2$, shows that $(U_{g_{n_k}}F)_k$ converges in $L^2(\mathcal{M}, \mu)$. Thus ΓF is relatively norm compact, i.e. for every $\epsilon > 0$ there are $U_{g_1}F, \dots, U_{g_n}F \in \Gamma F$ such that ΓF is covered by the open balls $B(U_{g_i}F, \epsilon)$.

Let $\xi \in L^2(\mathcal{M}, \mu)$ and let $\epsilon > 0$. Density of $C(\mathcal{M})$ in $L^2(\mathcal{M}, \mu)$ gives us $F \in C(\mathcal{M})$ such that $\|\xi - F\|_{L^2(\mathcal{M}, \mu)} \leq \frac{\epsilon}{3}$. For F , there are $U_{g_1}F, \dots, U_{g_n}F \in \Gamma F$ as in the first part of the proof of the claim for $\frac{\epsilon}{3}$. Using the identities

$$\|U_g\xi - U_gF\|_{L^2(\mathcal{M}, \mu)} = \|\xi - F\|_{L^2(\mathcal{M}, \mu)}$$

holding for every $g \in G$, we obtain that for every $g \in G$ there is some $i \leq n$ such that

$$\|U_g\xi - U_{g_i}\xi\|_{L^2(\mathcal{M}, \mu)} \leq \|U_g\xi - U_gF\|_{L^2(\mathcal{M}, \mu)} + \|U_gF - U_{g_i}F\|_{L^2(\mathcal{M}, \mu)}$$

$$+ \|U_{g_i}F - U_{g_i}\xi\|_{L^2(\mathcal{M}, \mu)} \\ \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Hence

$$C_\xi := \overline{\{U_g\xi : g \in G\}}^{\|\cdot\|}$$

is compact for every $\xi \in L^2(\mathcal{M}, \mu)$.

Regard Γ as a discrete space and let $\beta\Gamma$ be its Stone–Čech compactification, identified with the space of ultrafilters on Γ . Let $p \in \beta\Gamma$. Compactness of the spaces C_ξ gives limits

$$T_p(\xi) := \lim_{g \rightarrow p} U_g(\xi), \quad S_p(\xi) := \lim_{g \rightarrow p} U_g^{-1}(\xi),$$

where $\xi \in L^2(\mathcal{M}, \mu)$. Taking ultrafilter limits in the linearity and norm identities shows that T_p and S_p are linear isometries. Clearly, $\lim_{g \rightarrow p} (U_g, U_g^{-1}) = (T_p, S_p)$ in the product topology on $B(L^2(\mathcal{M}, \mu)) \times B(L^2(\mathcal{M}, \mu))$. Then, joint continuity of multiplication on norm-bounded sets gives $T_p S_p = S_p T_p = I$. Thus $T_p \in U(L^2(\mathcal{M}, \mu))$, $S_p = T_p^{-1}$, and p converges to T_p (in the strong operator topology).

The map

$$\beta\Gamma \longrightarrow U(L^2(\mathcal{M}, \mu)), \quad p \longmapsto T_p,$$

is continuous, because for every $\xi \in L^2(\mathcal{M}, \mu)$ the coordinate map $p \longmapsto T_p\xi$ is continuous. Its image is therefore compact. It contains Γ , and every T_p is a limit of elements of Γ , so its image is exactly K . Thus K is compact.

Since $U(L^2(\mathcal{M}, \mu))$ is a Hausdorff topological group, the closure of its subgroup Γ is a subgroup. Hence K is a compact Hausdorff group. $\square$ (claim)

Claim 2. There is a continuous semigroup epimorphism $w : E(X, G) \rightarrow K$ such that

$$\begin{array}{ccc} G & \xrightarrow{g \mapsto U_g} & K \\ & \searrow g \mapsto g \circ \text{id}_X & \nearrow w \\ & E(X, G) & \end{array}$$

commutes.

Proof. The map $G \longrightarrow E(X, G)$, given by $g \longmapsto g \circ \text{id}_X$, and the map $G \longrightarrow K$, given by $g \longmapsto U_g$, extend uniquely to continuous surjections

$$\pi_E : \beta G \longrightarrow E(X, G) \quad \text{and} \quad \pi_K : \beta G \longrightarrow K.$$

Define $\lambda : E(X, G) \longrightarrow E(\mathcal{M}, G)$ by $\lambda(\eta)(\xi) := \eta\xi$. For $\eta \in E(X, G)$, let

$$\widehat{\eta} := \text{res}^{-1}(\lambda(\eta)) \in E(\text{Prob}(\mathcal{M}), G),$$

where res is the isomorphism from Fact 2.6.

We claim that π_K is constant on every fibre of π_E . Let $\mathbf{p}, \mathbf{q} \in \beta G$ satisfy $\pi_E(\mathbf{p}) = \pi_E(\mathbf{q}) = \eta$. Fix a nonzero $F \in C(\mathcal{M})$ with $F \geq 0$, and put

$$c_F := \int_{\mathcal{M}} F d\mu, \quad \nu_F := \frac{F}{c_F} \mu.$$

Since $\text{supp}(\mu) = \mathcal{M}$, we have $c_F > 0$, and therefore $\nu_F \in \text{Prob}(\mathcal{M})$.

For every $g \in G$, invariance of μ gives

$$\begin{aligned} \langle U_g F, H \rangle_{L^2(\mathcal{M}, \mu)} &= \int_{\mathcal{M}} F(g^{-1}\xi) \overline{H(\xi)} d\mu(\xi) = \int_{\mathcal{M}} F(\xi) \overline{H(g\xi)} d\mu(\xi) \\ &= c_F \int_{\mathcal{M}} \overline{H(g\xi)} d\nu_F(\xi) = c_F \int_{\mathcal{M}} \overline{H} d(g_*\nu_F) \end{aligned}$$

for every $H \in C(\mathcal{M})$.

Since $\pi_E(\mathfrak{p}) = \eta$, continuity of λ and of res^{-1} implies that $g_*\nu_F \rightarrow \hat{\eta}(\nu_F)$ as $g \rightarrow \mathfrak{p}$. Consequently,

$$\langle \pi_K(\mathfrak{p})F, H \rangle_{L^2(\mathcal{M}, \mu)} = \lim_{g \rightarrow \mathfrak{p}} \langle U_g F, H \rangle_{L^2(\mathcal{M}, \mu)} = c_F \int_{\mathcal{M}} \overline{H} d\hat{\eta}(\nu_F).$$

The same argument applied to $\mathfrak{q}$ gives $\langle \pi_K(\mathfrak{q})F, H \rangle_{L^2(\mathcal{M}, \mu)} = c_F \int_{\mathcal{M}} \overline{H} d\hat{\eta}(\nu_F)$. Hence

$$\langle \pi_K(\mathfrak{p})F, H \rangle_{L^2(\mathcal{M}, \mu)} = \langle \pi_K(\mathfrak{q})F, H \rangle_{L^2(\mathcal{M}, \mu)}$$

for every nonnegative $F \in C(\mathcal{M})$ and every $H \in C(\mathcal{M})$. Note that the same equality holds for all $F, H \in C(\mathcal{M})$. Density of $C(\mathcal{M})$ in $L^2(\mathcal{M}, \mu)$ now yields

$$\pi_K(\mathfrak{p}) = \pi_K(\mathfrak{q}).$$

Thus π_K is constant on the fibres of π_E . Since π_E is a continuous surjection from the compact space βG onto $E(X, G)$, it is a quotient map. Therefore there is a unique continuous map

$$w : E(X, G) \rightarrow K$$

such that $\pi_K = w \circ \pi_E$. The surjectivity of π_K implies that w is surjective. Moreover, for every $g \in G$, $w(g \circ \text{id}_X) = U_g$.

It remains to verify that w is a semigroup homomorphism. Let $g \in G$ and $\xi \in E(X, G)$, and choose a net $(h_i)_i$ in G converging to ξ . Continuity of left multiplication by g on $E(X, G)$ gives $(gh_i)_i \rightarrow g\xi$. Hence, by continuity of w ,

$$w(g\xi) = \lim_i w(gh_i) = \lim_i U_{gh_i} = \lim_i U_g U_{h_i} = U_g w(\xi).$$

Indeed, $(U_{h_i})_i = (w(h_i))_i \rightarrow w(\xi)$ in the strong operator topology, and left multiplication by U_g is continuous (in the strong operator topology): for every $F \in L^2(\mathcal{M}, \mu)$,

$$\|U_g U_{h_i} F - U_g w(\xi) F\|_{L^2(\mathcal{M}, \mu)} = \|U_{h_i} F - w(\xi) F\|_{L^2(\mathcal{M}, \mu)} \rightarrow 0.$$

Now let $\eta, \xi \in E(X, G)$ and choose a net $(g_i)_i$ in G converging to η . Since right multiplication by ξ is continuous in $E(X, G)$,

$$w(\eta\xi) = \lim_i w(g_i\xi) = \lim_i U_{g_i} w(\xi) = w(\eta)w(\xi).$$

□(claim)

Combining the above two claims and the universality of the Bohr compactification $(B_2, \mathfrak{b}_2, w_2)$, we obtain a continuous epimorphism $\rho : B_2 \rightarrow K$ such that the diagram

$$\begin{array}{ccc} B_2 & \xrightarrow{\rho} & K \\ & \searrow \mathfrak{b}_2 & \nearrow g \mapsto U_g \\ & G & \end{array}$$

commutes. Thus, $\rho : B_2 \rightarrow U(L^2(\mathcal{M}, \mu))$ is the auxiliary unitary representation we are looking for.

We now apply Peter–Weyl, Fact 2.11, to the unitary representation ρ . Since B_2 is compact, that theorem says that $L^2(\mathcal{M}, \mu)$ is the Hilbert direct sum of irreducible $\rho(B_2)$ -invariant subspaces and that each of these subspaces is finite-dimensional. In particular, the finite-dimensional $\rho(B_2)$ -invariant subspaces have dense linear span in $L^2(\mathcal{M}, \mu)$. It is therefore enough to prove that every nonzero finite-dimensional $\rho(B_2)$ -invariant subspace W of $L^2(\mathcal{M}, \mu)$ is contained in $(w_2|_{\mathcal{M}})^*[C(B_2)]$.

Let $W \subseteq L^2(\mathcal{M}, \mu)$ be an arbitrary nonzero finite-dimensional $\rho(B_2)$ -invariant subspace, and set $n := \dim W$. Choose an orthonormal basis $f_1, \dots, f_n$ of W and write $\mathbf{f} := (f_1, \dots, f_n)$. For every $b \in B_2$, the operator $\rho(b)$ preserves W . Define the matrix $\pi(b) = (\pi_{ij}(b))_{i,j=1}^n$ by

$$\pi_{ij}(b) := \langle \rho(b)f_j, f_i \rangle_{L^2(\mathcal{M}, \mu)}.$$

Equivalently, $\rho(b)f_j = \sum_{i=1}^n f_i \pi_{ij}(b)$ for $j \leq n$. Thus, with $\rho(b)$ acting coordinate-wise on $\mathbf{f}$, these identities can be written as

$$\rho(b)\mathbf{f} = \mathbf{f} \pi(b).$$

Since $\rho(b)|_W$ is unitary, $\pi(b) \in U(n)$. Moreover, because $\rho : B_2 \rightarrow K$ is a continuous group epimorphism, we have $\pi(bc) = \pi(b)\pi(c)$ for every $b, c \in B_2$, and every matrix coefficient mapping

$$b \mapsto \pi_{ij}(b) = \langle \rho(b)f_j, f_i \rangle_{L^2(\mathcal{M}, \mu)}$$

is continuous. Hence $\pi : B_2 \rightarrow U(n)$ is a continuous unitary representation. Finally, since $\rho(b_2(g)) = U_g$ for all $g \in G$, we obtain

$$U_g \mathbf{f} = \mathbf{f} \pi(b_2(g)).$$

For every $g \in G$ and μ -almost every $\eta \in \mathcal{M}$, we have

$$\mathbf{f}(g^{-1}\eta)\pi(w_2(g^{-1}\eta)) = \mathbf{f}(\eta)\pi(b_2(g))\pi(b_2(g)^{-1}w_2(\eta)) = \mathbf{f}(\eta)\pi(w_2(\eta)).$$

Thus, every coordinate of $\mathcal{M} \ni \eta \mapsto \mathbf{f}(\eta)\pi(w_2(\eta))$ belongs to $L^2(\mathcal{M}, \mu)^G$. Since μ is ergodic, $L^2(\mathcal{M}, \mu)^G = \mathbb{C}\mathbf{1}$. Hence there is a constant row $c \in \mathbb{C}^n$ such that $\mathbf{f}(\eta)\pi(w_2(\eta)) = c$ for μ -almost every $\eta \in \mathcal{M}$. Consequently,

$$\mathbf{f}(\eta) = c\pi(w_2(\eta))^{-1}$$

for μ -almost every $\eta \in \mathcal{M}$. The map $B_2 \ni b \mapsto c\pi(b)^{-1} \in \mathbb{C}^n$ is continuous. Hence each f_j is the pullback under $w_2|_{\mathcal{M}}$ of a member of $C(B_2)$. It follows that $W \subseteq (w_2|_{\mathcal{M}})^*[C(B_2)]$. $\square$

The proof of Lemma 5.1 also yields the following corollary:

Corollary 5.2. *Let (X, G) be a tame flow and let $\mu \in \text{Prob}_G(X)$. Then*

$$L^2(X, \mu) = \overline{\text{span}}\{V : V \text{ is a finite-dimensional } G\text{-invariant subspace}\}.$$

Theorem 5.3. *Let (X, G) be tame, let $\mathcal{M} \trianglelefteq_m E(X, G)$, assume that $(\mathcal{M}, G)$ is amenable, and let $u^2 = u \in \mathcal{M}$. Then*

$$w_2|_{u\mathcal{M}} : u\mathcal{M} \longrightarrow B_2$$

is an isomorphism of compact Hausdorff topological groups (with the τ -topology on $u\mathcal{M}$).

Proof. Let μ be a G -invariant probability on $\mathcal{M}$. By Lemma 5.1, it is unique and minimality gives $\text{supp}(\mu) = \mathcal{M}$. Put $E := E(X, G)$.

We need to show that $\ker(w_2|_{u\mathcal{M}}) = \{u\}$. Take $\eta \in u\mathcal{M}$ such that $w_2(\eta) = 1_{B_2}$. Right multiplication

$$R_\eta : \mathcal{M} \longrightarrow E, \quad R_\eta(\xi) = \xi\eta,$$

is continuous and so an automorphism of the flow $(\mathcal{M}, G)$. Indeed, because $\eta \in \mathcal{M}$ we have that $\xi\eta \in \mathcal{M}$ for every $\xi \in \mathcal{M}$. Then, every $\xi \in \mathcal{M} = Eu$ satisfies $\xi u = \xi$, and right multiplication by the inverse of η in $u\mathcal{M}$ is the inverse of R_η . Moreover, $w_2(\eta) = 1_{B_2}$ implies $w_2|_{\mathcal{M}} \circ R_\eta = w_2|_{\mathcal{M}}$.

The measure $(R_\eta)_*\mu$ is G -invariant, so unique ergodicity gives $(R_\eta)_*\mu = \mu$. Since $(R_\eta)_*\mu = \mu$ and R_η is a homeomorphism, pullback by R_η defines a unitary operator

$$(R_\eta)^* : L^2(\mathcal{M}, \mu) \longrightarrow L^2(\mathcal{M}, \mu), \quad (R_\eta)^*(F) := F \circ R_\eta.$$

Furthermore, for every $h \in C(B_2)$, and its class $H = [h] \in L^2(B_2, \mathfrak{h}_{B_2})$,

$$(R_\eta)^*((w_2|_{\mathcal{M}})^*(H)) = ((w_2|_{\mathcal{M}}) \circ R_\eta)^*(H) = (w_2|_{\mathcal{M}})^*(H).$$

Thus $(R_\eta)^*$ fixes $(w_2|_{\mathcal{M}})^*[C(B_2)]$ pointwise. Since this subspace is dense in $L^2(\mathcal{M}, \mu)$ by Lemma 5.1, we obtain $(R_\eta)^* = \text{id}_{L^2(\mathcal{M}, \mu)}$. Consequently, for every $F \in C(\mathcal{M})$, $F \circ R_\eta = F$ happens μ -almost everywhere. Since $\text{supp}(\mu) = \mathcal{M}$ and both sides are continuous, this equality holds everywhere. Continuous functions separate points, so $R_\eta = \text{id}_{\mathcal{M}}$. Therefore $\eta = u\eta = R_\eta(u) = u$. $\square$

Proof of Theorem 4.5. By [HK26, Corollary 5.6], the Ellis flow $(E(X, G), G)$ is hereditarily amenable. Therefore, Theorem 5.3 applies to every $\mathcal{M} \trianglelefteq_m E(X, G)$ and every idempotent $u \in \mathcal{M}$, and yields GNC for (X, G) . $\square$

Corollary 5.4. *If a flow (X, G) is finite then B_2 is finite as well. Let (X, G) be minimal, tame and amenable. If B_2 is finite, then $|X| \leq |B_2|$. In particular, for a minimal, tame and amenable (X, G) , X is finite if and only if B_2 is finite.*

Proof. If X is finite then also $E(X, G)$ is finite and so B_2 as a surjective image of $E(X, G)$.

Now assume that B_2 is finite. Theorem 5.3 gives a topological group isomorphism $u\mathcal{M} \cong B_2$ and so $u\mathcal{M}$ is finite. Then, by Fact 2.8, $\mathcal{M} = u\mathcal{M}$. Since $X = \mathcal{M}x$, it follows that X is finite and bounded by the size of B_2 . $\square$

See Proposition 5.11 for a more general variant of the above corollary.

5.2. Measured structure of amenable tame flows. Now, we combine Lemma 5.1 with a Galois-correspondence result from [TT71] for von Neumann algebras, to characterize the measured structure of minimal, tame and amenable flows (in Theorem 5.7 and corresponding translation in Corollary 5.8).

Let (X, G) be a minimal flow. Put $E := E(X, G)$, choose $\mathcal{M} \trianglelefteq_m E$ and $u^2 = u \in \mathcal{M}$. Choose $x \in uX$, so $ux = x$. Moreover, set

$$\text{Gal}_X := \text{Gal}(X, x) \quad \text{and} \quad \text{Gal}_B := w_2[\text{Gal}_X].$$

Remark 5.5. Gal_X and Gal_B are closed (in the τ -topology and in the topology of B_2 , respectively).

Proof. By Remark 2.5, we have that Gal_X is τ -closed. Then Gal_B is closed in B_2 as the image of a closed subset in quasi-compact $u\mathcal{M}$ via a continuous map to a Hausdorff space. Here we use the τ -continuity of $w_2|_{u\mathcal{M}}$ from Remark 3.22. $\square$

Remark 5.6. The formula

$$\rho_x : \mathcal{M}x \longrightarrow B_2/\text{Gal}_B, \quad \rho_x(\xi x) := w_2(\xi) \text{Gal}_B,$$

where $\xi \in \mathcal{M}$, defines a continuous surjective G -equivariant map, where B_2/Gal_B carries the quotient topology and the natural left G -action (i.e. $g \cdot b \text{Gal}_B := \mathfrak{b}_2(g)b \text{Gal}_B$ for $g \in G$). Moreover, as X is minimal, we have $\mathcal{M}x = X$ and ρ_x is a factor map of compact (Hausdorff) flows.

Proof. If $\xi x = \zeta x$, then $(u\zeta)^{-1}(u\xi) \in \text{Gal}_X$, and, since $w_2(u) = 1_{B_2}$, this gives $w_2(\xi) \text{Gal}_B = w_2(\zeta) \text{Gal}_B$. Thus ρ_x is well-defined. If $\text{ev}_x : \mathcal{M} \rightarrow \mathcal{M}x$ is evaluation at x and $q : B_2 \rightarrow B_2/\text{Gal}_B$ is the quotient map, then $q \circ w_2|_{\mathcal{M}} = \rho_x \circ \text{ev}_x$ and so $\rho_x \circ \text{ev}_x$ is continuous. Since ev_x is a quotient map, ρ_x is continuous. An equivariant continuous surjective map between compact (Hausdorff) spaces is a factor map. $\square$

Theorem 5.7. *Let (X, G) be minimal, tame and amenable. Then:*

- (A) *The flows X and $\mathcal{M}$ carry unique G -invariant regular Borel probability measures, denoted by μ_X and $\mu_{\mathcal{M}}$, and we have $\text{supp}(\mu_X) = X$ and $\text{supp}(\mu_{\mathcal{M}}) = \mathcal{M}$. Thus, $(\text{ev}_x)_* \mu_{\mathcal{M}} = \mu_X$.*
- (B) *Pullback along $w_2|_{\mathcal{M}}$ is a normal trace-preserving $*$ -isomorphism*

$$(w_2|_{\mathcal{M}})^* : L^\infty(B_2, \mathfrak{h}_{B_2}) \longrightarrow L^\infty(\mathcal{M}, \mu_{\mathcal{M}}). \quad (2)$$

On both L^2 -level and L^∞ -level the map $\Phi_x := ((w_2|_{\mathcal{M}})^)^{-1} \text{ev}_x^*$ is G -equivariant and isometric, and on L^∞ -level it is a normal trace-preserving $*$ -monomorphism.*

- (C) *There exists a unique closed subgroup H_x of B_2 such that*

$$\begin{aligned} \Phi_x[L^\infty(X, \mu_X)] &= L^\infty(B_2, \mathfrak{h}_{B_2})^{H_x} \\ &:= \{F \in L^\infty(B_2, \mathfrak{h}_{B_2}) : R_h F = F \text{ for every } h \in H_x\}, \end{aligned}$$

where $(R_h F)(b) := F(bh)$. In fact, $H_x = \text{Gal}_B$. Consequently,

$$L^\infty(X, \mu_X) \cong_G L^\infty(B_2/\text{Gal}_B, \mathfrak{h}_{B_2/\text{Gal}_B}). \quad (3)$$

(Here G acts on B_2/Gal_B by $g \cdot b \text{Gal}_B = \mathfrak{b}_2(g)b \text{Gal}_B$ and $\cong_G$ denotes a normal trace-preserving G -equivariant $$ -isomorphism of commutative von Neumann algebras.)*

Proof. By Fact 2.8 and [HK26, Lemma 2.2(3)], $(\mathcal{M}, G)$ is amenable, minimal, and tame. Fact 2.1, applied to (X, G) and to $(\mathcal{M}, G)$, gives uniqueness of the G -invariant measures. The support of a G -invariant measure is a nonempty subflow, so minimality gives the assertions about support in (A).

We move on to the proof of (B). Lemma 5.1 gives

$$(w_2|_{\mathcal{M}})_* \mu_{\mathcal{M}} = \mathfrak{h}_{B_2} \quad (4)$$

and a unitary isomorphism

$$(w_2|_{\mathcal{M}})^* : L^2(B_2, \mathfrak{h}_{B_2}) \longrightarrow L^2(\mathcal{M}, \mu_{\mathcal{M}}). \quad (5)$$

If $F \in L^\infty(\mathcal{M}, \mu_{\mathcal{M}})$, surjectivity in (5) gives $f \in L^2(B_2, \mathfrak{h}_{B_2})$ with $F = f \circ w_2|_{\mathcal{M}}$ almost everywhere. For $C > \|F\|_\infty$, (4) yields

$$\mathfrak{h}_{B_2}(|f| > C) = \mu_{\mathcal{M}}(|f \circ w_2|_{\mathcal{M}} > C) = 0.$$

Thus, $f \in L^\infty(B_2, \mathfrak{h}_{B_2})$ and $\|f\|_\infty = \|f \circ w_2|_{\mathcal{M}}\|_\infty$. It follows that $(w_2|_{\mathcal{M}})^*$ restricted to $L^\infty(B_2, \mathfrak{h}_{B_2})$ is onto on $L^\infty(\mathcal{M}, \mu_{\mathcal{M}})$. Literal pullback and (4) make

it a trace-preserving $*$ -isomorphism. Every $*$ -isomorphism between von Neumann algebras is an order isomorphism, hence preserves suprema of bounded increasing nets of positive elements and is therefore normal [KR97, Corollary 7.1.16]. This proves (B).

Now, we move to the proof of (C). Set $\mathcal{A}_x := \Phi_x[L^\infty(X, \mu_X)]$. We will prove that

$$\mathcal{A}_x = L^\infty(B_2, \mathfrak{h}_{B_2})^{\text{Gal}_B}. \quad (6)$$

We start with showing that $\mathcal{A}_x \subseteq L^\infty(B_2, \mathfrak{h}_{B_2})^{\text{Gal}_B}$. For $\eta \in u\mathcal{M}$, let $R_\eta : \mathcal{M} \rightarrow \mathcal{M}$ be given by $R_\eta(\xi) := \xi\eta$. This is a G -automorphism with inverse $R_{\eta^{-1}}$ (see also the proof of Theorem 5.3). Hence $(R_\eta)_*\mu_{\mathcal{M}}$ is G -invariant, and uniqueness in (A) gives $(R_\eta)_*\mu_{\mathcal{M}} = \mu_{\mathcal{M}}$. Moreover, since w_2 is a semigroup homomorphism, $w_2|_{\mathcal{M}}(R_\eta(\xi)) = w_2(\xi)w_2(\eta)$ for every $\xi \in \mathcal{M}$, and therefore $(w_2|_{\mathcal{M}})^* \circ R_{w_2(\eta)} = (R_\eta)^* \circ (w_2|_{\mathcal{M}})^*$.

Claim 1. For every $\eta \in \text{Gal}_X$, for every $F \in \mathcal{A}_x$, we have $R_{w_2(\eta)}(F) = F$.

Proof. Suppose that $\eta(x) = x$, then $\text{ev}_x \circ R_\eta = \text{ev}_x$. Hence, for every $F \in L^\infty(X, \mu_X)$,

$$\begin{aligned} (w_2|_{\mathcal{M}})^*(R_{w_2(\eta)}\Phi_x(F)) &= (R_\eta)^*((w_2|_{\mathcal{M}})^*\Phi_x(F)) = (R_\eta)^*(\text{ev}_x^*(F)) \\ &= \text{ev}_x^*(F) = (w_2|_{\mathcal{M}})^*\Phi_x(F). \end{aligned}$$

Since $(w_2|_{\mathcal{M}})^*$ is injective, $R_{w_2(\eta)}$ fixes $\mathcal{A}_x$ pointwise. $\square$ (claim)

Since $\text{Gal}_B = w_2[\text{Gal}_X]$, the claim gives $\mathcal{A}_x \subseteq L^\infty(B_2, \mathfrak{h}_{B_2})^{\text{Gal}_B}$. For the reverse inclusion, let $p : B_2 \rightarrow B_2/\text{Gal}_B$ be the quotient map. By definition of ρ_x , $p \circ (w_2|_{\mathcal{M}}) = \rho_x \circ \text{ev}_x$. Together with (A) and (4), this gives $(\rho_x)_*\mu_X = p_*\mathfrak{h}_{B_2} = \mathfrak{h}_{B_2/\text{Gal}_B}$. Taking pullbacks, we obtain

$$(w_2|_{\mathcal{M}})^* \circ p^* = \text{ev}_x^* \circ \rho_x^*, \quad \text{hence} \quad p^* = \Phi_x \circ \rho_x^*.$$

Since Gal_B is closed (Remark 5.5), the final assertion of Fact 2.12 yields

$$L^\infty(B_2, \mathfrak{h}_{B_2})^{\text{Gal}_B} = p^*[L^\infty(B_2/\text{Gal}_B, \mathfrak{h}_{B_2/\text{Gal}_B})] \subseteq \Phi_x[L^\infty(X, \mu_X)] = \mathcal{A}_x.$$

This proves (6). The subgroup-recovery formula in Fact 2.12 shows that Gal_B is the unique closed subgroup of B_2 whose fixed-point algebra is $\mathcal{A}_x$. Taking $H_x := \text{Gal}_B$, we obtain the existence and uniqueness asserted in (C).

By the final assertion of Fact 2.12 and the identity $p_*\mathfrak{h}_{B_2} = \mathfrak{h}_{B_2/H_x}$, pullback gives a normal trace-preserving $*$ -isomorphism

$$p^* : L^\infty(B_2/H_x, \mathfrak{h}_{B_2/H_x}) \xrightarrow{\cong} \mathcal{A}_x = L^\infty(B_2, \mathfrak{h}_{B_2})^{H_x}.$$

Since p intertwines left translation by $\mathfrak{b}_2(g)$ with the stated G -action on B_2/H_x , the map

$$(p^*)^{-1} \circ \Phi_x : L^\infty(X, \mu_X) \longrightarrow L^\infty(B_2/\text{Gal}_B, \mathfrak{h}_{B_2/\text{Gal}_B})$$

is a normal trace-preserving G -equivariant $*$ -isomorphism. This proves (C). $\square$

Corollary 5.8. *For a probability space (Y, ν) , let $\text{MAlg}(Y, \nu)$ denote its complete Boolean algebra of measurable sets modulo null sets. Assertion (C) equivalently gives a measure-preserving G -isomorphism*

$$\text{MAlg}(X, \mu_X) \cong_G \text{MAlg}(B_2/\text{Gal}_B, \mathfrak{h}_{B_2/\text{Gal}_B})$$

for a minimal, tame and amenable flow (X, G) .

The next lemma is a consequence of Fact 2.8 and GNC (i.e., Theorem 4.5).

Lemma 5.9. *Let (X, G) be minimal, tame and amenable. The set uX is dense in X and the restriction of the map from Remark 5.6,*

$$\rho_x|_{uX} : uX \longrightarrow B_2/\text{Gal}_B$$

is a bijection.

Proof. First, we prove density. By minimality of X , $X = \mathcal{M}x$ and so $uX = (u\mathcal{M})x$. By Fact 2.8, $\overline{uX} = \overline{(u\mathcal{M})x} = \overline{u\mathcal{M}x} = \mathcal{M}x = X$.

The restriction $\rho_x|_{uX}$ is onto because $w_2|_{u\mathcal{M}}$ is onto. To prove injectivity, suppose that $\eta, \kappa \in u\mathcal{M}$ and $\rho_x(\eta x) = \rho_x(\kappa x)$. Then $w_2(\kappa^{-1}\eta) \in \text{Gal}_B = w_2[\text{Gal}_X]$. Since $w_2|_{u\mathcal{M}}$ is injective by Theorem 5.3, $\kappa^{-1}\eta \in \text{Gal}_X$, and consequently $\eta x = \kappa x$. Hence $\rho_x|_{uX}$ is a bijection. $\square$

There is a natural structure of a homogeneous $u\mathcal{M}$ -space on $u\mathcal{M}/\text{Gal}_X$, and similarly a structure of a homogeneous B_2 -space on B_2/Gal_B . If GNC holds, i.e. if $w_2|_{u\mathcal{M}} : u\mathcal{M} \rightarrow B_2$ is an isomorphism, then, using the definition of Gal_B , we conclude that $u\mathcal{M}/\text{Gal}_X$ and B_2/Gal_B are isomorphic as homogeneous spaces. Below, we provide an alternative proof which uses Lemma 5.9.

Corollary 5.10. *Let (X, G) be minimal, tame and amenable. Then*

$$B_2/\text{Gal}_B \cong u\mathcal{M}/\text{Gal}_X$$

as homogeneous spaces.

Proof. Consider the composition of maps ev_x and $\rho_x|_{uX}$ from Lemma 5.9, which is on the other hand equal to the composition of $w_2|_{u\mathcal{M}}$ with the quotient map $B_2 \rightarrow B_2/\text{Gal}_B$. It factors through $u\mathcal{M}/\text{Gal}_X$, giving a homeomorphism of homogeneous spaces $B_2/\text{Gal}_B \cong u\mathcal{M}/\text{Gal}_X$. $\square$

Proposition 5.11. *Let (X, G) be minimal, tame and amenable. If X is first countable (e.g. if X is metrizable) then $|X| = |B_2/\text{Gal}_B|$.*

Proof. By Lemma 5.9, $|B_2/\text{Gal}_B| = |uX| \leq |X|$. We need to argue for other inequality. In the case of finite $|B_2/\text{Gal}_B| = |uX|$, by density of uX in X , we obtain also $|uX| = |X|$ and so $|B_2/\text{Gal}_B| = |X|$. Let us assume that B_2/Gal_B is infinite.

By Lemma 5.9, we have that uX is dense in X . Thus, every point of $X = \overline{uX}$ is then the limit of a sequence in uX , so

$$|X| \leq |uX|^{\aleph_0} = |B_2/\text{Gal}_B|^{\aleph_0}.$$

By Remark 5.5, Gal_B is closed, so B_2/Gal_B is a compact Hausdorff homogeneous space.

By homogeneity, the minimum cardinality $\chi(b\text{Gal}_B, B_2/\text{Gal}_B)$ of a local base does not depend on the choice of point $b\text{Gal}_B$ in B_2/Gal_B . Thus, for the cardinal $w(B_2/\text{Gal}_B) = \chi(B_2/\text{Gal}_B) = \sup\{\chi(b\text{Gal}_B, B_2/\text{Gal}_B) : b\text{Gal}_B \in B_2/\text{Gal}_B\} + \omega$, Theorem 7.19 from [Hod84] gives us

$$2^{w(B_2/\text{Gal}_B)} = 2^{\chi(B_2/\text{Gal}_B)} \leq |B_2/\text{Gal}_B|.$$

As B_2/Gal_B is Hausdorff, we have $|B_2/\text{Gal}_B| \leq 2^{w(B_2/\text{Gal}_B)}$. Hence $2^{w(B_2/\text{Gal}_B)} = |B_2/\text{Gal}_B|$, and so $|B_2/\text{Gal}_B|^{\aleph_0} = |B_2/\text{Gal}_B|$. $\square$

Corollary 5.2 might be used in the weakly-mixing set-up. Recall that a probability-measure-preserving action $G \curvearrowright (X, \mu)$ is *weakly mixing* if the diagonal action of G on $(X \times X, \mu \otimes \mu)$ is ergodic. As the proof below recalls, weak mixing excludes nonzero finite-dimensional G -invariant subspaces of $(\mathbb{C}\mathbf{1})^\perp \subseteq L^2(X, \mu)$, whereas Corollary 5.2 says that, for tame flows, finite-dimensional invariant subspaces generate $L^2(X, \mu)$. Thus the only (non-zero) finite-dimensional G -invariant subspaces must be $L^2(X, \mu)^G = \mathbb{C}\mathbf{1}$ and so $L^2(X, \mu) = \mathbb{C}\mathbf{1}$:

Corollary 5.12. *Let (X, G) be minimal, tame and amenable. If the probability-measure-preserving action of G on (X, μ) is weakly mixing, then X is a singleton.*

Proof. Let $W \subseteq (\mathbb{C}\mathbf{1})^\perp \subseteq L^2(X, \mu)$ be a nonzero finite-dimensional G -invariant subspace, and let $f_1, \dots, f_n$ be an orthonormal basis of W . Then

$$F(x, y) := \sum_{i=1}^n f_i(x) \overline{f_i(y)}$$

defines an element of $L^2(X \times X, \mu \otimes \mu)$. For every $g \in G$, the functions $U_g f_1, \dots, U_g f_n$ form another orthonormal basis of W . Thus F is invariant under the diagonal action. Since every $f_i \in (\mathbb{C}\mathbf{1})^\perp$, every f_i has integral zero, and so we have

$$\int_{X \times X} F d(\mu \otimes \mu) = 0.$$

On the other hand, orthonormality gives $\|F\|_2^2 = n > 0$. Thus F is a nonzero invariant function orthogonal to the constants, contradicting weak mixing. Therefore $(\mathbb{C}\mathbf{1})^\perp \subseteq L^2(X, \mu)$ contains no nonzero finite-dimensional G -invariant subspace.

Now, Corollary 5.2 yields $L^2(X, \mu) = \mathbb{C}\mathbf{1}$. By Theorem 5.7(A), $\text{supp}(\mu) = X$, so the canonical map $C(X) \rightarrow L^2(X, \mu)$ is injective. Consequently $C(X) = \mathbb{C}\mathbf{1}$, and X is a singleton. $\square$

5.3. MEF for tame, minimal and amenable flows. In this subsection, we apply the results of the previous subsection to the maximal equicontinuous factor of a minimal, tame and amenable flow. We show that the factor map is proximal and induces isomorphisms on the von Neumann algebras level. We begin with a general observation for minimal flows.

We work with the set-up and notation of the previous subsection, in particular (X, G) is a minimal flow. Let $\pi : X \rightarrow Z$ be MEF of (X, G) , and let

$$\widehat{\pi} : E(X, G) \rightarrow E(Z, G)$$

be induced by π . Then set $z := \pi(x)$ and $K := u\mathcal{M}$.

Lemma 5.13. *There is a unique G -homeomorphism*

$$\gamma : B_2 / \text{Gal}_B \rightarrow Z$$

such that $\pi = \gamma \circ \rho_x$ (ρ_x as in Remark 5.6). It is given by $\gamma(w_2(\eta) \text{Gal}_B) = \widehat{\pi}(\eta)(z)$, where $\eta \in K$. Moreover, $\widehat{\pi}[K] = E(Z, G)$.

Proof. By Remark 5.6, the map $\rho_x : X \rightarrow B_2 / \text{Gal}_B$ is a factor map onto a minimal equicontinuous flow. The universal property of the Bohr compactification gives a continuous homomorphism

$$\beta : B_2 \rightarrow E(Z, G)$$

such that $\beta \circ w_2 = \widehat{\pi}$. Its image is compact and contains the dense image of G in $E(Z, G)$, so β is onto. Since $w_2[K] = B_2$, it follows that $\widehat{\pi}[K] = E(Z, G)$.

Let $p : B_2 \rightarrow B_2/\text{Gal}_B$ be the quotient map. Consider the following diagram:

$$\begin{array}{ccc} \mathcal{M}x & & B_2 \xrightarrow{\beta} E(Z, G) \\ & \searrow \rho_x & \downarrow p \quad \downarrow \text{ev}_z \\ & & B_2/\text{Gal}_B \quad Z \end{array}$$

Note that if $b \in \text{Gal}_B$ then $\beta(b)(z) = z$. Indeed, there is some $\eta \in \text{Gal}_X$ with $b = w_2(\eta)$, and so

$$\beta(b)(z) = (\beta w_2)(\eta)(\pi(x)) = \widehat{\pi}(\eta)(\pi(x)) = \pi(\eta(x)) = \pi(x) = z.$$

The formula $\gamma(b \text{Gal}_B) := \beta(b)(z)$ defines a continuous G -map from B_2/Gal_B to Z . By decoding the definitions, we obtain easily $\gamma \circ \rho_x = \pi$.

Since $\rho_x : X \rightarrow B_2/\text{Gal}_B$ is an equicontinuous factor of X and $\pi : X \rightarrow Z$ is MEF, there exists a factor map

$$\alpha : Z \longrightarrow B_2/\text{Gal}_B$$

such that $\rho_x = \alpha \circ \pi$. We have $\gamma \circ \alpha \circ \pi = \gamma \circ \rho_x = \pi$ and so, surjectivity of π gives $\gamma \circ \alpha = \text{id}_Z$. Since α is onto, it follows that α and γ are inverse G -homeomorphisms. Uniqueness of γ follows from the surjectivity of ρ_x . $\square$

From now on, assume additionally that (X, G) is tame and amenable, and let μ_X be its unique G -invariant measure. Let λ_Z be the unique G -invariant measure on the flow Z . We have $\pi_* \mu_X = \lambda_Z$.

Theorem 5.14. (1) *The pullback of π is a normal trace-preserving G -equivariant *-isomorphism of commutative von Neumann algebras:*

$$\pi^* : L^\infty(Z, \lambda_Z) \xrightarrow{\cong} L^\infty(X, \mu_X) \quad (7)$$

and a G -equivariant unitary isomorphism

$$\pi^* : L^2(Z, \lambda_Z) \xrightarrow{\cong} L^2(X, \mu_X).$$

(2) *For every $y \in Z$, $u[\pi^{-1}(y)]$ consists of the unique point of uX lying over y . In particular, π is proximal.*

Proof. We work in the notation of Theorem 5.7 and its proof, Remark 5.6 and Lemma 5.13.

Claim 1. $\text{ev}_x^* \circ \pi^* = (w_2|_{\mathcal{M}})^* \circ p^* \circ \gamma^*$.

Proof. Let $F \in L^\infty(Z, \lambda_Z)$ and $F = [f]$ for some $f \in C(Z)$. Then for every $\eta \in \mathcal{M}$

$$\begin{aligned} (w_2|_{\mathcal{M}}^* \circ p^* \circ \gamma^*)(f)(\eta) &= (f \circ \gamma \circ p \circ w_2|_{\mathcal{M}})(\eta) \\ &= (f \circ \text{ev}_z \circ \beta \circ w_2|_{\mathcal{M}})(\eta) \\ &= (f \circ \text{ev}_z \circ \widehat{\pi})(\eta) = f(\widehat{\pi}(\eta)(z)) \\ &= f(\pi(\eta(x))) = (f \circ \pi \circ \text{ev}_x)(\eta) \\ &= (\text{ev}_x^* \circ \pi^*)(f)(\eta) \end{aligned}$$

Thus $(\text{ev}_x^* \circ \pi^*)(F) = ((w_2|_{\mathcal{M}})^* \circ p^* \circ \gamma^*)(F)$. By the weak*-density of $C(Z)$ in $L^\infty(Z, \lambda_Z)$, we obtain $\text{ev}_x^* \circ \pi^* = (w_2|_{\mathcal{M}})^* \circ p^* \circ \gamma^*$ on the entire $L^\infty(Z, \lambda_Z)$. $\square$ (claim)

The map

$$\Psi_x := \left((p^*)^{-1} \circ (w_2|_{\mathcal{M}}^*)^{-1} \circ \text{ev}_x^* \right) : L^\infty(X, \mu_X) \longrightarrow L^\infty(B_2 / \text{Gal}_B, \mathfrak{h}_{B_2 / \text{Gal}_B})$$

is the normal trace-preserving G -equivariant $*$ -isomorphism from Theorem 5.7(C). Now, by applying $(p^*)^{-1} \circ (w_2|_{\mathcal{M}}^*)^{-1}$ to both sides of the equality from the above claim, we obtain $\Psi_x \circ \pi^* = \gamma^*$. Because γ is a G -homeomorphism and Ψ_x is a normal trace-preserving G -equivariant $*$ -isomorphism, we conclude that the map $\pi^* : L^\infty(Z, \lambda_Z) \xrightarrow{\cong} L^\infty(X, \mu_X)$ is a normal trace-preserving G -equivariant $*$ -isomorphism of commutative von Neumann algebras.

Pullback π^* is an isometry on the L^2 -level. By the just obtained L^∞ -level part, its range contains $L^\infty(X, \mu_X)$ and is therefore dense in $L^2(X, \mu_X)$. The range of an isometry is closed, so the extension is onto. This proves the first point.

We now prove the second point. By Lemmas 5.13 and 5.9, the map $\pi|_{uX} = \gamma \circ (\rho_x|_{uX})$ is bijective. Thus, $(\pi|_{uX})^{-1}(y)$ consists of a single point. Let $x' \in u[\pi^{-1}(y)]$, i.e. $x' = ux''$ and $\pi(x'') = y$ for some $x'' \in X$. Then

$$\pi(ux'') = \widehat{\pi}(u)\pi(x'') = \widehat{\pi}(u)y = y,$$

since $\widehat{\pi}(u)$ is the only idempotent in $E(Z, G)$. Thus, $ux'' \in \pi|_{uX}^{-1}(y)$.

Now, if $x' \in \pi|_{uX}^{-1}(y)$, then $x' = ux''$ for some $x'' \in X$ and $\pi(x'') = y$. We have $y = \pi(ux'')$ and so $ux'' \in \pi^{-1}(y)$. Hence, $x' = ux'' = u(ux'') \in u[\pi^{-1}(y)]$.

Consequently, u equalizes every pair of points in a common fibre of π , so π is proximal. $\square$

The above result will be used in a future paper by the first author and Andy Zucker to extend the main results of [Gla18] to nonmetrizable flows.

6. SPECIALIZATIONS TO MODEL-THEORETIC CONTEXTS

The goal of this section is to explain that the notions and results from this paper generalize the appropriate notions and results in the model-theoretic contexts of definable groups and of first order theories. We recover some known and obtain some new results in these contexts.

In Subsection 6.1, we explain that the GNC specializes in the context of definable NIP groups to the original Newelski's Conjecture (which justifies the name Generalized Newelski's Conjecture), and Theorem 4.4 implies Fact 1.8. In Subsection 6.2, we analyze the specialization of the GNC to first order theories and discuss the connection to Newelski's Conjecture for first order theories and Fact 1.9. In particular, we obtain a new result that the Bohr compactification of the group $\text{Aut}(\mathfrak{C})$ of automorphisms of the monster model $\mathfrak{C}$ below the ambit $(S_{\bar{c}}(\mathfrak{C}), \text{tp}(\bar{c}/\mathfrak{C}), \text{Aut}(\mathfrak{C}))$ (which is the core ambit for this context) is precisely the Kim-Pillay Galois group of the theory in question.

We will also point out some phenomena in the context of definable groups (and some also for first order theories) which fail drastically in the general abstract context, which may lead to some new (or existing) dynamical properties that should be assumed to get similar results in general. In Subsection 6.2.4, we state two natural problems that arise from these observations.

One of these phenomena is that for a group G definable in an NIP structure M , the Bohr compactification of G below the ambit $(S_G(M), \text{tp}(e/M), G)$ coincides with the Bohr compactification of G below the Ellis flow $(E(S_G(M), G), \text{id}, G)$. We leave as a question whether an analog holds for first order theories. This fails

in general for tame ambits as shown by Example 4.10. Another phenomenon is the transfer of amenability from an ambit to its Ellis flow, which holds both for definable groups and for first order theories (in the latter context, this is a new result proved in Subsection 6.2) under the NIP assumption, but fails for general tame ambits (we skip an example as it is pretty long).

In Subsection 6.3, using Theorem 4.4, we obtain a local version of Newelski's conjecture in the context of definable groups. The main point is a description of the Bohr compactification of G below an ambit of the form $(S_{G,\Delta}(M), \text{tp}_\Delta(e), G)$ for suitable families of formulas Δ , which in particular yields a description of the externally definable Bohr compactification of G .

6.1. Definable groups. For the notions and basic facts recalled below, see [New09; GPP14; KP17].

Let G be a group definable in a structure M . Let $N \succ M$ be $|M|^+$ -saturated, and $\mathfrak{C} \succ N$ a monster model (i.e. a κ -saturated and strongly κ -homogeneous model for a sufficiently large cardinal κ). Let $\bar{G} := G(\mathfrak{C})$, i.e. the interpretation of G in $\mathfrak{C}$. Now, we recall the main two flows considered in the context of definable groups.

By $S_G(M)$ we denote the space of complete types over M concentrated on G (i.e., containing the formula defining G). Then $(S_G(M), \text{tp}(e/M), G)$ is an ambit with the action defined by $g \cdot \text{tp}(a/M) := \text{tp}(ga/M)$ (where $a \in \bar{G}$). Note that the elements of $S_G(M)$ can be also treated as ultrafilters in the Boolean algebra of definable subsets of G ; then the above action is just the obvious action by left translation.

By $S_{G,M}(N)$ we denote the space of complete types over N concentrated on G and finitely satisfiable in M . Then $(S_{G,M}(N), \text{tp}(e/N), G)$ is an ambit with the action defined by $g \cdot \text{tp}(a/N) := \text{tp}(ga/N)$ (where $a \in G(\mathfrak{C})$ is such that $\text{tp}(a/N) \in S_{G,M}(N)$). This ambit is isomorphic to the ambit $(S_{G,\text{ext}}(M), \text{tp}_{\text{ext}}(e/M), G)$ of complete external types on G over M with the action given by left translation, where a complete external type on G over M is an ultrafilter in the Boolean algebra of externally definable subsets of G . We will be identifying these two ambits.

The advantage of $S_{G,M}(N)$ over $S_G(M)$ is that $S_{G,M}(N)$ carries a left continuous semigroup operation $*$ extending the action of G :

$$p * q = \text{tp}(ab/N),$$

where $b \models q$ and $a \models p$ is such that $\text{tp}(a/N, b)$ is finitely satisfiable in M . So $S_{G,M}(N)$ can and will be identified with $E(S_{G,M}(N), G)$ as a semigroup and G -ambit (see Fact 2.9).

The *definable Bohr compactification* of G is the universal group compactification of G such that the preimages of any two disjoint closed subsets of the target compact group can be separated by a definable subset of G . This is precisely the Bohr compactification of G below the ambit $(S_G(M), \text{tp}(e/M), G)$. Similarly, the *externally definable Bohr compactification* of G is the universal group compactification of G such that the preimages of any two disjoint closed subsets of the target compact group can be separated by an externally definable subset of G . This is precisely the Bohr compactification of G below the ambit $(S_{G,\text{ext}}(M), \text{tp}_{\text{ext}}(e/M), G)$.

So we have three natural Bohr compactifications of G :

- (1) $\mathfrak{b}_1: G \rightarrow B_1$ — the one below $(S_G(M), \text{tp}(e/M), G)$,
- (2) $\mathfrak{b}_2: G \rightarrow B_2$ — the one below $E(S_G(M), G)$,
- (3) $\mathfrak{b}_3: G \rightarrow B_3$ — the one below $(S_{G,\text{ext}}(M), \text{tp}_{\text{ext}}(e/M), G)$.

We have the obvious epimorphisms of G -ambits

$$S_{G,\text{ext}}(M) \longrightarrow E(S_G(M), G) \xrightarrow{\text{ev}_{\text{tp}(e/M)}} S_G(M),$$

which yield epimorphisms between the corresponding Bohr compactifications with the properties as in Remark 3.6

$$B_3 \xrightarrow{\mathfrak{g}} B_2 \xrightarrow{\mathfrak{f}} B_1.$$

Proposition 3.4 in [GPP14] yields $B_1 = \bar{G}/\bar{G}_M^{00}$ and $\mathfrak{b}_1(g) = g/\bar{G}_M^{00}$, where $\bar{G}_M^{00}$ is the smallest M -type-definable subgroup of $\bar{G}$ of bounded index (i.e., less than κ).

Let $u^2 = u \in \mathcal{M} \leq_m S_{G,M}(N)$.

Definition 6.1. *Newelski's map* is the group epimorphism $\rho: u\mathcal{M} \rightarrow \bar{G}/\bar{G}_M^{00}$ given by $\rho(\text{tp}(a/N)) := a/\bar{G}_M^{00}$.

Newelski's conjecture predicted that under suitable model-theoretic assumptions, the epimorphism ρ is an isomorphism. It was refuted in [GPP15] for groups definable in NIP theories, concretely for $G := \text{SL}_2(\mathbb{R})$ treated as a group definable in $M := (\mathbb{R}, +, \cdot)$. The main positive result is the following, where definable amenability of G means that the flow $(S_G(M), G)$ is amenable.

Fact 6.2 (Theorem 1.5 in [CS18]). *Newelski's conjecture holds for definably amenable groups definable in NIP theories.*

As discussed in the introduction, the various refinements and variants of Newelski's map [KP17], [KPR18], [KR20], [KP26] led to important contributions in model theory (e.g. to understand complexity of strong types) and to additive combinatorics (to construct generalized locally compact models for arbitrary approximate subgroups in a simpler and “more classical” way than originally done by Hrushovski).

Recall the map $\hat{\theta}: u\mathcal{M} \rightarrow B_3$ from Corollary 3.19 and Theorem 4.4.

Lemma 6.3. *The Newelski's map $\rho: u\mathcal{M} \rightarrow \bar{G}/\bar{G}_M^{00}$ coincides with the composition $\mathfrak{f} \circ \mathfrak{g} \circ \hat{\theta}$.*

Proof. Since $\hat{\theta}(p) := pe_{B_3}$ and both $\mathfrak{g}$ and $\mathfrak{f}$ are epimorphisms of G -ambits (see the comment after Definition 3.3), we have $(\mathfrak{f} \circ \mathfrak{g} \circ \hat{\theta})(p) = pe^{B_1} = p(e/\bar{G}_M^{00})$. In order to finish the proof, it is enough to check that the unique left continuous action of $S_{G,M}(N)$ on $\bar{G}/\bar{G}_M^{00}$ extending the action of G is given by $p(b/\bar{G}_M^{00}) = ab/\bar{G}_M^{00}$ for $a \models p$ such that $\text{tp}(a/N, b)$ is finitely satisfiable in M . The easiest way to see it is by showing that the formula $p \cdot (b/\bar{G}_M^{00}) := ab/\bar{G}_M^{00}$ for $a \models p$ such that $\text{tp}(a/N, b)$ is finitely satisfiable in M yields a well-defined left continuous function extending the action of G . It is well-defined, because whenever $a \equiv_M a'$, then a and a' have the same Lascar strong type over M (see Subsection 6.2.1) so they are in the same coset modulo $\bar{G}_M^{00}$ (as lying in the same coset is a bounded M -invariant equivalence relation on $\bar{G}$). The fact that it extends the action of G is obvious. To show continuity, for a basic open set $[\varphi(x, c)]$ in $S_{G,M}(N)$ (so $c \in N$) and an element $b/\bar{G}_M^{00} \in \bar{G}/\bar{G}_M^{00}$, we need to show that the set $U := \{p \in S_{G,M}(N) : p \cdot (b/\bar{G}_M^{00}) \in [\varphi(x, c)]\}$ is open. Since the restriction map $r: S_{G,M}(N, b) \rightarrow S_{G,M}(N)$ is a homeomorphism (by $|M|^+$ -saturation of N), it is enough to see that $r^{-1}[U]$

is open. And this is true, because for $a \in \bar{G}$ such that $\text{tp}(a/N, b) \in S_{G,M}(N, b)$ we have

$$\text{tp}(a/N, b) \in r^{-1}[U] \iff \text{tp}(ab/N) \in [\varphi(x, c)] \iff \text{tp}(a/N, b) \in [\varphi(xb, c)].$$

□

Thus, we see the GNC (i.e., Conjecture 4.1, equivalently 4.3) applied to the ambit $S_{G,M}(N)$ is weaker than the original Newelski's conjecture. Note that Example 4.10 shows that working in the general abstract context of an ambit (X, x_0, G) , even under the very strong assumptions of minimality and equicontinuity, it is not true that $\mathfrak{f} \circ \hat{\theta}: u\mathcal{M} \rightarrow B_1$ is an isomorphism, because the canonical epimorphism $\mathfrak{f}: B_2 \rightarrow B_1$ need not be an isomorphism. Thus, it seems hard to find reasonable assumptions which would imply that $\mathfrak{f} \circ \hat{\theta}$ is an isomorphism (cf. Problem 6.21).

On the other hand, in the NIP context (i.e., when the theory of M has NIP) Theorem 4.15 of [CPS14] implies that $\mathfrak{f} \circ \mathfrak{g}$ is an isomorphism, so

Corollary 6.4. *In the NIP context, the conclusion of the GNC applied to $S_{G,M}(N)$ implies the conclusion of Newelski's conjecture.*

Note that Example 4.17 in [KLM26] shows that without NIP (still in the model-theoretic context of the ambit $(S_G(M), \text{tp}(e/M), G)$), $\mathfrak{f}$ need not be an isomorphism.

By [CPS14, Theorem 3.17], under NIP, definable amenability of G implies amenability of the flow $(S_{G,M}(N), G)$. Moreover NIP together with Shelah's theorem from [She09] (saying that if M has NIP, then the Shelah expansion M^{Sh} of M has quantifier elimination and NIP) implies that $(S_{G,M}(N), G)$ is tame (see Remark 1.6 in [CS18]). Therefore,

Corollary 6.5. *Theorem 4.4 implies Fact 6.2.*

Fact 2.8 and Lemma 4.12 show that whenever $\mathcal{A}$ is a tame amenable ambit admitting a left continuous semigroup operation extending the action of G , then amenability of $\mathcal{A}$ implies strong amenability of $\mathcal{A}$. This yields the left-to-right implication in the following

Fact 6.6 (Theorem 0.6 in [KP17]). *If G is a group definable in a NIP theory, then G is definably amenable (equivalently externally definable amenable) if and only if it is externally definably strongly amenable.*

6.2. First order theories. Let $\mathfrak{C}$ be a monster model of a first order theory T . Let $\bar{c}$ be an enumeration of $\mathfrak{C}$ and $S_{\bar{c}}(\mathfrak{C})$ the space of complete types over $\mathfrak{C}$ (in a very long tuple of variables corresponding to $\bar{c}$) extending $\text{tp}(\bar{c}/\emptyset)$. Then $(S_{\bar{c}}(\mathfrak{C}), \text{tp}(\bar{c}/\mathfrak{C}), \text{Aut}(\mathfrak{C}))$ is an ambit with respect to the obvious action. As mentioned in the introduction, this ambit plays a central role in the topological dynamics of first order theories, as it captures a lot of information about the underlying theory T (e.g., see [KPR18; KNS19; Hru19; KR20; HKP26; GHK25]).

6.2.1. Bohr compactification of $\text{Aut}(\mathfrak{C})$ below $(S_{\bar{c}}(\mathfrak{C}), \text{tp}(\bar{c}/\mathfrak{C}), \text{Aut}(\mathfrak{C}))$. The goal is to prove that the Bohr compactification of $\text{Aut}(\mathfrak{C})$ below $(S_{\bar{c}}(\mathfrak{C}), \text{tp}(\bar{c}/\mathfrak{C}), \text{Aut}(\mathfrak{C}))$ coincides with the quotient map from $\text{Aut}(\mathfrak{C})$ to the Kim-Pillay Galois group $\text{Gal}_{\text{KP}}(T)$ of the theory T . This is counterpart for first order theories of Proposition 3.4 in [GPP14] which says that for a group G definable in a structure M the quotient map $G \rightarrow \bar{G}/G_M^{00}$ is the definable Bohr compactification of G .

First, we briefly recall basic knowledge on Lascar strong types and Galois groups. A longer exposition can be found for instance in Subsection 2.6 of [GHK25]. For details the reader may consult [LP01; Cas+01].

The relation of having the same *Lascar strong type*, denoted by E_L or by $\equiv_{Ls}$, is the finest bounded (i.e. with less than κ classes) invariant equivalence relation on a given product of sorts. The classes of E_L are called Lascar strong types. It turns out that E_L is the transitive closure of the relation of having the same type over a model (where the model is varying). The group of *Lascar strong automorphisms*, denoted by $\text{Autf}_L(\mathfrak{C})$, is the subgroup of $\text{Aut}(\mathfrak{C})$ consisting of the automorphisms preserving all Lascar strong types in all (possibly infinite) numbers of variables. The *Lascar Galois group* of T , denoted by $\text{Gal}_L(T)$, is the quotient $\text{Aut}(\mathfrak{C})/\text{Autf}_L(\mathfrak{C})$.

The relation of having the same *Kim-Pillay strong type*, denoted by E_{KP} or by $\equiv_{KP}$, is the finest bounded $\emptyset$ -type-definable equivalence relation (on a given product of sorts). The classes of E_{KP} are called *Kim-Pillay strong types*. The group of *Kim-Pillay strong automorphisms*, denoted by $\text{Autf}_{KP}(\mathfrak{C})$, is the subgroup of $\text{Aut}(\mathfrak{C})$ consisting of the automorphisms preserving all Kim-Pillay strong types in all (possibly infinite) numbers of variables. The *Kim-Pillay Galois group* of T , denoted by $\text{Gal}_{KP}(T)$, is the quotient $\text{Aut}(\mathfrak{C})/\text{Autf}_{KP}(\mathfrak{C})$. We have that $\text{Autf}_L(\mathfrak{C}) \leq \text{Autf}_{KP}(\mathfrak{C})$, so we have the obvious epimorphism $\text{Gal}_L(T) \rightarrow \text{Gal}_{KP}(T)$. The theory T is called *G-compact* if $\text{Autf}_L(\mathfrak{C}) = \text{Autf}_{KP}(\mathfrak{C})$.

Both groups $\text{Gal}_L(T)$ and $\text{Gal}_{KP}(T)$ do not depend on the choice of $\mathfrak{C}$, up to a natural isomorphism.

Both groups $\text{Gal}_L(T)$ and $\text{Gal}_{KP}(T)$ are equipped with the logic topology which make these groups respectively a quasi-compact and compact Hausdorff topological group. To define the logic topologies, choose any small $M \prec \mathfrak{C}$ and its enumeration $\bar{m}$. Let $\pi_L: \text{Aut}(\mathfrak{C}) \rightarrow \text{Aut}(\mathfrak{C})/\text{Autf}_L(\mathfrak{C})$ and $\pi_{KP}: \text{Aut}(\mathfrak{C}) \rightarrow \text{Aut}(\mathfrak{C})/\text{Autf}_{KP}(\mathfrak{C})$ be the quotient maps, and $t: \text{Aut}(\mathfrak{C}) \rightarrow S_{\bar{m}}(M)$ be given by $t(\sigma) := \text{tp}(\sigma(\bar{m})/M)$. We have the following commutative diagram:

$$\begin{array}{ccccc} \text{Aut}(\mathfrak{C}) & \xrightarrow{\pi_L} & \text{Gal}_L(T) & \longrightarrow & \text{Gal}_{KP}(T) \\ & \searrow t & \nearrow \pi_L, \bar{m} & \nearrow \pi_{KP}, \bar{m} & \\ & & S_{\bar{m}}(M) & & \end{array}$$

The *logic topologies* on $\text{Gal}_L(T)$ and $\text{Gal}_{KP}(T)$ are defined as the quotient topologies induced by the maps $\pi_L, \bar{m}$ and $\pi_{KP}, \bar{m}$, respectively. This definition does not depend on the choice of $\mathfrak{C}$ and M . It follows from the definition that for $* \in \{L, KP\}$, a subset C of $\text{Gal}_*(T)$ is closed if and only if $\pi_*^{-1}[C] = \{\sigma \in \text{Aut}(\mathfrak{C}) : \mathfrak{C} \models \pi(\sigma(\bar{a}))\}$ for some small (i.e., the number of formulas in $\pi(\bar{x})$ is bounded) partial type $\pi(\bar{x})$ and a tuple $\bar{a}$ from $\mathfrak{C}$ (without loss of generality $|\bar{a}| = |\bar{x}|$ is small here).

Now, we turn to Bohr compactifications. As in Section 3.1, $\mathbf{b}_1: \text{Aut}(\mathfrak{C}) \rightarrow B_1$ denotes the Bohr compactification of $\text{Aut}(\mathfrak{C})$ below $(S_{\bar{c}}(\mathfrak{C}), \text{tp}(\bar{c}/\mathfrak{C}), \text{Aut}(\mathfrak{C}))$, and $\mathbf{b}_2: \text{Aut}(\mathfrak{C}) \rightarrow B_2$ the Bohr compactification of $\text{Aut}(\mathfrak{C})$ below the Ellis flow $(E(S_{\bar{c}}(\mathfrak{C}), \text{Aut}(\mathfrak{C})), \text{id}, \text{Aut}(\mathfrak{C}))$. Let $\mathbf{f}: B_2 \rightarrow B_1$ be the epimorphism from Remark 3.6, so $\mathbf{f} \circ \mathbf{b}_2 = \mathbf{b}_1$. Analogously, since $\pi_{KP}: \text{Aut}(\mathfrak{C}) \rightarrow \text{Gal}_{KP}(T)$ is a group compactification of $\text{Aut}(\mathfrak{C})$ below $S_{\bar{c}}(\mathfrak{C})$, there exists a unique continuous group epimorphism $\mathbf{g}: B_1 \rightarrow \text{Gal}_{KP}(T)$ such that $\mathbf{g} \circ \mathbf{b}_1 = \pi_{KP}$. Therefore,

Remark 6.7. $\text{Aut}(\mathfrak{C}) \geq \text{Autf}_{KP}(\mathfrak{C}) \geq \ker(\mathbf{b}_1) \geq \ker(\mathbf{b}_2)$.

Let $u^2 = u \in \mathcal{M} \trianglelefteq_m E := E(S_{\bar{c}}(\mathfrak{C}), \text{Aut}(\mathfrak{C}))$.

Lemma 6.8. $|B_1| \leq |B_2| \leq \beth_3(|T|)$. Thus, $|\text{Aut}(\mathfrak{C})/\ker(\mathfrak{b}_1)| \leq |\text{Aut}(\mathfrak{C})/\ker(\mathfrak{b}_2)| \leq \beth_3(|T|)$.

Proof. The first inequality follows from the existence of the epimorphism $f: B_2 \rightarrow B_1$. By Corollary 3.19 there is an epimorphism $\hat{\theta}: u\mathcal{M} \rightarrow B_2$, so $|B_2| \leq |u\mathcal{M}|$. On the other hand, by [Hru19, Corollary A.8], $|u\mathcal{M}| \leq \beth_3(|T|)$ (it was first proved in [KNS19] with the bigger bound equal to $\beth_5(|T|)$ which is sufficient to realize the goal of this subsection). $\square$

Lemma 6.9. $\ker(\mathfrak{b}_1) = \text{Autf}_{KP}(\mathfrak{C})$ and $\text{Autf}_L(\mathfrak{C}) \leq \ker(\mathfrak{b}_2)$.

Proof. Let us first prove that $\ker(\mathfrak{b}_1) = \text{Autf}_{KP}(\mathfrak{C})$. By Remark 6.7, $\text{Autf}_{KP}(\mathfrak{C}) \geq \ker(\mathfrak{b}_1)$, hence it remains to prove $\text{Autf}_{KP}(\mathfrak{C}) \leq \ker(\mathfrak{b}_1)$. By Lemma 6.8, $|\text{Aut}(\mathfrak{C})/\ker(\mathfrak{b}_1)|$ is bounded, so, by virtue of [HKP26, Proposition 4.5], it is enough to show that $\ker(\mathfrak{b}_1)$ is relatively type-definable in the sense of [HKP26].

Since $\mathfrak{b}_1$ is a compactification of $\text{Aut}(\mathfrak{C})$ below $S_{\bar{c}}(\mathfrak{C})$, there is a unique $\text{Aut}(\mathfrak{C})$ -flow epimorphism $w_1: S_{\bar{c}}(\mathfrak{C}) \rightarrow B_1$ such that $\mathfrak{b}_1 = w_1 \circ t_1$, where $t_1: \text{Aut}(\mathfrak{C}) \rightarrow S_{\bar{c}}(\mathfrak{C})$ is given by $t_1(\sigma) := \text{tp}(\sigma(\bar{c})/\mathfrak{C})$. By boundedness of B_1 and continuity of w_1 , we get that $w_1^{-1}(e_{B_1}) = [\pi]$ for some small partial type π (i.e. the number of formulas in π is bounded). Thus, $\ker(\mathfrak{b}_1) = t_1^{-1}[w_1^{-1}(e_{B_1})] = \{\sigma \in \text{Aut}(\mathfrak{C}) : \mathfrak{C} \models \pi(\sigma(\bar{c}))\}$, which is a relatively type-definable subset (by smallness of π).

Now, we show that $\text{Autf}_L(\mathfrak{C}) \leq \ker(\mathfrak{b}_2)$. By Lemma 6.8, $|\text{Aut}(\mathfrak{C})/\ker(\mathfrak{b}_2)|$ is bounded, so, by virtue of [KNS19, Proposition 5.2], it is enough to show that $\ker(\mathfrak{b}_2)$ is closed in the pointwise convergence topology.

As above, we have a (continuous) epimorphism $w_2: E \rightarrow B_2$ such that $\mathfrak{b}_2 = w_2 \circ t_2$, where $t_2: \text{Aut}(\mathfrak{C}) \rightarrow E$ is given by $t_2(\sigma) := \sigma \text{id}_{S_{\bar{c}}(\mathfrak{C})}$. So closedness of $\ker(\mathfrak{b}_2)$ follows from continuity of t_2 and w_2 . $\square$

Question 6.10. Is it true that $\ker(\mathfrak{b}_2) = \text{Autf}_{KP}(\mathfrak{C})$?

We expect a negative answer, but by Lemma 6.9 and Remark 6.7, we get

Corollary 6.11. *If T is G -compact, then $\ker(\mathfrak{b}_1) = \ker(\mathfrak{b}_2) = \text{Autf}_{KP}(\mathfrak{C})$. Thus, $\mathfrak{b}_1$ and $\mathfrak{b}_2$ coincides (by which we mean that they are isomorphic).*

Lemma 6.12. *The topology on $\text{Aut}(\mathfrak{C})/\ker(\mathfrak{b}_1)$ inherited from B_1 is weaker than the logic topology.*

Proof. We have the map $h_1 := \pi_{KP, \bar{c}}: S_{\bar{c}}(\mathfrak{C}) \rightarrow \text{Gal}_{KP}(T) = \text{Aut}(\mathfrak{C})/\ker(\mathfrak{b}_1)$ which restricted to $S := \text{Aut}(\mathfrak{C}) \cdot \text{tp}(\bar{c}/\mathfrak{C})$ is given by $h_1(\text{tp}(\sigma(\bar{c})/\mathfrak{C})) = \sigma/\ker(\mathfrak{b}_1)$. We also have the factorization $i_1: \text{Aut}(\mathfrak{C})/\ker(\mathfrak{b}_1) \rightarrow B_1$ of $\mathfrak{b}_1$, which is a group embedding. Let $t_1: \text{Aut}(\mathfrak{C}) \rightarrow S$ be given by $\sigma \mapsto \text{tp}(\sigma(\bar{c})/\mathfrak{C})$, and $q_1 = \pi_{KP}: \text{Aut}(\mathfrak{C}) \rightarrow \text{Aut}(\mathfrak{C})/\ker(\mathfrak{b}_1)$ be the quotient map. Let $w_1: S_{\bar{c}}(\mathfrak{C}) \rightarrow B_1$ be the unique $\text{Aut}(\mathfrak{C})$ -flow epimorphism with $\mathfrak{b}_1 = w_1 \circ t_1$. It is clear that the following diagram commutes:

$$\begin{array}{ccccc}
 & & w_1|_S & & \\
 & \nearrow & & \searrow & \\
 S & \xrightarrow{h_1|_S} & \text{Aut}(\mathfrak{C})/\ker(\mathfrak{b}_1) & \xrightarrow{i_1} & B_1 \\
 \uparrow t_1 & \nearrow q_1 & \nearrow \mathfrak{b}_1 & & \\
 \text{Aut}(\mathfrak{C}) & & & &
 \end{array}$$

Consider any closed subset C of $\text{Aut}(\mathfrak{C})/\ker(\mathfrak{b}_1)$ in the topology inherited from B_1 via the embedding i_1 , i.e. $i_1[C] = C' \cap \text{im}(i_1)$ for some closed $C' \subseteq B_1$. By boundedness of B_1 and continuity of w_1 , we get that $w_1^{-1}[C'] = [\pi]$ for some small partial type π . So $(h_1|_S)^{-1}[C] = (w_1|_S)^{-1}[C' \cap \text{im}(i_1)] = [\pi] \cap S$. Therefore, $q_1^{-1}[C] = \{\sigma \in \text{Aut}(\mathfrak{C}) : \mathfrak{C} \models \pi(\sigma(\bar{c}))\}$ which is relatively type-definable (by smallness of π), and so C is closed in the logic topology. $\square$

Proposition 6.13. *The map $\mathfrak{b}_1$ is surjective and the induced $i_1 : \text{Aut}(\mathfrak{C})/\ker(\mathfrak{b}_1) \rightarrow B_1$ is a group isomorphism. Moreover, the topology on $\text{Aut}(\mathfrak{C})/\ker(\mathfrak{b}_1)$ inherited from B_1 coincides with the logic topology. In consequence, the quotient map $q_1 : \text{Aut}(\mathfrak{C}) \rightarrow \text{Gal}_{KP}(T)$ is the Bohr compactification of $\text{Aut}(\mathfrak{C})$ below the ambit $(S_{\bar{c}}(\mathfrak{C}), \text{tp}(\bar{c}/\mathfrak{C}), \text{Aut}(\mathfrak{C}))$.*

Proof. By Lemma 6.12, compactness of $\text{Aut}(\mathfrak{C})/\ker(\mathfrak{b}_1) = \text{Gal}_{KP}(T)$, and Hausdorffness of B_1 , we get that the topology on $\text{Aut}(\mathfrak{C})/\ker(\mathfrak{b}_1)$ inherited from B_1 is compact (Hausdorff) and coincides with the logic topology. This implies that $\mathfrak{b}_1[\text{Aut}(\mathfrak{C})] = i_1[\text{Aut}(\mathfrak{C})/\ker(\mathfrak{b}_1)]$ is closed in B_1 , but it is also dense, and so it is equal to B_1 . $\square$

Question 6.14. Do we have a counterpart of Proposition 6.13 for $\mathfrak{b}_2$, i.e.: Is $\mathfrak{b}_2$ surjective (equivalently, is the induced $i_2 : \text{Aut}(\mathfrak{C})/\ker(\mathfrak{b}_2) \rightarrow B_2$ a group isomorphism)? Does the topology on $\text{Aut}(\mathfrak{C})/\ker(\mathfrak{b}_2)$ inherited from B_2 coincide with the logic topology? A positive answer to the last question would imply that $\text{Aut}(\mathfrak{C})/\ker(\mathfrak{b}_2)$ is compact (Hausdorff), hence $\ker(\mathfrak{b}_2) = \text{Autf}_{KP}(\mathfrak{C})$ and $\mathfrak{b}_2$ would be surjective. Thus, the quotient map $q_2 : \text{Aut}(\mathfrak{C}) \rightarrow \text{Gal}_{KP}(T)$ would be the Bohr compactification of $\text{Aut}(\mathfrak{C})$ below E (equivalently, $\mathfrak{b}_1$ and $\mathfrak{b}_2$ would coincide).

Note that the only difficulty in adapting the above proofs to $\mathfrak{b}_2$ is in Lemma 6.12. Most likely it is false. In the context of definable groups, Example 4.17 in [KLM26] yields a definable group $G = M$ (even ω -categorical) such that the Bohr compactification of G below $S_G(M)$ is trivial, whereas the Bohr compactification below $E(S_G(M))$ is non-trivial. Maybe by adding an affine sort we could produce a counter-example to the above question, but it does not seem to be straightforward, as we consider compactifications of a much bigger group after adding the affine sort.

Question 6.15. Do $\mathfrak{b}_1$ and $\mathfrak{b}_2$ coincide under NIP?

By [CPS14, Theorem 4.15], we know that a counterpart of this statement for groups definable in NIP theories holds.

6.2.2. Transfer of amenability. The goal is to show that if T is NIP and amenable in the sense of [HKP26], i.e. the flow $(S_{\bar{c}}(\mathfrak{C}), \text{Aut}(\mathfrak{C}))$ is amenable, then so is the Ellis flow $(E(S_{\bar{c}}(\mathfrak{C}), \text{Aut}(\mathfrak{C})), \text{Aut}(\mathfrak{C}))$. This is a variant for first order theories of Theorem 3.17 in [CPS14] which says that if G is a group definable in a structure M , then definable amenability of G implies definable amenability of G in the Shelah expansion of M which in particular implies that the flow $(E(S_G(M), G), G)$ is amenable.

As before, let $E := E(S_{\bar{c}}(\mathfrak{C}), \text{Aut}(\mathfrak{C}))$.

Lemma 6.16. *Assume that T is NIP and amenable. Then there exists $\eta \in E$ which is $\text{Autf}_{KP}(\mathfrak{C})$ -invariant.*

Proof. By assumption and [HKP26, Corollary 2.24], the empty set is an extension base for forking. So, by [KNS19, Theorem 7.7], the minimal left ideals in E are of bounded size. By [KNS19, Corollary 7.2], this implies that there exists $\eta \in E$ whose image is contained in the set of types which do not fork over $\emptyset$. On the other hand, by [HP11, Proposition 2.11], a global type p does not fork over $\emptyset$ if and only if p is invariant under $\text{Autf}_{\text{KP}}(\mathfrak{C})$. Thus, η is $\text{Autf}_{\text{KP}}(\mathfrak{C})$ -invariant. $\square$

Proposition 6.17. *Assume that T is NIP and amenable. Then the flow $(E, \text{Aut}(\mathfrak{C}))$ is also amenable.*

Proof. By Lemma 6.16, we can choose $\eta \in E$ which is $\text{Autf}_{\text{KP}}(\mathfrak{C})$ -invariant. The map $\text{Aut}(\mathfrak{C}) \ni \sigma \mapsto \sigma\eta \in \overline{\text{Aut}(\mathfrak{C})\eta}$ factors through $\text{Aut}(\mathfrak{C})/\text{Autf}_{\text{KP}}(\mathfrak{C})$:

$$\begin{array}{ccc} \text{Aut}(\mathfrak{C}) & \xrightarrow{\sigma \mapsto \sigma\eta} & \overline{\text{Aut}(\mathfrak{C})\eta} \\ & \searrow \sigma \mapsto \sigma/\text{Autf}_{\text{KP}}(\mathfrak{C}) & \nearrow f \\ & \text{Aut}(\mathfrak{C})/\text{Autf}_{\text{KP}}(\mathfrak{C}) & \end{array}$$

Note that f is an $\text{Aut}(\mathfrak{C})$ -equivariant bijection on $\overline{\text{Aut}(\mathfrak{C})\eta}$, i.e. $f(\tau\sigma/\text{Autf}_{\text{KP}}(\mathfrak{C})) = \tau \cdot f(\sigma/\text{Autf}_{\text{KP}}(\mathfrak{C}))$ for every $\tau, \sigma \in \text{Aut}(\mathfrak{C})$. Moreover, f is Borel due to Lemma 7.15 from [KNS19]. Indeed, consider a basic open set

$$U := \{\xi \in E : \bigwedge_{k \leq n} \xi(p_k) \in [\theta_k(\bar{x}; \bar{c})]\},$$

where $p_1, \dots, p_n \in S_{\bar{c}}(\mathfrak{C})$ and $\theta_1(\bar{x}, \bar{y}), \dots, \theta_n(\bar{x}, \bar{y})$ are formulas without parameters. Then $\sigma/\text{Autf}_{\text{KP}}(\mathfrak{C}) \in f^{-1}[U]$ if and only if for every $k \leq n$ we have $\sigma\eta(p_k) \in [\theta_k(\bar{x}; \bar{c})]$. By the choice of η , each $\eta(p_k)$ is $\text{Autf}_{\text{KP}}(\mathfrak{C})$ -invariant, so its $\text{Aut}(\mathfrak{C})$ -orbit is of bounded size. Thus, Lemma 7.15 from [KNS19] implies that

$$U_k := \{\sigma/\text{Autf}_{\text{KP}}(\mathfrak{C}) : \sigma\eta(p_k) \in [\theta_k(\bar{x}; \bar{c})]\}$$

is constructible. Hence, $f^{-1}[U] = \bigcap_{k \leq n} U_k$ is constructible, too.

Since $\text{Aut}(\mathfrak{C})/\text{Autf}_{\text{KP}}(\mathfrak{C})$ is a compact Hausdorff group, it has a normalized Haar measure, say $\mathfrak{h}$. Then the pushforward $f_*\mathfrak{h}$ is an $\text{Aut}(\mathfrak{C})$ -invariant, regular, Borel probability measure on E . $\square$

6.2.3. Newelski's map for first order theories. Let $u^2 = u \in \mathcal{M}^{\leq_m} E := E(S_{\bar{c}}(\mathfrak{C}), \text{Aut}(\mathfrak{C}))$. Let $\mathfrak{C}' \succ \mathfrak{C}$ be a monster model in which $\mathfrak{C}$ is small. A counterpart of Newelski's map in the context of first order theories was introduced in [KPR18, Section 2]. This is the continuous group epimorphism $\rho: u\mathcal{M} \rightarrow \text{Gal}_{\text{KP}}(T) = \text{Aut}(\mathfrak{C}')/\text{Autf}_{\text{KP}}(\mathfrak{C}')$ given by $\rho(\eta) := \sigma'/\text{Autf}_{\text{KP}}(\mathfrak{C}')$ where $\sigma'(\bar{c}) \models \eta(\text{tp}(\bar{c}/\mathfrak{C}))$. Theorem 0.7 from [KNS19] is a counterpart of Fact 6.2; it is a counterpart of Newelski's conjecture for NIP amenable theories.

Fact 6.18 (Theorem 0.7 in [KNS19]). *If T is NIP and amenable, then ρ is an isomorphism.*

Let us use the notation from Subsection 6.2.1. Namely, using Proposition 6.13, we have Bohr compactifications $\mathfrak{b}_1: \text{Aut}(\mathfrak{C}) \rightarrow B_1 = \text{Gal}_{\text{KP}}(T)$ and $\mathfrak{b}_2: \text{Aut}(\mathfrak{C}) \rightarrow B_2$, the group epimorphism $\mathfrak{f}: B_2 \rightarrow B_1$ satisfying $\mathfrak{f} \circ \mathfrak{b}_2 = \mathfrak{b}_1$, and the epimorphism $\hat{\theta}: u\mathcal{M} \rightarrow B_2$.

Lemma 6.19. *The Newelski's map ρ recalled above coincides with $\mathfrak{f} \circ \hat{\theta}$.*

Proof. Since $\widehat{\theta}(\eta) := \eta e_{B_2}$ and $\mathfrak{f}$ is an epimorphism of $\text{Aut}(\mathfrak{C})$ -ambits, we have $(\mathfrak{f} \circ \widehat{\theta})(\eta) = \eta(\text{id}_{\mathfrak{C}'} / \text{Autf}_{\text{KP}}(\mathfrak{C}'))$. So it is enough to show that the unique left continuous action of E on $\text{Gal}_{\text{KP}}(T)$ extending the action of $\text{Aut}(\mathfrak{C})$ is given by $\eta(\tau' / \text{Autf}_{\text{KP}}(\mathfrak{C}')) = (\sigma'\tau') / \text{Autf}_{\text{KP}}(\mathfrak{C}')$ where $\sigma'(\bar{c}) \models \eta(\text{tp}(\bar{c}/\mathfrak{C}))$ (and $\sigma', \tau' \in \text{Aut}(\mathfrak{C}')$). For that it remains to show that the last formula defines a left continuous function extending the action of $\text{Aut}(\mathfrak{C})$. It is well-defined as $\text{tp}(\sigma'_1(\bar{c})/\mathfrak{C}) = \text{tp}(\sigma'_2(\bar{c})/\mathfrak{C})$ implies $\sigma'_1 / \text{Autf}_{\text{KP}}(\mathfrak{C}') = \sigma'_2 / \text{Autf}_{\text{KP}}(\mathfrak{C}')$. The fact that it extends the action of $\text{Aut}(\mathfrak{C})$ is obvious. For continuity consider any $\tau' / \text{Autf}_{\text{KP}}(\mathfrak{C}') \in \text{Gal}_{\text{KP}}(T)$ and any closed $C \subseteq \text{Gal}_{\text{KP}}(T)$. Then, for the quotient map $\pi'_{\text{KP}}: \text{Aut}(\mathfrak{C}') \rightarrow \text{Gal}_{\text{KP}}(T)$, by the definition of the logic topology, there exists a partial type $\pi(\bar{x})$ over $\mathfrak{C}$ such that $\pi'^{-1}_{\text{KP}}[C] = \{\sigma' \in \text{Aut}(\mathfrak{C}') : \mathfrak{C}' \models \pi(\sigma'(\bar{c}))\}$. So $\{\eta \in E : \eta(\tau' / \text{Autf}_{\text{KP}}(\mathfrak{C}')) \in C\} = \{\eta \in E : \eta(\text{tp}(\tau'(\bar{c})/\mathfrak{C})) \in [\pi]\}$ which is closed (to see the last equality, one should note that for any $\eta \in E$ there is $\sigma' \in \text{Aut}(\mathfrak{C}')$ with $\sigma'(\bar{c}) \models \eta(\text{tp}(\bar{c}/\mathfrak{C}))$ and $\sigma'(\tau'(\bar{c})) \models \eta(\text{tp}(\tau'(\bar{c})/\mathfrak{C}))$; see [KPR18, Proposition 2.2]). $\square$

So the conclusion of GNC applied to the ambit $(S_{\bar{c}}(\mathfrak{C}), \text{tp}(\bar{c}/\mathfrak{C}), \text{Aut}(\mathfrak{C}))$ is weaker than the conclusion of Fact 6.18, similarly to the case of definable groups (see the comments after Lemma 6.3). However, for definable groups, assuming NIP, both conclusions are equivalent by Theorem 4.15 from [CPS14], whereas for theories a counterpart of [CPS14, Theorem 4.15] is left as Question 6.15. Note that assuming NIP and amenability of T , Fact 6.18 together with Lemma 6.19 imply that $\mathfrak{b}_1$ and $\mathfrak{b}_2$ coincide, but it would be very interesting to find a simple direct proof. Then we would get that Theorem 4.4 implies Fact 6.18. Recall that amenability of T implies G-compactness by [HKP26, Theorem 4.1] (in the NIP case, this is a much easier [HKP26, Corollary 2.26]), so by Corollary 6.11, we get $\ker(\mathfrak{b}_2) = \ker(\mathfrak{b}_1)$, but it is still weaker than the equality (or isomorphism) of $\mathfrak{b}_1$ and $\mathfrak{b}_2$.

6.2.4. Two problems.

- (1) By [CPS14, Theorem 3.17], for a group G definable in a NIP structure M we have that $(S_G(M), G)$ is amenable if and only if $(E(S_G(M), G), G)$ is amenable.
- (2) By Proposition 6.17, for a first order theory T with NIP we have that $(S_{\bar{c}}(\mathfrak{C}), \text{Aut}(\mathfrak{C}))$ is amenable if and only if $(E(S_{\bar{c}}(\mathfrak{C}), \text{Aut}(\mathfrak{C})), \text{Aut}(\mathfrak{C}))$ is amenable.

In a general context of a tame flow (X, G) , Fact 2.7 tells us that amenability of $(E(X, G), G)$ is equivalent to hereditary amenability of (X, G) . A simple example discussed after Conjecture 4.3 shows that hereditary amenability cannot be weakened to amenability. It is also possible to give an example of a tame ambit (not just a flow) where amenability is not sufficient for the above equivalence, but it is much more complicated so we skip it. However, in the model-theoretic situations (1) and (2), we do not need hereditary amenability. A posteriori (i.e. using (1) and (2)), hereditary amenability follows from amenability in those two contexts. This leads to

Problem 6.20. *Find a dynamical property of a tame ambit (X, x_0, G) which holds for ambits in (1) and (2) and which implies that amenability of (X, G) is equivalent to hereditary amenability of (X, G) (and so to amenability of $(E(X, G), G)$)?*

By [CPS14], if G is a group definable in a NIP structure M , then the Bohr compactifications of G below $(S_G(M), \text{tp}(e/M), G)$ and $(E(S_G(M), G), \text{id}_{S_G(M)}, G)$ coincide. Question 6.15 asks whether for a NIP theory T the Bohr compactifications of $\text{Aut}(\mathfrak{C})$ below $(S_{\bar{e}}(\mathfrak{C}), \text{tp}(\bar{e}/\mathfrak{C}), \text{Aut}(\mathfrak{C}))$ and $(E(S_{\bar{e}}(\mathfrak{C}), \text{Aut}(\mathfrak{C})), \text{id}_{S_{\bar{e}}(\mathfrak{C})}, \text{Aut}(\mathfrak{C}))$ coincide, with expected positive answer. This leads to

Problem 6.21. *Find a dynamical property of a tame ambit (X, x_0, G) which holds for the above two model-theoretic ambits and which implies that the Bohr compactifications of G below (X, x_0, G) and $(E(X, G), \text{id}_X, G)$ coincide⁵.*

6.3. Local Newelski's Conjecture. Let G be a group definable in a structure M , $\mathfrak{C} \succ M$ a monster model, and $\bar{G} := G(\mathfrak{C})$. Let Δ be a (finite or infinite) collection of formulas $\varphi(x, y)$, each of which implies $G(x)$, and such that $\{\varphi(G, b) : b \in M\}$ is closed under the left translations by the elements of G . By a Δ -formula we mean a Boolean combination of instances of formulas from Δ . Finally, $S_{G, \Delta}(M)$ denotes the space of complete Δ -types over M (i.e., complete consistent collections of Δ -formulas over M) concentrated on G . Then G acts on the left on $S_{G, \Delta}(M)$ via $g \cdot \text{tp}_{\Delta}(a/M) := \text{tp}_{\Delta}(ga/M)$, and $(S_{G, \Delta}(M), \text{tp}_{\Delta}(e/M), G)$ is an ambit. Let $E := E(S_{G, \Delta}(M), G)$ and $u^2 = u \in \mathcal{M} \leq_m E$.

We consider our usual map $\hat{\theta} = w_2|_{u\mathcal{M}} : u\mathcal{M} \rightarrow B_2$, where $\mathfrak{b}_2 : G \rightarrow B_2$ is the Bohr compactification of G below $(E, \text{id}_{S_{G, \Delta}(M)}, G)$.

Theorem 6.22. *If each formula in Δ has NIP and $(S_{G, \Delta}(M), G)$ is hereditarily amenable, then $\hat{\theta}$ is an isomorphism of topological groups.*

Proof. Since each formula in Δ has NIP, the characteristic functions of clopens in $S_{G, \Delta}(M)$ are all tame which implies that the flow $(S_{G, \Delta}(M), G)$ is tame. Thus, the result follows from Theorem 4.5. $\square$

The key thing is to describe B_2 in a more explicit way. For that we first recall a piece of basic general theory developed by Newelski.

Let $\mathcal{A}$ be a *Boolean G -algebra* of subsets of G , i.e. a Boolean algebra closed under the left translations by the elements of G . By $S(\mathcal{A})$ we denote the Stone space of $\mathcal{A}$. The group G acts naturally on $S(\mathcal{A})$ by left translations, turning $S(\mathcal{A})$ into a G -ambit with the principal ultrafilter $p_e := \{A \in \mathcal{A} : e \in A\}$ as the distinguished point. For $p \in S(\mathcal{A})$ the operator $d_p : \mathcal{A} \rightarrow \mathcal{P}(G)$ is defined by:

$$d_p(A) := \{g \in G : g^{-1}A \in p\}.$$

Each d_p is a homomorphism of Boolean G -algebras. The algebra $\mathcal{A}$ is called *d-closed* if $d_p[\mathcal{A}] = \mathcal{A}$ for every $p \in S(\mathcal{A})$. The next fact was observed by Newelski, but appeared in Adam Malinowski's thesis [Mal23].

Fact 6.23. [Mal23, Fact 2.22 and Theorem 2.24] *Let $\mathcal{A}$ be a Boolean G -algebra of subsets of G . There exists a smallest d-closed G -algebra of subsets of G containing $\mathcal{A}$ which is denoted by $\mathcal{A}^d$. The d-closure $\mathcal{A}^d$ is realized in one step, i.e. $\mathcal{A}^d$ is precisely the Boolean algebra generated by $\{d_p(A) : A \in \mathcal{A}, p \in S(\mathcal{A})\}$. Moreover, $E(S(\mathcal{A})) \cong S(\mathcal{A}^d)$ as semigroups and as G -ambits (with respect to a unique left-continuous semigroup operation $*$ on $S(\mathcal{A}^d)$ extending the action of G), with $\text{id}_{S(\mathcal{A})}$*

⁵The first author discussed a similar problem with Eli Glasner, who provided a solution. Let us comment that here we would like to recognize the general mechanism from topological dynamics which is behind the model-theoretic techniques.

and p_e as the distinguished points, respectively (where p_e is the principal ultrafilter at e).

Applying this fact in our situation to $\mathcal{A}$ defined as the Boolean algebra generated by $\{\varphi(G, b) : b \in M\}$, we produce $\mathcal{A}^d$ as above, and we can replace $(E, \text{id}_{S_{G, \Delta}(M)}, G)$ by $(S(\mathcal{A}^d), p_e, G)$. Now, let M' be the structure with the universe G equipped with the group operation and all members of $\mathcal{A}^d$ as predicates. Let Δ' consist of these predicates. Then $(S(\mathcal{A}^d), p_e, G)$ is naturally identified with $(S_{\Delta'}(M'), \text{tp}_{\Delta'}(e/M'), G)$ and Δ' is easily seen to be closed under the left and right translations by the elements of G (i.e., $\mathcal{A}^d$ has this property). We describe the Bohr compactification of G below $(S_{\Delta'}(M'), \text{tp}_{\Delta'}(e/M'), G)$ under the additional assumptions that Δ' is closed under group inversion. More generally, we describe the Bohr compactification of G below the ambit of all complete Σ -types over M for suitable collections of formulas Σ . The proof follows the standard method as in [GPP14], but there are some new issues to take care of in the local version.

Proposition 6.24. *Let Σ be a (finite or infinite) collection of formulas $\varphi(x, y)$, each of which implies $G(x)$, and such that $\{\varphi(G, b) : b \in M\}$ is closed under the left translations by the elements of G and under group inversion. Let $\bar{G}_{\Sigma, M}^{00}$ be the smallest bounded index Σ -type-definable over M subgroup of $\bar{G}$ (where Σ -type-definable over M means that $\bar{G}_{\Sigma, M}^{00} = \pi(\bar{G})$ for some partial Σ -type π over M). Then the quotient map $G \rightarrow \bar{G}/\bar{G}_{\Sigma, M}^{00}$ is the Bohr compactification of G below the ambit $(S_{G, \Sigma}(M), \text{tp}_{\Sigma}(e/M), G)$.*

Proof. Let $\mathfrak{b}: G \rightarrow B$ be the Bohr compactification of G below $(S_{G, \Sigma}(M), \text{tp}_{\Sigma}(e/M), G)$. Note that $\mathfrak{b}$ is Σ -definable in the sense that the preimages of any two disjoint closed subsets of B can be separated by a Σ -definable subset of G . Adapting the proof of Lemma 3.2 from [GPP14], we get that the unique G -flow epimorphism $w: S_{G, \Sigma}(M) \rightarrow B$ such that $w \circ t = \mathfrak{b}$ for $t: G \rightarrow S_{G, \Sigma}(M)$ given by $t(g) := \text{tp}_{\Sigma}(g/M)$ satisfies the formula

$$w(p) = \bigcap_{\varphi(x) \in p} \overline{\mathfrak{b}[\varphi(G)]}.$$

We also have the function $\mathfrak{b}^*: \bar{G} \rightarrow B$ given by $\mathfrak{b}^*(a) := w(\text{tp}_{\Sigma}(a/M))$. A slight elaboration of the proof of [GPP14, Proposition 3.4] shows that $\mathfrak{b}^*$ is a group homomorphism. An elaboration, which we give below, is needed, because Σ -definable sets are not closed under the group products.

Claim 1. $\mathfrak{b}^*$ is a group homomorphism.

Proof. Suppose for a contradiction that $\mathfrak{b}^*(ab) \neq \mathfrak{b}^*(a)\mathfrak{b}^*(b)$ for some $a, b \in \bar{G}$. Then $\mathfrak{b}^*(ab) \in U$ and $\mathfrak{b}^*(a)\mathfrak{b}^*(b) \in V$ for some open, disjoint subsets U and V of B . So there is a Σ -formula $\varphi(x) \in \text{tp}_{\Sigma}(ab/M)$ such that $\overline{\mathfrak{b}[\varphi(G)]} \subseteq U$. Set $p := \text{tp}_{\Sigma}(a/M)$ and $q := \text{tp}_{\Sigma}(b/M)$. Then

$$\begin{aligned} \bigcap_{\psi_1 \in p, \psi_2 \in q} \overline{\mathfrak{b}[\psi_1(G) \cdot \psi_2(G)]} &= \bigcap_{\psi_1 \in p, \psi_2 \in q} \overline{\mathfrak{b}[\psi_1(G)]} \cdot \overline{\mathfrak{b}[\psi_2(G)]} \\ &= \bigcap_{\psi_1 \in p} \overline{\mathfrak{b}[\psi_1(G)]} \cdot \bigcap_{\psi_2 \in q} \overline{\mathfrak{b}[\psi_2(G)]} \\ &= \{\mathfrak{b}^*(a) \cdot \mathfrak{b}^*(b)\}. \end{aligned}$$

Hence, there are $\psi_1(x) \in p$ and $\psi_2(x) \in q$ such that $\overline{\mathbf{b}[\psi_1(G) \cdot \psi_2(G)]} \subseteq V$. Since $U \cap V = \emptyset$, we conclude that $\varphi(G) \cap (\psi_1(G) \cdot \psi_2(G)) = \emptyset$ which contradicts the fact that $\varphi(\bar{G}) \cap (\psi_1(\bar{G}) \cdot \psi_2(\bar{G})) \neq \emptyset$ witnessed by the element ab . $\square$ (claim)

To finish the proof, it is enough to show that $\ker \mathbf{b}^* = \bar{G}_{\Sigma, M}^{00}$. By the claim and the fact that $\mathbf{b}^*$ naturally factors through $S_{G, \Sigma}(M)$, we get that $\ker \mathbf{b}^*$ is a Σ -type-definable over M subgroup of $\bar{G}$ which is of bounded index (as $|B| \leq 2^{2^{|G|}}$). Thus, $\ker \mathbf{b}^* \supseteq \bar{G}_{\Sigma, M}^{00}$. The opposite inclusion is very easy in the global case, but in our local case it requires a proof in which we will use the additional assumption that the family of the instances of the formulas from Σ is closed under inversion.

First, observe that it is enough to show that the quotient map $\mathbf{b}': G \rightarrow \bar{G}/\bar{G}_{\Sigma, M}^{00}$ is a compactification of G below $(S_{G, \Sigma}(M), \text{tp}_{\Sigma}(e/M), G)$. Indeed, by the inclusion $\ker \mathbf{b}^* \supseteq \bar{G}_{\Sigma, M}^{00}$, we have the obvious epimorphism $h: \bar{G}/\bar{G}_{\Sigma, M}^{00} \rightarrow \bar{G}/\ker \mathbf{b}^*$ commuting with the quotient maps from G . Composing h with the isomorphism $f: \bar{G}/\ker \mathbf{b}^* \rightarrow B$ induced by $\mathbf{b}^*$, we obtain an epimorphism $f \circ h: \bar{G}/\bar{G}_{\Sigma, M}^{00} \rightarrow B$ commuting with $\mathbf{b}'$ and $\mathbf{b}$. Since $\mathbf{b}'$ is a compactification of G below the ambit $(S_{G, \Sigma}(M), \text{tp}_{\Sigma}(e/M), G)$ and $\mathbf{b}$ is the universal such compactification, we conclude that h is an isomorphism, so $\ker \mathbf{b}^* = \bar{G}_{\Sigma, M}^{00}$.

To see that $\mathbf{b}': G \rightarrow \bar{G}/\bar{G}_{\Sigma, M}^{00}$ is a compactification of G below the ambit $(S_{G, \Sigma}(M), \text{tp}_{\Sigma}(e/M), G)$, it remains to show that for $a, b \in \bar{G}$:

$$\text{if } \text{tp}_{\Sigma}(a/M) = \text{tp}_{\Sigma}(b/M), \text{ then } a/\bar{G}_{\Sigma, M}^{00} = b/\bar{G}_{\Sigma, M}^{00}. \quad (*)$$

Let $\Psi := \{\psi_{\varphi}(x, z; y) : \varphi(x, y) \in \Sigma\}$, where $\psi_{\varphi}(x, z; y) := \varphi(x^{-1}z, y)$.

Claim 2. If $\text{tp}_{\Sigma}(a/M) = \text{tp}_{\Sigma}(b/M)$, then there exists a sequence $(c_1, c_2, \dots)$ such that both sequences $a^{\frown}(c_i)$ and $b^{\frown}(c_i)$ are Ψ -indiscernible over M (meaning that for every $i < j$ and $k < l$, $\text{tp}_{\Psi}(a_i, a_j/M) = \text{tp}_{\Psi}(a_k, a_l/M)$) and Σ -indiscernible over M (meaning that for every i, k , $\text{tp}_{\Sigma}(a_i/M) = \text{tp}_{\Sigma}(a_k/M)$).

Proof. As in the usual construction of coheirs, choose an ultrafilter $\mathcal{U}$ on the Boolean algebra of all subsets of G which extends the type $p := \text{tp}_{\Sigma}(a/M)$ (treated as an ultrafilter on the Boolean algebra of Σ -definable subsets of G). Set

$$p_{\mathcal{U}} := \{\varphi(x) : \varphi(x) \text{ is a } \Sigma\text{-formula such that } \varphi(\bar{G}) \cap G \in \mathcal{U}\}.$$

Then $p \subseteq p_{\mathcal{U}}$ and $p_{\mathcal{U}}$ is finitely satisfiable in G .

We claim that for every Ψ -formula $\delta(x, z)$

$$\text{if } \text{tp}_{\Sigma}(a/M) = \text{tp}_{\Sigma}(b/M), \text{ then } \delta(a, z) \in p_{\mathcal{U}} \text{ iff } \delta(b, z) \in p_{\mathcal{U}}. \quad (\dagger)$$

It is clear that for any $c \in M$, $\delta(c, z)$ is a Σ -formula. As the set of instances of the formulas from Σ is closed under the left translations and under *inversion*, we see that for any $c \in M$, $\delta(x, c)$ is a Σ -formula, and so the condition $\text{tp}_{\Sigma}(a/M) = \text{tp}_{\Sigma}(b/M)$ implies that for any $c \in M$, $\models \delta(a, c)$ iff $\models \delta(b, c)$. Therefore, we have the following sequence of equivalences:

$$\delta(a, z) \in p_{\mathcal{U}} \iff \delta(a, \bar{G}) \cap G \in \mathcal{U} \iff \delta(b, \bar{G}) \cap G \in \mathcal{U} \iff \delta(b, z) \in p_{\mathcal{U}},$$

so $(\dagger)$ is proved.

Now, let $(c_1, c_2, \dots)$ be a Morley sequence in $p_{\mathcal{U}}$ over M, a, b , that is $c_i \models p_{\mathcal{U}}|_{M, a, b, c_{<i}}$ for every i . The fact that $a^{\frown}(c_i)$ and $b^{\frown}(c_i)$ are both Σ -indiscernible over M is obvious. Thus, Ψ -indiscernibility over M follows from $(\dagger)$. $\square$ (claim)

The next claim follows by compactness.

Claim 3. If E is a Ψ -type-definable over M bounded equivalence relation on $\bar{G}$ and a sequence $(c_i)_{i < \omega}$ is Ψ -indiscernible over M , then $E(c_i, c_j)$ for all $i < j < \omega$.

The next claim is a direct consequence of Claims 2 and 3.

Claim 4. If $\text{tp}_\Sigma(a/M) = \text{tp}_\Sigma(b/M)$ and E is a Ψ -type-definable over M bounded equivalence relation on $\bar{G}$, then $E(a, b)$.

The desired condition (*) follows from Claim 4, as the relation of lying in the same coset modulo $\bar{G}_{\Sigma, M}^{00}$ is clearly a Ψ -type-definable over M bounded equivalence relation on $\bar{G}$. $\square$

We finish with a list of corollaries, comments, and questions around Theorem 6.22 and Proposition 6.24.

- Unlike in the case of the ambits $S_G(M)$ and $S_{\bar{e}}(\mathfrak{C})$, Theorem 6.22 cannot be strengthened by saying that $u\mathcal{M}$ is isomorphic to the Bohr compactification of G below $(S_{G, \Delta}(M), \text{tp}_\Delta(e/M), G)$. Indeed, take $G := S_3$, $M := (G, \cdot, H)$ where $H := \langle (1, 2) \rangle$, and $\Delta := \{\varphi(x, y)\}$ where $\varphi(x, y)$ is the formula “ $x \in yH$ ”. Then $(S_{G, \Delta}(M), \text{tp}_\Delta(\text{id}/M), G)$ is isomorphic to $(G/H, \text{id}/H, G)$ and by Example 4.10, we get that $u\mathcal{M} \cong G$ while the Bohr compactification below $(G/H, \text{id}/H, G)$ is trivial.
- Unlike in the case of the ambits $S_G(M)$ and $S_{\bar{e}}(\mathfrak{C})$, the assumption that $(S_{G, \Delta}(M), G)$ is hereditarily amenable cannot be weakened to amenability. We skip an example, as it is pretty long. One could ask if there is a counterexample with finite Δ .
- Proposition 6.24 applied to Δ being the collection of all formulas $\varphi(x, y)$ implying $G(x)$ recovers Proposition 3.4 from [GPP14].
- Consider the Shelah expansion M^{Sh} of M , i.e. the universe M equipped with predicates for all externally definable subsets in any finite numbers of variables (for our purpose we could only consider externally definable sets in the variables corresponding to G). The externally definable Bohr compactification of G considered in [KP17] is precisely the one below the ambit $(S_G^{\text{qf}}(M^{\text{Sh}}), \text{tp}^{\text{qf}}(e/M^{\text{Sh}}), G)$ (where “qf” stands for “quantifier free”). Proposition 6.24 applied to M^{Sh} and Δ being the collection of all quantifier free formulas $\varphi(x, y)$ implying $G(x)$ (or just the collection of all predicates for externally definable subsets of G) yields a new description of the externally definable Bohr compactification of G in terms of a quotient of $\bar{G}$ by a suitable model-theoretic connected component.
- In Proposition 6.24, can the assumption that the family of the instances of the formulas from Σ is closed under inversion be dropped?
- Is the property that a Boolean G -algebra $\mathcal{A}$ of subsets of G is closed under inversion preserved under passing to $\mathcal{A}^d$? For an abelian or a finite group G it is clear, but in general there is no obvious reason why it should be true.

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