HAAR DECOMPRESSION AND AMENABILITY OF ELLIS FLOWS

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Abstract. Let (X, G) be a tame flow and let K be an Ellis group of its enveloping semigroup $E(X, G)$. Although K is a compact Hausdorff topological group in its τ-topology, the inclusion $K \hookrightarrow E(X, G)$ need not be Borel. We show that normalized Haar measure on K nevertheless determines, via the Riesz–Markov theorem, a canonical regular Borel probability measure μ_K on $E(X, G)$, called its Haar decompression.

Haar decompression gives an exact ergodicity description. For arbitrary flow (X, G), amenability of its Ellis flow is equivalent to hereditary amenability of all finite powers X^n. In the tame setting

$$\big(E(X, G), G\big) \text{ is amenable} \iff (X, G) \text{ is hereditarily amenable.}$$

Moreover, for the minimal left ideal $\mathcal{M}$ in $E(X, G)$ containing K, μ_K is G-invariant if and only if $(\mathcal{M}, G)$ is amenable; when this holds, μ_K is the unique ergodic measure on $\mathcal{M}$ and $\mathcal{M} = \overline{K}$. Then, we conclude that the ergodic measures of $E(X, G)$ are precisely the Haar decompressions associated with amenable minimal left ideals.

We also prove that every minimal tame amenable flow is uniquely ergodic and that every ergodic invariant measure on a tame flow has minimal support. Consequently, every ergodic measure on a tame ambit arises by evaluating a suitable Haar decompression.

1. Introduction

Haar decompression and its guiding principle. Let G be a discrete group. A G-flow is a compact Hausdorff space equipped with an action of G by homeomorphisms, and an ambit is a pointed flow (X, x_0, G) such that $\overline{Gx_0} = X$.

For a minimal equicontinuous flow, the unique invariant probability measure is induced by the normalized Haar measure on the compact group governing the action. In the paper, we ask how far this compact-group origin of invariant measures survives beyond equicontinuity. We show that it exists in tame dynamics, but in a less direct form: the relevant compact groups are Ellis groups inside the enveloping semigroup, and their Haar measures must first be regularized before they can be transferred to the original flow. In this sense, normalized Haar measures are the *ergodic protoplasts* of all ergodic, and in consequence of all invariant, measures.

The enveloping semigroup $E(X, G)$ is the pointwise closure of G in X^X. Every minimal left ideal $\mathcal{M} \trianglelefteq_m E(X, G)$ and every idempotent $u^2 = u \in \mathcal{M}$ determine an Ellis group $K = u\mathcal{M}$, equipped with the τ-topology [Aus88; Gla76]. We work

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mainly with tame flows, whose enveloping semigroups are characterized by the fragmentedness of their elements; see [GMU08; Gla06; GM18b; GM22; KL07; Hua06; CH23]. Tameness does not turn $E(X,G)$ into a topological semigroup, but it provides enough measure-theoretic regularity to connect its algebraic structure with invariant measures.

For a tame flow, every Ellis group (K,τ) is a compact Hausdorff topological group (Lemma 3.3, using [Gla18; CGK26; BZ24]), and hence carries normalized Haar measure $\mathfrak{h}_K$. The principal obstacle is that the inclusion

$$j : (K,\tau) \longrightarrow E(X,G)$$

need not be Borel. Thus, the ordinary Borel pushforward $j_*\mathfrak{h}_K$ might not work for our purposes. Appendix A.1 gives a minimal metrizable tame flow for which j is not Borel.

We prove nevertheless that, for every $f \in C(E(X,G),\mathbb{C})$, the restriction $f \circ j$ is measurable for the completion of $\mathfrak{h}_K$. Consequently,

$$f \longmapsto \int_K f(j(k))\, d\mathfrak{h}_K(k)$$

defines a state on $C(E(X,G),\mathbb{C})$. The Riesz–Markov theorem therefore yields a unique regular Borel probability measure μ_K on $E(X,G)$ representing this state. We call μ_K the *Haar decompression* associated with K (Lemma 3.7). It follows from the definition that $\operatorname{supp}(\mu_K) \subseteq \overline{K} \subseteq \mathcal{M}$.

This construction leads to two complementary problems. The first is to determine when μ_K is invariant under the left action of G on $E(X,G)$. The second is to determine which ergodic measures on an ambit arise by evaluating Haar decompressions at its distinguished point.

Main results. Recall that a flow is *amenable* if it carries an invariant probability measure and is *hereditarily amenable* if every (nonempty) subflow is amenable. We say that (X,G) is *finitely power-hereditarily amenable* if every subflow of every finite power X^n is amenable.

Our first result characterizes amenability of the Ellis flow:

Theorem (Theorems 5.3 and 5.5, and Corollary 5.6). For every flow (X,G),

$$E(X,G) \text{ is amenable} \quad \Longleftrightarrow \quad X \text{ is finitely power-hereditarily amenable.}$$

If (X,G) is tame, these conditions are further equivalent to hereditary amenability of (X,G).

The first equivalence above reflects the finite-coordinate nature of continuous functions on $E(X,G)$: functions depending on finitely many coordinates form a uniformly dense subalgebra of $C(E(X,G),\mathbb{C})$. Then, the tame part of the theorem above shows that ordinary hereditary amenability already passes to all finite powers. Its proof goes first through metrizable tame factors and then removes separability. Moreover, amenability of the Ellis flow is automatically hereditary (Remark 5.2).

The above theorem decides when the Ellis flow admits some invariant probability measure. On the other hand, Haar decompression gives a corresponding criterion at the level of an individual minimal left ideal:

Theorem (Theorem 5.9 and Remark 5.10). Let (X,G) be tame, let $\mathcal{M} \trianglelefteq_m E(X,G)$, let $u^2 = u \in \mathcal{M}$, and put $K = u\mathcal{M}$. Then

$$\mu_K \text{ is } G\text{-invariant} \quad \Longleftrightarrow \quad (\mathcal{M},G) \text{ is amenable.}$$

When these equivalent conditions hold, μ_K, regarded as a measure on $\mathcal{M}$, is its unique invariant probability measure. Moreover,

$$\mu_K \in \mathrm{Erg}_G\big(E(X,G)\big), \qquad \mathrm{supp}(\mu_K) = \mathcal{M} = \overline{K}.$$

Here, for a flow (Y,G), $\mathrm{Erg}_G(Y)$ denotes the collection of all ergodic G-invariant regular Borel probability measures on Y. Together with the minimal-support result below, the preceding theorem gives an exact description of the ergodic measures on the Ellis flow:

$$\mathrm{Erg}_G\left(E(X,G)\right) = \left\{\mu_{u\mathcal{M}} : \mathcal{M} \trianglelefteq_m E, \quad u^2 = u \in \mathcal{M}, \quad (\mathcal{M},G) \text{ is amenable}\right\}.$$

If (X,G) is hereditarily amenable, then every minimal left ideal is amenable, so all Haar decompressions occur in the above description. The tameness assumption is essential: Appendix A.3 constructs a metrizable hereditarily amenable flow whose Ellis flow is not amenable.

A second group of results establishes rigidity of invariant measures on tame flows:

Theorem (Theorem 4.2, Proposition 4.4, and Corollary 4.6). Every minimal tame amenable flow is uniquely ergodic. Moreover, if (X,G) is tame, then the support of every $\mu \in \mathrm{Erg}_G(X)$ is a minimal subflow, and the support map

$$\mathrm{Erg}_G(X) \longrightarrow \mathrm{Min}_G^{\mathrm{am}}(X), \qquad \mu \longmapsto \mathrm{supp}(\mu),$$

is a bijection onto the collection of amenable minimal subflows of X. In particular, distinct ergodic invariant measures have disjoint supports.

In the metrizable minimal case, unique ergodicity follows from the almost one-to-one structure theorem for minimal tame flows [Gla07; Gla18] and the regularity theorem of [Fuh+21]. We also identify the unique invariant measure directly as the evaluation of every Haar decompression. A separable C^*-algebra reduction then yields unique ergodicity without metrizability. The minimal-support theorem uses a related reduction together with an integration identity supplied by tameness, which plays the role of dominated convergence for limits indexed by general nets. A special case of the minimal-support phenomenon for tame semicascades, under additional assumptions, was obtained in [Rom16, Theorem 3.6].

We now make the ergodic protoplast principle precise. For an ambit (X,x_0,G), evaluation at the distinguished point is the continuous G-map

$$\mathrm{ev}_{x_0} : E(X,G) \longrightarrow X, \qquad \eta \longmapsto \eta(x_0).$$

There are two notions of ergodic transfer. We say that an ambit (X,x_0,G) has the *ergodic Ellis-transfer property* if

$$(\mathrm{ev}_{x_0})_*\mu_K \in \mathrm{Erg}_G(X)$$

for every Ellis group K of $E(X,G)$. We say that it has the *ergodic Ellis-realization property* if every $\mu \in \mathrm{Erg}_G(X)$ is equal to $(\mathrm{ev}_{x_0})_*\mu_K$ for at least one Ellis group K.

Theorem (Corollaries 6.3 and 6.1). Every tame ambit has the ergodic Ellis-realization property, and every tame hereditarily amenable ambit has the ergodic Ellis-transfer property.

Examples. The Appendix shows that our results are sharp. Appendix A.1 gives a minimal metrizable tame flow for which $j : (K, \tau) \to E(X, G)$ is not Borel. Appendix A.3 gives a counterexample to the hereditary-amenability characterization of the Ellis flow, i.e. a hereditarily amenable non-tame flow with non-amenable Ellis flow.

In the split-circle dihedral flow of Appendix A.2, two Ellis groups are interchanged by the action, whereas their Haar decompressions coincide and are invariant; decompression therefore restores invariance which is not visible on either group separately.

Model-theoretic motivation. The construction is motivated by model-theoretic dynamics. Enveloping semigroups of type-space flows have played a central role since the work of Newelski [New09; New12]. Model-theoretic tameness (NIP) corresponds to dynamical tameness [Sim15; Iba16; KR20; CH23], and Ellis groups are closely related to canonical compact quotients such as G/G^{00} and the Kim–Pillay Galois group; see [KP17; CS18; KNS19; KP19; KPR18; HKP26].

In [HR25], jointly with Rzepecki, following ideas of [CS18], the first author described ergodic measures of $(S_{\bar{m}}(M), \mathrm{Aut}(M))$ for countable NIP theories in terms of the Haar measure on the Kim–Pillay Galois group. The present paper isolates a similar, purely dynamical mechanism: the Haar measure on an Ellis group is first regularized on the enveloping semigroup and then transferred by evaluation (cf. Subsection 6.4).

The idea of inducing a measure on the enveloping semigroup from the Haar measure on an Ellis group appeared already in [CGK26, Section 4] in the context of definable groups, but the situation there did not require regularization which we need in the abstract context (cf. Subsection 6.4).

Organization of the paper. Section 2 fixes the conventions and recalls the required facts about Ellis semigroups, the τ-topology, fragmented maps, and tame flows. Section 3 establishes the required measurability and constructs Haar decompressions. Section 4 proves unique ergodicity of minimal tame amenable flows and minimality of supports of ergodic measures. Section 5 proves the general finite-power amenability criterion, its tame hereditary-amenability consequence, and the invariance criterion for individual Haar decompressions. The final section develops ergodic Ellis transfer and realization, proves the ergodic Ellis-realization property for every tame ambit, obtains the Choquet representation in the metrizable hereditarily amenable case and links our general results with the model-theoretic dynamics. Appendix A contains the non-Borel inclusion example, the split-circle computation, and the counterexample showing that tameness is essential.

2. Preliminaries

Throughout, G denotes a discrete group. All compact spaces are assumed to be Hausdorff, all probability measures are regular Borel measures.

2.1. **Flows, invariant measures and Gelfand duality.** A *G-flow* is a compact space X equipped with an action of G by homeomorphisms. A *subflow* is a nonempty closed G-invariant subset, and a *factor map* is a continuous surjective G-map. A pointed flow (X, x_0, G) is an *ambit* if $\overline{Gx_0} = X$, and a flow is *minimal* if every orbit is dense.

For a compact space X, let $C(X) := C(X, \mathbb{C})$, endowed with the uniform norm and the involution $f^*(x) := \overline{f(x)}$. Its positive cone is

$$\begin{aligned} C(X)_+ &:= \{h^*h : h \in C(X)\} \\ &= \{f \in C(X) : f = f^* \text{ and } f(x) \geqslant 0 \text{ for every } x \in X\}. \end{aligned}$$

A *state* on a unital C^*-algebra A is a complex-linear functional $\Lambda : A \to \mathbb{C}$ such that $\Lambda(a^*a) \geqslant 0$ for every $a \in A$ and $\Lambda(1_A) = 1$.

We write $\mathrm{Prob}(X)$ for the compact convex space of regular Borel probability measures on X, equipped with the weak-$*$ topology inherited from $C(X)^*$. If $\pi : X \to Y$ is Borel and $\mu \in \mathrm{Prob}(X)$, then $\pi_*\mu$ denotes the pushforward of μ. For $g \in G$, we write $g_*\mu$ for the pushforward under $x \mapsto gx$, and put

$$\mathrm{Prob}_G(X) := \{\mu \in \mathrm{Prob}(X) : g_*\mu = \mu \text{ for every } g \in G\}.$$

An invariant measure is *ergodic* if it is an extreme point of $\mathrm{Prob}_G(X)$, and we write

$$\mathrm{Erg}_G(X) := \mathrm{ext}\big(\mathrm{Prob}_G(X)\big).$$

The flow (X, G) is *amenable* if $\mathrm{Prob}_G(X) \neq \varnothing$, and it is *uniquely ergodic* if $\mathrm{Prob}_G(X)$ is a singleton. It is *hereditarily amenable* if every subflow of X is amenable. The action of G on $C(X)$ is given by $(g \cdot f)(x) := f(g^{-1}x)$.

We briefly recall the form of Gelfand duality used below (see, for example, [Mur90]). For a unital commutative C^*-algebra A, its spectrum $\mathrm{Spec}(A)$ is the set of characters, i.e. unital $*$-homomorphisms $\chi : A \to \mathbb{C}$, equipped with the weak-$*$ topology. It is compact Hausdorff, and the Gelfand transform identifies A isometrically with $C(\mathrm{Spec}(A), \mathbb{C})$. If $A \subseteq C(Y, \mathbb{C})$ is a unital C^*-subalgebra, then

$$q_A : Y \longrightarrow \mathrm{Spec}(A), \qquad q_A(y)(a) := a(y),$$

is a continuous surjection; it is equivariant whenever A is invariant under the acting group. An inclusion $A \subseteq B$ induces the continuous surjection $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ given by restriction of characters. Moreover, $\mathrm{Spec}(A)$ is metrizable when A is separable. Moreover, if A is a subset of a C^*-algebra $\mathcal{C}$, then $C^*(A)$ stands for the C^*-subalgebra of $\mathcal{C}$ generated by a A.

We also recall the barycentric terminology used in the proof of Lemma 4.3. Let Q be a compact convex subset of a locally convex real topological vector space. Every $\Lambda \in \mathrm{Prob}(Q)$ has a unique *barycenter* $b(\Lambda) \in Q$, characterized by

$$f\big(b(\Lambda)\big) = \int_Q f(q)\, d\Lambda(q)$$

for every continuous affine function $f : Q \to \mathbb{R}$. A universally measurable function $F : Q \to \mathbb{R}$ is said to satisfy the *barycentric formula* if

$$F\big(b(\Lambda)\big) = \int_Q F(q)\, d\Lambda(q) \qquad \text{for every } \Lambda \in \mathrm{Prob}(Q).$$

By [Luk+10, Theorem 4.21], every real-valued fragmented affine function on a compact convex set satisfies the barycentric formula.

2.2. **Ellis semigroups, fragmented maps and tameness.** The *enveloping semigroup* of a flow (X, G) is

$$E(X, G) := \overline{\{g_X : g \in G\}}^{X^X}, \qquad g_X(x) := gx,$$

where X^X carries the product topology. Multiplication is composition: $(pq)(x) := p(q(x))$. For every fixed $q \in E(X,G)$, the map $p \mapsto pq$ is continuous. For $x \in X$, evaluation is the continuous map

$$\mathrm{ev}_x : E(X,G) \longrightarrow X, \qquad \mathrm{ev}_x(p) := p(x),$$

and $\mathrm{ev}_x[E(X,G)] = \overline{Gx}$. A factor map $\pi : X \to Y$ induces a continuous surjective G-equivariant semigroup homomorphism

$$\pi_E : E(X,G) \longrightarrow E(Y,G)$$

satisfying $\pi \circ p = \pi_E(p) \circ \pi$ for every $p \in E(X,G)$.

Put $E := E(X,G)$. We write $\mathcal{M} \trianglelefteq_m E$ for a minimal left ideal. If $u^2 = u \in \mathcal{M}$, then $K := u\mathcal{M}$ is the corresponding *Ellis group*. For $p \in E$ and $A \subseteq E$, define the Ellis circle operation by

$$p \circ A := \big\{q \in E : \text{there are nets } (g_i)_i \subseteq G \text{ and } (a_i)_i \subseteq A \\ \text{such that } g_i \to p \text{ and } g_i a_i \to q\big\}.$$

For $A \subseteq K$, put

$$\mathrm{cl}_\tau(A) := K \cap (u \circ A) = u(u \circ A).$$

The operator cl_τ is a closure operator on K. The topology determined by this closure operator is called the *τ-topology*. The τ-topology is coarser than the topology on K inherited from E. Moreover, (K,τ) is always a quasi-compact T_1 semitopological group, that is, multiplication is separately continuous. No Hausdorffness is asserted here. See [Gla76] for the classical construction in the βG setting, and [Rze18, Appendix A] for the general formulation used here. Whenever (K,τ) is a compact Hausdorff topological group, $\mathfrak{h}_K$ denotes its normalized Haar measure.

Fact 2.1. *Let (X,G) and (Y,G) be flows, and let $\Phi : E(X,G) \longrightarrow E(Y,G)$ be a continuous surjective G-equivariant semigroup homomorphism. Let $\mathcal{M} \trianglelefteq_m E(X,G)$, $u^2 = u \in \mathcal{M}$, and put $\mathcal{N} := \Phi[\mathcal{M}]$, $v := \Phi(u)$, $K := u\mathcal{M}$, $L := v\mathcal{N} = \Phi[K]$. Then $\mathcal{N} \trianglelefteq_m E(Y,G)$, $v^2 = v \in \mathcal{N}$, and the restriction*

$$\Phi|_K : (K,\tau) \longrightarrow (L,\tau)$$

is a continuous group epimorphism and a quotient map for the τ-topologies. (In particular, the conclusion applies when $\Phi = \pi_E$ is induced by a factor map $\pi : X \to Y$, and when $Y \subseteq X$ is a subflow and $\Phi : E(X,G) \to E(Y,G)$ is the restriction map.) If (K,τ) and (L,τ) are compact Hausdorff topological groups, then $(\Phi|_K)_*\mathfrak{h}_K = \mathfrak{h}_L$.

Proof. The assertions concerning minimal left ideals, idempotents, Ellis groups, and the quotient property are [KLM22, Facts 2.3 and 2.5]. The last assertion follows from the uniqueness of normalized Haar measure on a compact Hausdorff topological group. □

For $\eta \in E(X,G)$ and $\bar{x} = (x_1, \dots, x_n) \in X^n$, we write

$$\eta(\bar{x}) := \big(\eta(x_1), \dots, \eta(x_n)\big).$$

Let X be a compact space and let Y be a uniform space. A map $f : X \to Y$ is *fragmented* if, for every nonempty set $A \subseteq X$ and every entourage V of Y, there is a nonempty relatively open set $U \subseteq A$ such that

$$(f(x), f(y)) \in V$$

for all $x, y \in U$. When Y is metrizable, this is equivalent to requiring that, for every $\varepsilon > 0$ and every nonempty closed $A \subseteq X$, there is a nonempty relatively open $U \subseteq A$ with

$$\operatorname{diam} f[U] < \varepsilon.$$

In particular, a bounded function $f : X \to \mathbb{C}$ is fragmented if and only if both $\operatorname{Re} f$ and $\operatorname{Im} f$ are fragmented. A flow (X, G) is *tame* if every element of $E(X, G)$ is a fragmented self-map of X. This is one of the standard equivalent definitions of tameness; see [GM18b, Definition 3.1]. We use this characterization throughout.

We equip every compact Hausdorff space with its unique compatible uniformity. A flow (X, G) is *equicontinuous* if the family

$$\{g_X : g \in G\} \subseteq X^X$$

is equicontinuous with respect to this uniformity, where $g_X(x) := gx$.

Lemma 2.2. *Let (X, G) be a flow.*

(1) If (X, G) is equicontinuous, then, for every cardinal κ, the flow X^κ, equipped with the diagonal G-action, is equicontinuous.
(2) If (X, G) is tame, then, for every cardinal κ, the flow X^κ, equipped with the diagonal G-action, is tame.
(3) If (X, G) is tame, then its enveloping semigroup $E(X, G)$, equipped with the natural left G-action, is tame.

Consequently, if X is metrizable and (X, G) is tame, then every subflow of every finite power X^n is metrizable and tame.

Proof. (1) Let V be an entourage of X^κ. By the definition of the product uniformity, there are a finite set $F \subseteq \kappa$ and entourages V_α of X, for $\alpha \in F$, such that

$$(x_\alpha, y_\alpha) \in V_\alpha \text{ for every } \alpha \in F \quad \Longrightarrow \quad (x, y) \in V.$$

For every $\alpha \in F$, equicontinuity of (X, G) gives an entourage U_α of X such that

$$(s, t) \in U_\alpha \quad \Longrightarrow \quad (gs, gt) \in V_\alpha \text{ for every } g \in G.$$

The basic entourage of X^κ determined by the U_α, $\alpha \in F$, witnesses equicontinuity of the diagonal action on X^κ.

For (2) and (3), see [GM12, Lemma 5.4 and the sentence immediately following its proof]. □

Fact 2.3 (Theorem 8.2 in [GM13]). *If (Y, G) is tame, then the canonical restriction homomorphism*

$$\operatorname{res} : E(\operatorname{Prob}(Y), G) \longrightarrow E(Y, G)$$

is a topological semigroup (and G-flow) isomorphism. We set $\eta^\# := \operatorname{res}^{-1}(\eta)$ for every $\eta \in E(Y, G)$.

For further background on tame flows and Ellis semigroups, see [Aus88; Gla76; Gla06; GM18b; CH23].

3. Haar decompression on the Ellis semigroup

We first establish the measurability input needed to define Haar decompression. The argument combines a descent property for fragmented functions with the measurability of bounded fragmented functions for the completion of every regular Borel probability measure.

Lemma 3.1. *Let $\pi : L \to K$ be a continuous surjection of compact Hausdorff spaces. Let $f : K \to \mathbb{C}$ be bounded. If $f \circ \pi : L \to \mathbb{C}$ is fragmented, then f is fragmented.*

Proof. This is a standard fact (see, for example, [Sim15, Lemma 2.4]). Suppose that f is not fragmented. Then there are $\varepsilon > 0$ and a nonempty closed set $F \subseteq K$ such that $\operatorname{diam} f[U] \geqslant \varepsilon$ for every nonempty relatively open subset $U \subseteq F$.

By Zorn's lemma choose a minimal closed set $L_0 \subseteq \pi^{-1}(F)$ such that $\pi[L_0] = F$: the existence follows because if $(L_i)_{i\in I}$ is a decreasing chain of closed subsets of $\pi^{-1}(F)$ mapping onto F, then for each $x \in F$ the compact sets $L_i \cap \pi^{-1}(\{x\})$ have the finite intersection property, so $\bigcap_{i\in I} L_i$ still maps onto F. Since $f \circ \pi$ is fragmented on L, and hence also on the closed subspace L_0, there is a nonempty relatively open set $V \subseteq L_0$ such that $\operatorname{diam}(f \circ \pi)[V] < \varepsilon$. If $L_0 \setminus V$ still mapped onto F, this would contradict the minimality of L_0. Hence,

$$U := F \setminus \pi[L_0 \setminus V]$$

is nonempty. Moreover, since $L_0 \setminus V$ is compact, $\pi[L_0 \setminus V]$ is closed in F, so U is relatively open in F.

For every $x \in U$, $L_0 \cap \pi^{-1}(\{x\}) \subseteq V$. Therefore, $f[U] \subseteq (f \circ \pi)[V]$. Consequently,

$$\operatorname{diam} f[U] \leqslant \operatorname{diam}(f \circ \pi)[V] < \varepsilon,$$

contradicting the choice of F. Thus, f is fragmented. □

By [Luk+10, Theorem A.121], bounded fragmented functions on compact spaces are $\Sigma_1(H_s)$-measurable, where H_s denotes the family of resolvable sets. By [Luk+10, Proposition A.118], resolvable sets are universally measurable. Thus, the following fact is standard. For the convenience of the reader, we include a self-contained proof.

Fact 3.2. *Let K be a compact Hausdorff space and let $f : K \to \mathbb{C}$ be bounded and fragmented. Then, for every regular Borel probability measure λ on K, there are a Borel function $f_\lambda : K \to \mathbb{C}$ and a Borel set $N_\lambda \subseteq K$ with $\lambda(N_\lambda) = 0$ such that*

$$f = f_\lambda \quad \text{on } K \setminus N_\lambda.$$

In particular, f is λ-measurable (i.e., measurable for the λ-completion of the Borel σ-algebra on K). Consequently, if K is a compact Hausdorff group, then f is measurable for the completion of the Borel σ-algebra on K with respect to the normalized Haar measure on K.

Proof. For every $\varepsilon > 0$ and every nonempty closed subset $F \subseteq K$, there is a nonempty relatively open subset $V \subseteq F$ such that $\operatorname{diam} f[V] < \varepsilon$. Fix $\varepsilon > 0$. We construct, by transfinite recursion, a decreasing family of closed sets $(F_\gamma)_\gamma$ and disjoint Borel pieces $(P_\gamma)_\gamma$. Put $F_0 := K$. If $F_\gamma \neq \varnothing$, choose an open set $O_\gamma \subseteq K$ such that

$$P_\gamma := F_\gamma \cap O_\gamma \neq \varnothing \quad \text{and} \quad \operatorname{diam} f[P_\gamma] < \varepsilon,$$

and set

$$F_{\gamma+1} := F_\gamma \setminus P_\gamma = F_\gamma \cap (K \setminus O_\gamma).$$

At limit ordinals δ, put $F_\delta := \bigcap_{\gamma<\delta} F_\gamma$. Since the closed subsets of K form a set, this strictly decreasing process stops: there is an ordinal θ such that $F_\theta = \varnothing$.

Hence,

$$K = \bigsqcup_{\gamma<\theta} P_\gamma.$$

Indeed, if $x \in K$, then the least ordinal γ such that $x \notin F_\gamma$ cannot be a limit ordinal, and therefore $\gamma = \delta + 1$ with $x \in P_\delta$. Each P_γ is Borel, being the intersection of a closed set and an open set.

Now, fix a regular Borel probability measure λ on K. We first recall the following continuity property for regular Borel probability measures on compact spaces: if $(C_i)_{i\in I}$ is a downward directed family of closed subsets of K, then

$$\lambda\Big(\bigcap_{i\in I} C_i\Big) = \inf_{i\in I} \lambda(C_i).$$

For $\gamma \leqslant \theta$, put

$$s_\gamma := \sum_{\delta<\gamma} \lambda(P_\delta),$$

where the sum means the supremum of all finite subsums. Since the sets P_δ are pairwise disjoint and $\lambda(K) = 1$, only countably many summands are positive, so this agrees with the usual countable sum over those positive summands. We claim that $\lambda(F_\gamma) = 1 - s_\gamma$, for all $\gamma \leqslant \theta$. This is proved by transfinite induction. The case $\gamma = 0$ is immediate. For the successor step, use $F_\gamma = F_{\gamma+1} \sqcup P_\gamma$, thus

$$\lambda(F_{\gamma+1}) = \lambda(F_\gamma) - \lambda(P_\gamma) = 1 - s_\gamma - \lambda(P_\gamma) = 1 - s_{\gamma+1}.$$

At a limit ordinal δ, the closed sets $(F_\gamma)_{\gamma<\delta}$ are downward directed and $F_\delta = \bigcap_{\gamma<\delta} F_\gamma$. By the continuity property given above,

$$\lambda(F_\delta) = \lambda\Big(\bigcap_{\gamma<\delta} F_\gamma\Big) = \inf_{\gamma<\delta} \lambda(F_\gamma) = \inf_{\gamma<\delta}(1 - s_\gamma) = 1 - \sup_{\gamma<\delta} s_\gamma = 1 - s_\delta.$$

Since $F_\theta = \varnothing$, we obtain $\sum_{\gamma<\theta} \lambda(P_\gamma) = 1$.

Let

$$\Gamma_\varepsilon = \{\gamma < \theta : \lambda(P_\gamma) > 0\}.$$

This set is countable: for each $n \geqslant 1$, there are only finitely many γ with $\lambda(P_\gamma) \geqslant 1/n$. Put

$$B_\varepsilon := \bigcup_{\gamma\in\Gamma_\varepsilon} P_\gamma.$$

Then B_ε is Borel and $\lambda(B_\varepsilon) = 1$. For each $\gamma \in \Gamma_\varepsilon$, choose a point $x_\gamma \in P_\gamma$, and set

$$c_\gamma := f(x_\gamma).$$

Define a Borel step function $g_\varepsilon : K \to \mathbb{C}$ by

$$g_\varepsilon = \sum_{\gamma\in\Gamma_\varepsilon} c_\gamma\, \mathbf{1}_{P_\gamma},$$

with value 0 off B_ε. Since the sets P_γ are pairwise disjoint and Γ_ε is countable, g_ε is Borel. For $x \in B_\varepsilon$, say $x \in P_\gamma$, we have

$$|f(x) - g_\varepsilon(x)| = |f(x) - c_\gamma| < \varepsilon,$$

because $\operatorname{diam} f[P_\gamma] < \varepsilon$.

Apply the construction with $\varepsilon = 1/n$. Put $N := \bigcup_{n=1}^\infty (K \setminus B_{1/n})$. Then N is Borel and $\lambda(N) = 0$. Let

$$h_n := g_{1/n} \cdot \mathbf{1}_{K\setminus N}.$$

The functions h_n are Borel, and for every $x \in K \setminus N$,

$$|h_n(x) - f(x)| < \frac{1}{n}.$$

Thus, $h_n \to f$ pointwise on $K \setminus N$, while $h_n = 0$ on N for all n. Consequently,

$$f_\lambda := \lim_{n\to\infty} h_n$$

is Borel and agrees with f on $K \setminus N$. Therefore, f is measurable for the λ-completion of the Borel σ-algebra. $\square$

Let (X, G) be a tame flow, $u^2 = u \in \mathcal{M} \trianglelefteq_m E := E(X, G)$, $K := u\mathcal{M}$, let $j : K \to E$ be the inclusion map. The next lemma is essentially a general abstract form of the so-called *Revised Newelski's Conjecture* stated in [KP26] for definable groups, proved in [CGK26] for countable definable groups, and then in [BZ24] in a general abstract context (and without the countability assumption) for minimal flows. Here, we remove the assumption on minimality of the flow.

Lemma 3.3. *K is a compact Hausdorff group in the τ-topology.*

Proof. By Lemma 2.2(3), the flow $(E(X, G), G)$ is tame, and hence its minimal subflow $(\mathcal{M}, G)$ is tame. By [CGK26, Lemma 5.16], (K, τ) is isomorphic as a semitopological group to an Ellis group K' of $E(\mathcal{M}, G)$. Since $(\mathcal{M}, G)$ is minimal and tame, [BZ24, Theorem 11.7] implies that K' is Hausdorff, and therefore so is K.

By the basic properties of the τ-topology recalled in Section 2, (K, τ) is a quasi-compact T_1 semitopological group. Since by the first paragraph K is Hausdorff, the Ellis joint continuity theorem implies that it is a compact Hausdorff topological group. $\square$

Lemma 3.4. *If $F \in C(E, \mathbb{C})$, then $F \circ j : K \to \mathbb{C}$ is a fragmented map with respect to the τ-topology on K.*

Proof. Consider the map $\lambda_u : E \to E$ given by $\lambda_u(\eta) := u\eta$. By Lemma 2.2(3), the flow $(E(X, G), G)$ is tame. Moreover, $\lambda_u \in E(E(X, G), G)$. Therefore, $\lambda_u : E \to E$ is fragmented. In particular, its restriction $\lambda_u|_{\overline{K}} : \overline{K} \to \overline{K}$ is a fragmented map, and thus $F \circ \lambda_u|_{\overline{K}}$ is fragmented as well.

Because (X, G) is tame, we have that (K, τ) is a compact Hausdorff topological group by Lemma 3.3. By Lemma 3.1 from [KPR18], the map $\lambda_u|_{\bar{K}} : \overline{K} \to K$ is a continuous surjection (from the induced topology to the τ-topology).

Now, the composition $(F \circ j) \circ \lambda_u|_{\overline{K}}$ is equal to $F \circ \lambda_u|_{\overline{K}}$, hence it is fragmented. We apply Lemma 3.1, to get the conclusion. $\square$

Let $\mathfrak{h}_K$ be the normalized Haar measure on the compact Hausdorff group K.

Corollary 3.5. *If $F \in C(E, \mathbb{C})$, then $F \circ j : K \to \mathbb{C}$ is $\mathfrak{h}_K$-measurable.*

Proof. By Lemma 3.4, we have that $F \circ j : K \to \mathbb{C}$ is fragmented. The conclusion follows by Fact 3.2. $\square$

Fact 3.6 (Riesz–Markov representation)**.** *Let Y be a compact Hausdorff space. If*

$$\Lambda : C(Y, \mathbb{C}) \to \mathbb{C}$$

is a positive unital complex-linear functional, then there is a unique regular Borel probability measure μ_Λ *on* Y *such that*

$$\Lambda(F) = \int_Y F \, d\mu_\Lambda \quad \textit{for every } F \in C(Y, \mathbb{C}).$$

Lemma 3.7. *The formula*

$$\Lambda_K(F) := \int_K F(j(k)) \, d\mathfrak{h}_K(k) \qquad (F \in C(E, \mathbb{C}))$$

defines a positive unital complex-linear functional on $C(E, \mathbb{C})$*. Consequently, by Fact 3.6, there is a unique regular Borel probability measure* μ_K *on* E *such that*

$$\int_E F \, d\mu_K = \int_K F(j(k)) \, d\mathfrak{h}_K(k) \quad \textit{for every } F \in C(E, \mathbb{C}).$$

We call μ_K *the decompression of* $\mathfrak{h}_K$*.*

Proof. The integral is well defined by Corollary 3.5. Complex linearity follows immediately from linearity of the integral. Moreover,

$$\Lambda_K(1_E) = \int_K 1 \, d\mathfrak{h}_K = 1.$$

If $F \in C(E)_+$, then $F(j(k)) \geqslant 0$ for every $k \in K$. Therefore, $F \circ j$ is real-valued and non-negative, and

$$\Lambda_K(F) = \int_K F(j(k)) \, d\mathfrak{h}_K(k) \geqslant 0.$$

Thus, Λ_K is positive. The Riesz–Markov representation theorem now yields the unique regular Borel probability measure μ_K with the displayed property. □

Remark 3.8. The measure μ_K is not defined as a raw pushforward. Although $F \circ j$ is $\mathfrak{h}_K$-measurable for every $F \in C(E)$, the inclusion $j : (K, \tau) \to E$ need not be Borel; see Appendix A.1. It is not known whether j is always measurable from the $\mathfrak{h}_K$-completion of the Borel σ-algebra on K to the Borel σ-algebra on E. Even when this measurability holds, it is not known whether the Borel pushforward $j_*\mathfrak{h}_K$ must be regular. The Riesz–Markov construction instead produces a canonical regular Borel probability measure on E. If j is measurable in the above sense and $j_*\mathfrak{h}_K$ is regular, then

$$\mu_K = j_*\mathfrak{h}_K,$$

since the two measures have the same integrals against every function in $C(E)$.

Question 3.9. (1) Does there exist a tame flow (X, G) for which the inclusion $j : (K, \tau) \longrightarrow E(X, G)$ is not measurable with respect to the $\mathfrak{h}_K$-completion?
(2) Suppose that j is measurable for the $\mathfrak{h}_K$-completion. Must the Borel pushforward $j_*\mathfrak{h}_K$ be regular?

Remark 3.10. (1) $\operatorname{supp}(\mu_K) \subseteq \overline{u\mathcal{M}} \subseteq \mathcal{M}$.
(2) If μ_K is G-invariant, then $\overline{u\mathcal{M}} = \mathcal{M}$.
(3) For every $x \in X$, $\operatorname{supp}((\mathrm{ev}_x)_*\mu_K) \subseteq \overline{u\mathcal{M}}\, x \subseteq \mathcal{M}x$.
(4) If $(\mathrm{ev}_x)_*\mu_K$ is G-invariant, then $\operatorname{supp}((\mathrm{ev}_x)_*\mu_K) = \mathcal{M}x$ is a minimal flow.

Proof. (1) If not, then there is $F \in C(E)_+$ and $\epsilon > 0$ such that $\mu_K(\{\eta \in E : F(\eta) > \epsilon\}) > 0$ and $F|_{\mathrm{cl}(u\mathcal{M})} = 0$. Then $0 < \int_E F d\mu_K = \int_K F(j(k)) d\mathfrak{h}_K(k) = 0$, a contradiction.

(2) If μ_K is G-invariant, then so is its support which is also closed, so the conclusion follows from (1) by minimality of $\mathcal{M}$. Also (3) follows from (1).

(4) By G-invariance, $\mathrm{supp}((\mathrm{ev}_x)_*\mu_K)$ is a subflow of (X, G) which, by (3), is contained in the minimal subflow $\mathcal{M}x$, and so it must be equal to $\mathcal{M}x$. □

For every $x \in X$, $\mathrm{supp}\big((\mathrm{ev}_x)_*\mu_K\big) \subseteq \mathcal{M}x$, and $\mathcal{M}x$ is a minimal subflow of X. Consequently, whenever $(\mathrm{ev}_x)_*\mu_K$ is G-invariant, its support is minimal. This already shows that evaluation of Haar decompressions cannot produce invariant measures with nonminimal support. Proposition 4.4 below shows that an ergodic invariant measure with nonminimal support cannot occur on a tame flow. For comparison, an ergodic Bernoulli measure on a full shift has nonminimal support; however, the full shift is not tame.

An interesting feature of decompression is that μ_K may become G-invariant on $E(X, G)$ even when invariance is not visible on the original Ellis group. The split-circle dihedral example in Appendix A.2 exhibits this phenomenon.

4. Unique ergodicity and minimal supports in tame flows

Tameness imposes strong rigidity on invariant measures. We prove (without metrizability assumptions) that every minimal tame amenable flow is uniquely ergodic and that every ergodic measure on a tame flow has minimal support. Consequently, the ergodic measures are canonically parametrized by the amenable minimal subflows.

4.1. **Unique ergodicity.** We begin with the metrizable case, where the almost one-to-one description of minimal tame flows identifies the evaluation of every Haar decompression with the unique invariant probability measure. Then, we remove metrizability and obtain unique ergodicity for arbitrary minimal tame amenable flows.

We first recall the relevant terminology. Let $\pi : X \to Z$ be a factor map of compact metrizable flows, and put

$$Z_0 := \{z \in Z : |\pi^{-1}(z)| = 1\}.$$

The extension π is *almost one-to-one* if Z_0 is dense in Z. If (Z, G) is uniquely ergodic, with invariant probability measure λ_Z, then π is called *regular* if $\lambda_Z(Z_0) = 1$.

Lemma 4.1. *Let (X, G) be a metrizable minimal tame amenable flow, let $x_0 \in X$, and let $\mathcal{M} \trianglelefteq_m E := E(X, G)$, $u^2 = u \in \mathcal{M}$ and $K := u\mathcal{M}$. Then $\nu_K := (\mathrm{ev}_{x_0})_*\mu_K$ is a unique G-invariant regular Borel probability measure on X.*

Proof. Let

$$\pi : X \longrightarrow Z, \qquad z_0 := \pi(x_0),$$

be the maximal equicontinuous factor of (X, G). By [Gla18, Corollary 5.4(2)], the map π is almost one-to-one and X is uniquely ergodic. Moreover, by [Fuh+21, Theorem 1.2], the extension is regular. The minimal equicontinuous flow (Z, G) is uniquely ergodic. Thus, if λ_Z denotes the unique invariant probability measure on Z, then

$$\lambda_Z(Z_0) = 1, \qquad Z_0 := \{z \in Z : |\pi^{-1}(z)| = 1\}. \tag{1}$$

Put $L := E(Z, G)$. Since (Z, G) is minimal and equicontinuous, L is a compact Hausdorff topological group, and

$$\lambda_Z = (\mathrm{ev}_{z_0})_* \mathfrak{h}_L, \tag{2}$$

where $\mathfrak{h}_L$ is the normalized Haar measure on L. The factor map π induces a continuous surjective semigroup homomorphism $\pi_E : E(X, G) \longrightarrow E(Z, G) = L$ such that $\pi \circ \eta = \pi_E(\eta) \circ \pi$ for every $\eta \in E$. We have $\pi_E[\mathcal{M}] = L$ and $\pi_E(u)$ is equal to the identity of L, and the restriction $\pi_E|_K : K \longrightarrow L$ is a continuous epimorphism of compact Hausdorff groups. Hence,

$$(\pi_E|_K)_* \mathfrak{h}_K = \mathfrak{h}_L. \tag{3}$$

We now compute the projection of ν_K to Z. For every $\varphi \in C(Z, \mathbb{C})$, the function

$$E \ni \eta \longmapsto \varphi\big(\pi(\eta(x_0))\big) \in \mathbb{C},$$

is continuous. Using decompression, (3), and (2), we obtain

$$\begin{aligned} \int_Z \varphi \, d(\pi_* \nu_K) &= \int_E \varphi\big(\pi(\eta(x_0))\big) \, d\mu_K(\eta) = \int_K \varphi\big(\pi(k(x_0))\big) \, d\mathfrak{h}_K(k) \\ &= \int_K \varphi\big(\pi_E|_K(k) z_0\big) \, d\mathfrak{h}_K(k) = \int_L \varphi(\ell z_0) \, d\mathfrak{h}_L(\ell) \\ &= \int_Z \varphi \, d\lambda_Z. \end{aligned}$$

Hence,

$$\pi_* \nu_K = \lambda_Z. \tag{4}$$

It remains to use regularity of the almost one-to-one extension. By a standard argument the set Z_0 is a G_δ subset of Z, and hence is Polish. The set $X_0 := \pi^{-1}(Z_0)$ is likewise a G_δ subset of X, and hence is Polish. The restriction

$$\pi|_{X_0} : X_0 \longrightarrow Z_0$$

is a continuous bijection. By the Lusin–Souslin theorem, its inverse

$$s : Z_0 \longrightarrow X_0$$

is Borel. Now, let ρ be a Borel probability measure on X such that $\pi_* \rho = \lambda_Z$. Since $\lambda_Z(Z_0) = 1$, we have $\rho(X_0) = 1$. Therefore, for every Borel set $B \subseteq X$,

$$\rho(B) = \lambda_Z\big(s^{-1}(B \cap X_0)\big).$$

Thus, ρ is uniquely determined by λ_Z, and consequently there is exactly one Borel probability measure on X which projects to λ_Z.

By (4), the measure ν_K is such a lift. The unique invariant measure, say μ_X, is also such a lift, because $\pi_* \mu_X$ is G-invariant on the uniquely ergodic flow (Z, G) and hence equals λ_Z. □

Theorem 4.2. *Every minimal tame amenable flow is uniquely ergodic.*

Proof. Let (Y, G) be minimal and tame, and suppose that $\mu_0, \mu_1 \in \mathrm{Prob}_G(Y)$ are distinct. Choose $f \in C(Y)$ such that

$$\int_Y f\, d\mu_0 \neq \int_Y f\, d\mu_1.$$

We inductively construct a countable subgroup $G_0 \leqslant G$ and a metrizable minimal G_0-factor through which f descends.

Put $A_0 = C^*(1, f) \subseteq C(Y)$ and $G_{0,0} = \{1\}$. Now, suppose that a countable subgroup $G_{0,n} \leqslant G$ and a separable unital $G_{0,n}$-invariant C^*-algebra $A_n \subseteq C(Y)$ have been chosen. Let $Y_n = \mathrm{Spec}(A_n)$, let $\pi_n : Y \to Y_n$ be the Gelfand dual of $A_n \subseteq C(Y)$, i.e. $\pi_n(y)(a) := a(y)$, and choose a countable base $\mathcal{U}_n$ for Y_n. For each nonempty $U \in \mathcal{U}_n$, minimality and compactness give a finite set $F_U \subseteq G$ such that

$$Y = \bigcup_{g \in F_U} g\pi_n^{-1}[U].$$

Let $G_{0,n+1}$ be generated by $G_{0,n}$ together with all F_U for $\emptyset \neq U \in \mathcal{U}_n$. Let $A_{n+1} := C^*\big(G_{0,n+1} \cdot A_n\big)$.

Set

$$G_0 = \bigcup_{n<\omega} G_{0,n}, \qquad A = \overline{\bigcup_{n<\omega} A_n}, \qquad Z = \mathrm{Spec}(A),$$

and denote the Gelfand dual of $A \subseteq C(Y)$ by $\pi : Y \to Z$. For $y \in Y$ and $a \in A$, we have $\pi(y)(a) = a(y)$. The algebra A is separable and G_0-invariant; hence Z is a metrizable G_0-flow.

For each $n < \omega$, let $r_n : Z \to Y_n$ be the restriction map induced by $A_n \subseteq A$; thus $\pi_n = r_n \circ \pi$. If $n \leqslant m$, let $r_{m,n} : Y_m \to Y_n$ be the restriction map induced by $A_n \subseteq A_m$. Then $r_n = r_{m,n} \circ r_m$. The compatible family $(r_n)_{n<\omega}$ induces a continuous map

$$R : Z \longrightarrow \prod_{n<\omega} Y_n, \qquad R(z) = (r_n(z))_{n<\omega}.$$

This map is injective. Indeed, if $R(z) = R(z')$, then the characters z and z' agree on every A_n, hence on the norm-dense subalgebra $\bigcup_n A_n$ of A, and therefore on all of A. Since Z is compact and $\prod_n Y_n$ is Hausdorff, R is a topological embedding. Thus the topology of Z is the initial topology induced by the family $(r_n)_{n<\omega}$.

Consequently, if $\emptyset \neq V \subseteq Z$ is open, there are $n < \omega$ and a nonempty $U \in \mathcal{U}_n$ such that $r_n^{-1}[U] \subseteq V$. For $g \in F_U \subseteq G_0$, equivariance of π gives

$$\pi^{-1}\big[g r_n^{-1}[U]\big] = g\pi_n^{-1}[U].$$

We have

$$\pi^{-1}\Big[\bigcup_{g \in F_U} g r_n^{-1}[U]\Big] = \bigcup_{g \in F_U} \pi^{-1}\big[g r_n^{-1}[U]\big] = \bigcup_{g \in F_U} g\pi_n^{-1}[U] = Y.$$

Since $\pi : Y \to Z$ is surjective, it follows that $Z = \bigcup_{g \in F_U} g r_n^{-1}[U]$, and hence

$$Z = \bigcup_{g \in F_U} g r_n^{-1}[U] \subseteq \bigcup_{g \in F_U} gV.$$

Thus (Z, G_0) is minimal.

Since $f \in A$, its Gelfand transform $\hat{f} : Z \to \mathbb{C}$, ($\hat{f}(z) = z(f)$ for $z \in Z$) belongs to $C(Z)$. Moreover, for every $y \in Y$, $\hat{f}(\pi(y)) = \pi(y)(f) = f(y)$. Thus $f = \hat{f} \circ \pi$.

Note that (Z, G_0) is tame, $\pi_*\mu_0$ and $\pi_*\mu_1$ are G_0-invariant and are separated by $\hat{f}$:

$$\int \hat{f} d\pi_*\mu_0 = \int (\hat{f} \circ \pi) d\mu_0 = \int f d\mu_0 \neq \int f d\mu_1 = \int \hat{f} d\pi_*\mu_1.$$

The metrizable minimal tame flow (Z, G_0) is therefore amenable but not uniquely ergodic. This contradicts Lemma 4.1. □

4.2. **Minimality of supports.** We start with a very useful lemma which is our technique to replace the Lebesgue dominated convergence theorem when working with general nets. It will be crucial in Subsection 5.3, and here it shortens a bit the proof of Proposition 4.4. More precisley, let $\eta \in E(Y, G)$ be a limit of a net $(g_i)_{i \in I}$ in G, let $\mu \in \mathrm{Prob}_G(Y)$ and $f \in C(Y)$. For each $i \in I$ we have

$$\int_Y f \circ g_i \, d\mu = \int_Y f \, d(g_i)_*\mu = \int_Y f \, d\mu,$$

but it is not clear whether we can pass with the above equalities to the limit of $(g_i)_{i \in I} = \eta$, to obtain $\int f \circ \eta \, d\mu = \int f \, d\mu$. Tameness helps as follows:

Lemma 4.3. *Let (Y, G) be tame, $\nu \in \mathrm{Prob}(Y)$, $f \in C(Y)$. Let $\eta \in E(Y, G)$ and let $\eta^\# \in E(\mathrm{Prob}(Y), G)$ be as in Fact 2.3. Then, we have*

$$\int_Y f \circ \eta \, d\nu = \int_Y f \, d\eta^\#(\nu).$$

Proof. Without loss of generality we may assume that $f \in C(Y, \mathbb{R})$. By Lemma 2.2(3), Fact 2.3, and preservation of tameness under taking factors, $(\mathrm{Prob}(Y), G)$ is tame, and thus

$$\eta^\# : \mathrm{Prob}(Y) \to \mathrm{Prob}(Y)$$

is a fragmented map. The map $\hat{f} : \mathrm{Prob}(Y) \to \mathbb{R}$, given by $\hat{f}(\lambda) = \int f \, d\lambda$, is continuous. Thus, the map $\hat{f} \circ \eta^\# : \mathrm{Prob}(Y) \to \mathbb{R}$ is fragmented, bounded and affine. Thus, by [Luk+10, Theorem 4.21], $\hat{f} \circ \eta^\#$ satisfies the barycentric formula (cf. the last paragraph of Subsection 2.1).

Let $\delta : Y \to \mathrm{Prob}(Y)$ denote the embedding via Dirac measures, i.e. $\delta(y) = \delta_y$. Consider the measure $\delta_*\nu \in \mathrm{Prob}\big(\mathrm{Prob}(Y)\big)$.

Claim 1. The barycenter $b(\delta_*\nu) \in \mathrm{Prob}(Y)$ of $\delta_*\nu$ is equal to ν.

Proof. Let $h \in C(Y, \mathbb{R})$ and consider $\hat{h} : \mathrm{Prob}(Y) \to \mathbb{R}$, given by $\hat{h}(\lambda) := \int h d\lambda$. Applying the barycentric formula for $\hat{h}$, we have

$$\begin{aligned} \hat{h}\big(b(\delta_*\nu)\big) &= \int\limits_{\mu \in \mathrm{Prob}(Y)} \hat{h}(\mu) \, d(\delta_*\nu)(\mu) = \int\limits_Y \hat{h}(\delta_y) \, d\nu(y) \\ &= \int\limits_Y h(y) \, d\nu(y) = \hat{h}(\nu) \end{aligned}$$

Because $h \in C(Y, \mathbb{R})$ was arbitrary, we conclude $b(\delta_*\nu) = \nu$. □(claim)

Now, we use Claim 1 and apply the barycentric formula to $\hat{f} \circ \eta^\#$:

$$\int\limits_Y f \, d(\eta^\#\nu) = (\hat{f} \circ \eta^\#)(\nu) = (\hat{f} \circ \eta^\#)\big(b(\delta_*\nu)\big)$$

$$
\begin{aligned}
&= \int\limits_{\mu \in \operatorname{Prob}(Y)} (\hat{f} \circ \eta^{\#})(\mu)\, d(\delta_* \nu)(\mu) \\
&= \int\limits_{Y} (\hat{f} \circ \eta^{\#} \circ \delta)(y)\, d\nu(y) \\
&= \int\limits_{Y} \hat{f}(\delta_{\eta(y)})\, d\nu(y) = \int\limits_{Y} f \circ \eta\, d\nu
\end{aligned}
$$

□

A special instance of the minimal-support phenomenon below was obtained for tame semicascades, under additional hypotheses, in [Rom16, Theorem 3.6]. We establish the result here for arbitrary tame flows, without metrizability assumptions and for general acting groups.

Proposition 4.4. *Let (X, G) be a tame flow and let $\mu \in \operatorname{Erg}_G(X)$. Then* $\operatorname{supp}(\mu)$ *is a minimal subflow of X.*

Proof. Set $Y := \operatorname{supp}(\mu)$, and regard μ as a probability measure on Y. Then (Y, G) is tame, $\mu \in \operatorname{Erg}_G(Y)$, and μ has full support.

Suppose, towards a contradiction, that Y is not minimal. Choose a proper nonempty subflow $C \subsetneq Y$, a point $y_0 \in Y \setminus C$, and a function $f \in C(Y, [0, 1])$ such that $f|_C = 0$ and $f(y_0) = 1$. As in the proof of Theorem 4.2, we shall construct a countable subgroup $G_0 \leqslant G$ and a metrizable tame G_0-factor through which f descends. The construction will also ensure that every nonempty open subset of the factor has conull G_0-orbit with respect to the projected measure.

Put $A_0 := C^*(1, f) \subseteq C(Y)$, $G_{0,0} := \{1_G\}$. Assume that a countable subgroup $G_{0,n} \leqslant G$ and a separable unital $G_{0,n}$-invariant C^*-algebra $A_n \subseteq C(Y)$ have been chosen. Let

$$Y_n := \operatorname{Spec}(A_n),$$

let $\pi_n : Y \to Y_n$ be the Gelfand dual of $A_n \subseteq C(Y)$, and choose a countable base $\mathcal{U}_n$ for Y_n. Fix a nonempty $U \in \mathcal{U}_n$ and put $W_U := \pi_n^{-1}[U]$. The set W_U is nonempty and open, so $\mu(W_U) > 0$. Consider an open and G-invariant set

$$GW_U = \bigcup_{g \in G} gW_U.$$

By [Phe01, Proposition 12.4], ergodicity of μ implies $\mu(GW_U) = 1$. By regularity and compactness, there is a countable set $F_U \subseteq G$ such that

$$\mu\Big(\bigcup_{g \in F_U} gW_U\Big) = 1.$$

Indeed, for every $m \geqslant 1$, choose a compact set $C_m \subseteq GW_U$ with $\mu(C_m) > 1 - 1/m$ and then a finite set $F_{U,m} \subseteq G$ such that

$$C_m \subseteq \bigcup_{g \in F_{U,m}} gW_U.$$

We set $F_U := \bigcup_{m \geqslant 1} F_{U,m}$.

Let $G_{0,n+1}$ be generated by $G_{0,n}$ together with all the sets F_U, where $\emptyset \neq U \in \mathcal{U}_n$, and let $A_{n+1} := C^*\big(G_{0,n+1} \cdot A_n\big)$. Set

$$G_0 := \bigcup_{n<\omega} G_{0,n}, \qquad A := \overline{\bigcup_{n<\omega} A_n}, \qquad Z := \operatorname{Spec}(A),$$

and let $\pi : Y \to Z$ be the Gelfand dual of $A \subseteq C(Y)$. The flow (Z, G_0) is tame and metrizable. We will show that it is also a minimal flow.

For each $n < \omega$, let $r_n : Z \longrightarrow Y_n$ be the restriction map induced by $A_n \subseteq A$; thus $\pi_n = r_n \circ \pi$. If $n \leqslant m$, let $r_{m,n} : Y_m \to Y_n$ be induced by $A_n \subseteq A_m$. Then $r_n = r_{m,n} \circ r_m$. Exactly as in the proof of Theorem 4.2, the map

$$R : Z \longrightarrow \prod_{n<\omega} Y_n, \qquad R(z) := (r_n(z))_{n<\omega},$$

is a topological embedding. Consequently, if $\emptyset \neq V \subseteq Z$ is open, then there are $n < \omega$ and a nonempty $U \in \mathcal{U}_n$ such that

$$r_n^{-1}[U] \subseteq V.$$

Put $\nu := \pi_*\mu$. For every nonempty $U \in \mathcal{U}_n$ and every $g \in F_U \subseteq G_0$, G_0-equivariance of π gives $\pi^{-1}\big[g r_n^{-1}[U]\big] = g\pi_n^{-1}[U]$. It follows from the choice of F_U that

$$\begin{aligned} \nu\big(\bigcup_{g\in F_U} g r_n^{-1}[U]\big) &= \mu\Big(\pi^{-1}\big[\bigcup_{g\in F_U} g r_n^{-1}[U]\big]\Big) \\ &= \mu\big(\bigcup_{g\in F_U} g\pi_n^{-1}[U]\big) = 1. \end{aligned}$$

Therefore, for every nonempty open subset $V \subseteq Z$, $\nu(G_0 V) = 1$.

Choose a countable base $(V_m)_{m<\omega}$ of nonempty open subsets of Z and put

$$T := \bigcap_{m<\omega} G_0 V_m.$$

Then $\nu(T) = 1$, and every point of T has dense G_0-orbit in Z.

We now use Lemma 4.3 to complete the minimality argument. Let $E_Z := E(Z, G_0)$, choose a minimal left ideal $\mathcal{M}_Z \trianglelefteq_m E_Z$, and let $u \in \mathcal{M}_Z$ be an idempotent. Since ν is G_0-invariant, it is fixed by every element of $E(\operatorname{Prob}(Z), G_0)$. In particular, $u^{\#}(\nu) = \nu$ (notation as in Fact 2.3). Since Z is compact metrizable and u is fragmented, u is Borel measurable (by [GM12, Lemma 2.5(2)], u is of Baire class one). Because Z is compact metrizable, $u_*\nu$ is regular [Kec95, Theorem 17.10], and in consequence, since by Lemma 4.3 for every $f \in C(Z)$ we have

$$\int_Z f du^{\#}\nu = \int_Z (f \circ u) d\nu = \int_Z f du_*\nu,$$

we get $u^{\#}\nu = u_*\nu$. In particular, for every Borel set $A \subseteq Z$,

$$\big(u^{\#}(\nu)\big)(A) = \nu\big(u^{-1}[A]\big).$$

Let $F := \{z \in Z : uz = z\}$. The set F is Borel, and $u^2 = u$ implies that $u^{-1}[F] = Z$. Therefore,

$$\nu(F) = \big(u^{\#}(\nu)\big)(F) = \nu\big(u^{-1}[F]\big) = 1.$$

Choose $z \in F \cap T$. Then $uz = z$ and $\overline{G_0 z} = Z$, so

$$Z = E_Z z = E_Z uz = (E_Z u)z = \mathcal{M}_Z z.$$

Thus, (Z, G_0) is minimal.

Finally, since $f \in A$, its Gelfand transform $\widehat{f} \in C(Z)$ satisfies $f = \widehat{f} \circ \pi$. The set $\pi(C)$ is a nonempty closed G_0-invariant subset of the minimal flow Z, and hence $\pi(C) = Z$. Since $f|_C = 0$, it follows that $\widehat{f} = 0$ (on Z), contradicting $f(y_0) = 1$. Therefore, $Y = \operatorname{supp}(\mu)$ is minimal. □

Remark 4.5. Let (X, G) be a flow. Recall that a G-invariant measure μ is ergodic if it is an extreme point of $\operatorname{Prob}_G(X)$. Alternatively (by [Phe01, Proposition 12.4]), if for every measurable $A \subseteq X$ such that $\mu(gA \triangle A) = 0$ holds for each $g \in G$, we have that $\mu(A) \in \{0, 1\}$. In the literature, there is a weaker notion of ergodicity:

(‡) μ is G-invariant and for every measurable G-invariant $A \subseteq X$ we have $\mu(A) \in \{0, 1\}$.

We claim that (‡) is equivalent to ergodicity for tame flows: Let (X, G) be tame and let μ satisfy condition (‡). Set $Y := \operatorname{supp}(\mu)$ and note that the proof of Proposition 4.4 might be repeated using only condition (‡). Thus (Y, G) is tame, minimal and amenable. By Theorem 4.2, $\mu \in \operatorname{Erg}_G(Y)$. Now, if $\nu_1, \nu_2 \in \operatorname{Prob}_G(X)$ and $\mu = a_1\nu_1 + a_2\nu_2$ for some $0 < a_1, a_2$ with $a_1 + a_2 = 1$, then $\operatorname{supp}(\nu_1), \operatorname{supp}(\nu_2) \subseteq Y$. Then, by Theorem 4.2, $\nu_1 = \mu = \nu_2$.

Corollary 4.6. *Let (X, G) be tame, and let $\operatorname{Min}_G^{\mathrm{am}}(X)$ denote the collection of all amenable minimal subflows of X. Then the support map*

$$\operatorname{Erg}_G(X) \longrightarrow \operatorname{Min}_G^{\mathrm{am}}(X), \qquad \mu \longmapsto \operatorname{supp}(\mu),$$

is a bijection. Its inverse assigns to each $Y \in \operatorname{Min}_G^{\mathrm{am}}(X)$ its unique invariant probability measure μ_Y, regarded as a measure on X.

Consequently, distinct ergodic invariant probability measures on X have disjoint supports. If (X, G) is hereditarily amenable, then $\operatorname{Min}_G^{\mathrm{am}}(X)$ is the collection of all minimal subflows of X.

Proof. Let $\mu \in \operatorname{Erg}_G(X)$. By Proposition 4.4, $\operatorname{supp}(\mu)$ is minimal. Thus, $\operatorname{supp}(\mu) \in \operatorname{Min}_G^{\mathrm{am}}(X)$.

Conversely, let $Y \in \operatorname{Min}_G^{\mathrm{am}}(X)$. Since (Y, G) is minimal, tame and amenable, Theorem 4.2 gives a unique invariant probability measure μ_Y on Y. We have $\operatorname{supp}(\mu_Y) = Y$. Moreover, μ_Y is ergodic when regarded as a measure on X. Indeed, if

$$\mu_Y = t\nu_0 + (1-t)\nu_1, \qquad 0 < t < 1, \qquad \nu_0, \nu_1 \in \operatorname{Prob}_G(X),$$

then concentration of μ_Y on Y forces both ν_0 and ν_1 to be concentrated on Y. The uniqueness of the invariant probability measure on Y therefore gives $\nu_0 = \nu_1 = \mu_Y$. □

5. Invariant measures on the Ellis semigroup

We first characterize amenability of the Ellis flow in terms of hereditary amenability of all finite powers of X; this criterion does not require tameness (Theorem 5.3). We then prove that, for tame flows, ordinary hereditary amenability implies the above finite-power condition (Theorem 5.5), which yields Corollary 5.6. Finally, Theorem 5.9 shows that the Haar decompression is G-invariant whenever the corresponding minimal left ideal $\mathcal{M}$ is amenable as a G-flow. (We do not assume that (X, G) has a dense orbit.)

5.1. **Finite-power hereditary amenability.** Theorem 5.3 below shows, without any tameness assumption, that amenability of the Ellis flow is equivalent to hereditary amenability of all finite powers of X. Its proof uses the dense algebra of finite-coordinate functions introduced below. This motivates the following definition.

Definition 5.1. The flow X is *finitely power-hereditarily amenable* if, for every $n \geqslant 1$, every subflow of X^n, equipped with the diagonal action

$$g(x_1, \dots, x_n) := (gx_1, \dots, gx_n),$$

carries a G-invariant regular Borel probability measure.

We shall use the algebra of finite-coordinate functions

$$\mathcal{A}_{\mathrm{fin}} := \left\{ F \in C(E(X,G), \mathbb{C}) : \begin{array}{l} \text{for some } n \geqslant 1,\ \bar{x} \in X^n, \text{ and } \varphi \in C(X^n, \mathbb{C}), \\ F(\eta) = \varphi(\eta \bar{x}) \text{ for every } \eta \in E(X,G) \end{array} \right\}.$$

By concatenating the tuples on which two functions depend, one sees that $\mathcal{A}_{\mathrm{fin}}$ is closed under sums and products; it is also closed under complex conjugation, scalar multiplication and contains the constants. It separates points of $E(X,G)$: if $\eta \neq \xi$, choose $x \in X$ with $\eta(x) \neq \xi(x)$ and then choose $f \in C(X, \mathbb{C})$ separating these two points. The function

$$\zeta \longmapsto f(\zeta(x))$$

belongs to $\mathcal{A}_{\mathrm{fin}}$ and separates η and ξ. Hence, by the complex Stone–Weierstrass theorem,

$$\overline{\mathcal{A}_{\mathrm{fin}}}^{\|\cdot\|_\infty} = C(E(X,G), \mathbb{C}).$$

Remark 5.2. $(E(X,G), G)$ is amenable if and only if $(E(X,G), G)$ is hereditarily amenable.

Proof. Only the left-to-right implication requires proof. Put $E := E(X,G)$, and let ν be a G-invariant regular Borel probability measure on E.

Let $Y \subseteq E$ be a nonempty subflow and fix $q \in Y$. By the left-continuity in E, the right translation

$$R_q : E \longrightarrow E, \qquad R_q(p) := pq,$$

is continuous and G-equivariant (i.e. $R_q(gp) = gR_q(p)$). Moreover, $R_q[E] = Eq = \overline{Gq}$. Since Y is closed and G-invariant and $q \in Y$, one has $\overline{Gq} \subseteq Y$. Thus, $(R_q)_*\nu$ is a probability measure on Y. Because R_q is G-equivariant, $(R_q)_*\nu$ is G-invariant. □

Theorem 5.3. *$\big(E(X,G), G\big)$ is amenable if and only if (X,G) is finitely power-hereditarily amenable.*

Proof. Assume first that $\big(E(X,G), G\big)$ is amenable. Fix $n \geqslant 1$, and let $W \subseteq X^n$ be a nonempty subflow. Choose a minimal subflow $V \subseteq W$ and a point $\bar{x} = (x_1, \dots, x_n) \in V$. Consider the evaluation map

$$e_{\bar{x}} : E(X,G) \longrightarrow V, \qquad e_{\bar{x}}(\eta) := \eta(\bar{x}).$$

It is continuous and G-equivariant. Moreover, the image of $e_{\bar{x}}$ is compact and contains $G\bar{x}$, so minimality of V gives $e_{\bar{x}}[E(X,G)] = V$.

The pushforward of a G-invariant probability measure on $E(X,G)$ is therefore a G-invariant regular Borel probability measure on V, and hence, can be viewed as a measure on W which is supported on V.

Now, we prove that if (X,G) is finitely power-hereditarily amenable then the flow $\big(E(X,G),G\big)$ is amenable. Let $E := E(X,G)$. For $g \in G$, $F \in C(E,\mathbb{C})$, and $\varepsilon > 0$, consider

$$D(g,F,\varepsilon) := \left\{\rho \in \mathrm{Prob}(E) : \left|\int_E F(g\eta)\,d\rho(\eta) - \int_E F(\eta)\,d\rho(\eta)\right| \leqslant \varepsilon\right\}.$$

Each $D(g,F,\varepsilon)$ is weak-$*$ closed in $\mathrm{Prob}(E)$. As $\mathrm{Prob}(E)$ is weak-$*$ compact, it is enough to show that these sets have the finite-intersection property (see the end of the proof).

Fix $g_1,\dots,g_r \in G$, $F_1,\dots,F_r \in C(E,\mathbb{C})$ and $\epsilon_1,\dots,\epsilon_r > 0$. By the density of the algebra $\mathcal{A}_{\mathrm{fin}}$ in $C(E,\mathbb{C})$ (see the paragraph after Definition 5.1), for every $j \leqslant r$ choose $A_j \in \mathcal{A}_{\mathrm{fin}}$ such that

$$\|F_j - A_j\|_\infty < \frac{\varepsilon_j}{3}. \tag{5}$$

After concatenating the finitely many tuples on which the functions A_j depend, there are a single finite tuple $\bar{x} = (x_1,\dots,x_n) \in X^n$ and functions $\varphi_j \in C(X^n,\mathbb{C})$ such that

$$A_j(\eta) = \varphi_j(\eta\bar{x}) \qquad (\eta \in E,\ 1 \leqslant j \leqslant r). \tag{6}$$

Consider $e_{\bar{x}} : E \longrightarrow X^n$ given by $e_{\bar{x}}(\eta) := \eta(\bar{x})$, and set $Y_{\bar{x}} := e_{\bar{x}}[E]$. The set $Y_{\bar{x}}$ is a subflow of X^n, and $Y_{\bar{x}} = \overline{G\bar{x}}$.

Since X is finitely power-hereditarily amenable, there is a G-invariant regular Borel probability measure λ on $Y_{\bar{x}}$.

Claim 1. λ has a lift ρ to E.

Proof. Since $e_{\bar{x}} : E \to Y_{\bar{x}}$ is a continuous surjection, the formula

$$L_0(f \circ e_{\bar{x}}) := \int_{Y_{\bar{x}}} f\,d\lambda$$

defines a state on the unital C^*-subalgebra $e_{\bar{x}}^* C(Y_{\bar{x}},\mathbb{C}) \subseteq C(E,\mathbb{C})$. By Krein extension theorem, we can extend L_0 to a state on $C(E,\mathbb{C})$. By the Riesz–Markov theorem, the extension is integration against some regular Borel probability measure ρ on E, and, by construction,

$$(e_{\bar{x}})_*\rho = \lambda. \tag{7}$$

□(claim)

Using (6), (7), and the G-invariance of λ, we obtain, for every $1 \leqslant j \leqslant r$,

$$\begin{aligned}\int_E A_j(g_j\eta)\,d\rho(\eta) &= \int_{Y_{\bar{x}}} \varphi_j(g_j y)\,d\lambda(y) = \int_{Y_{\bar{x}}} \varphi_j(y)\,d\lambda(y)\\ &= \int_E A_j(\eta)\,d\rho(\eta).\end{aligned}$$

Consequently, by (5),

$$\begin{aligned}&\left|\int_E F_j(g_j\eta)\,d\rho(\eta) - \int_E F_j(\eta)\,d\rho(\eta)\right|\\ &\quad\leqslant \left|\int_E \big(F_j - A_j\big)(g_j\eta)\,d\rho(\eta)\right| + \left|\int_E \big(F_j - A_j\big)(\eta)\,d\rho(\eta)\right|\\ &\quad\leqslant 2\|F_j - A_j\|_\infty < \frac{2\varepsilon_j}{3} < \varepsilon_j.\end{aligned}$$

Thus,

$$\rho \in \bigcap_{j=1}^{r} D(g_j, F_j, \varepsilon_j),$$

which proves the finite-intersection property.

By compactness of $\mathrm{Prob}(E)$, there is

$$\rho_0 \in \bigcap_{\substack{g\in G,\ F\in C(E,\mathbb{C})\\ m\geqslant 1}} D\left(g, F, \frac{1}{m}\right).$$

Therefore, for every $g \in G$ and every $F \in C(E, \mathbb{C})$,

$$\int_E F(g\eta)\, d\rho_0(\eta) = \int_E F(\eta)\, d\rho_0(\eta).$$

Equivalently, $g_*\rho_0 = \rho_0$ for every $g \in G$. Hence, the Ellis flow $(E(X, G), G)$ carries a G-invariant regular Borel probability measure. □

5.2. **Hereditary amenability for tame flows.** Tameness will be needed in order to pass from hereditary amenability of X to hereditary amenability of its finite powers. We first prove a joining lemma for minimal metrizable tame flows and then use a separable reduction to treat arbitrary compact tame flows.

Lemma 5.4. *Let H be a group, let $Y_1, \dots, Y_n$ be minimal metrizable tame H-flows, each carrying an H-invariant regular Borel probability measure, and let*

$$V \subseteq Y_1 \times \cdots \times Y_n$$

be a minimal H-subflow. Then V carries an H-invariant regular Borel probability measure.

Proof. For each i, let

$$\pi_i : Y_i \longrightarrow Z_i$$

be the maximal equicontinuous factor. As in the proof of Lemma 4.1, by [Gla18, Corollary 5.4(2)] and [Fuh+21, Theorem 1.2], if $\mathfrak{m}_i$ denotes the unique invariant probability measure on Z_i, then

$$\mathfrak{m}_i(Z_i^0) = 1, \qquad Z_i^0 := \left\{ z \in Z_i : |\pi_i^{-1}(z)| = 1 \right\}.$$

Define

$$\pi_V : V \longrightarrow Z_1 \times \cdots \times Z_n$$

as $\pi_V := \pi_1 \times \ldots \times \pi_n$, and put $Z := \pi_V[V]$. By Lemma 2.2(1), the product $Z_1 \times \cdots \times Z_n$ is equicontinuous. Hence, its subflow Z is equicontinuous. Since Z is minimal, it has a unique invariant probability measure $\mathfrak{m}_Z$.

For each i, the image of the coordinate projection $V \longrightarrow Y_i$ is a nonempty closed H-invariant subset of the minimal flow Y_i, thus equal to Y_i. It follows that the induced projection $p_i : Z \longrightarrow Z_i$ is also onto. Therefore,

$$(p_i)_*\mathfrak{m}_Z = \mathfrak{m}_i.$$

Consequently, $\mathfrak{m}_Z(Z^0) = 1$ for

$$Z^0 := Z \cap \left(Z_1^0 \times \cdots \times Z_n^0\right) = \bigcap_{i=1}^{n} p_i^{-1}[Z_i^0].$$

Now, if $z = (z_1, \dots, z_n) \in Z^0$, then

$$\pi_V^{-1}(z) \subseteq \pi_1^{-1}(z_1) \times \cdots \times \pi_n^{-1}(z_n).$$

The set on the right is a singleton. Since $z \in \pi_V[V]$, the fibre $\pi_V^{-1}(z)$ is nonempty and hence is itself a singleton.

A probability lift of $\mathfrak{m}_Z$ to V exists by applying the Krein extension theorem to extend the state

$$f \circ \pi_V \mapsto \int_Z f \, d\mathfrak{m}_Z$$

from the unital C^*-subalgebra $\pi_V^* C(Z)$ to a state on $C(V)$, and subsequently applying the Riesz–Markov representation theorem.

This lift is unique. Indeed, each Z_i^0 is a G_δ subset of Z_i, and hence Z^0 is a G_δ subset of the compact metrizable space Z. Put $V^0 := \pi_V^{-1}[Z^0]$. Then V^0 and Z^0 are Polish spaces, and $\pi_V|_{V^0} : V^0 \longrightarrow Z^0$ is a continuous bijection. By the Lusin–Souslin theorem, its inverse is Borel. Every probability measure on V which projects to $\mathfrak{m}_Z$ is concentrated on V^0, and is therefore the pushforward of $\mathfrak{m}_Z|_{Z^0}$ under this inverse. Thus, the lift is unique.

For every $h \in H$, the translate of this lift is again a lift of $\mathfrak{m}_Z$. By uniqueness, the lift is H-invariant. □

Theorem 5.5. *Let (X, G) be tame. If X is hereditarily amenable, then X is finitely power-hereditarily amenable.*

Proof. Fix $n \geqslant 1$, and let $W \subseteq X^n$ be a nonempty subflow. It is enough to construct an invariant probability measure on a minimal subflow of W. Replacing W by such a subflow, we assume that W is minimal.

For $1 \leqslant i \leqslant n$, put

$$Y_i := \mathrm{pr}_i[W].$$

Each Y_i is a minimal subflow of X. By hereditary amenability, choose a G-invariant regular Borel probability measure μ_i on Y_i. For $g \in G$ and $f \in C(W, \mathbb{C})$, we consider

$$C(g, f) := \left\{ \nu \in \mathrm{Prob}(W) : \int_W f(gw)\, d\nu(w) = \int_W f(w)\, d\nu(w) \right\}.$$

Each $C(g, f)$ is weak-$*$ closed, and

$$\mathrm{Prob}_G(W) = \bigcap_{\substack{g \in G \\ f \in C(W, \mathbb{C})}} C(g, f).$$

We verify that the above family has the finite-intersection property.

Fix $g_1, \dots, g_r \in G$ and $f_1, \dots, f_r \in C(W, \mathbb{C})$ and let $H_0 \leqslant G$ be the subgroup generated by $g_1, \dots, g_r$. Since W is a closed subset of the compact Hausdorff, and hence normal, space

$$Y_1 \times \cdots \times Y_n,$$

the Tietze extension theorem, applied separately to the real and imaginary parts, gives, for every $j \leqslant r$, an extension $\widetilde{f}_j \in C(Y_1 \times \cdots \times Y_n, \mathbb{C})$ of f_j.

By the complex Stone–Weierstrass theorem, each $\widetilde{f}_j$ is a uniform limit of finite sums of products of coordinate functions. Choose a sequence of such approximations for every j, and, for $1 \leqslant i \leqslant n$, let $\mathcal{D}_i \subseteq C(Y_i, \mathbb{C})$ be the countable family of all i-th coordinate functions which occur in these approximations.

We now construct recursively an increasing sequence of countable subgroups

$$H_0 \leqslant H_1 \leqslant H_2 \leqslant \cdots \leqslant G$$

and, simultaneously for every $1 \leqslant i \leqslant n$, an increasing sequence of separable unital H_m-invariant C^*-subalgebras

$$A_{i,m} \subseteq C(Y_i, \mathbb{C}).$$

The action on functions is given by $(h \cdot a)(y) := a(h^{-1}y)$. We already have H_0. For each i, let $A_{i,0}$ be the unital C^*-algebra generated by $\{h \cdot a : h \in H_0,\ a \in \mathcal{D}_i\}$. Now, assume that H_m and all the algebras $A_{i,m}$ have been constructed. Let

$$q_{i,m} : Y_i \longrightarrow Y_{i,m} := \operatorname{Spec}(A_{i,m})$$

be the associated factor maps of H_m-flows, with metrizable targets $Y_{i,m}$, and put

$$q_m := (q_{1,m}, \ldots, q_{n,m})|_W, \qquad W_m := q_m[W].$$

Choose a countable basis $\mathcal{B}_m$ consisting of nonempty open subsets of W_m. For $B \in \mathcal{B}_m$, the set $U_B := q_m^{-1}[B]$ is a nonempty open subset of the minimal G-flow W, and so the family $\{gU_B : g \in G\}$ covers W. By compactness, choose a finite set $F_B \subseteq G$ such that

$$W = \bigcup_{g \in F_B} gU_B.$$

Let H_{m+1} be the subgroup generated by

$$H_m \cup \bigcup_{B \in \mathcal{B}_m} F_B.$$

This group is countable. For every i, let $A_{i,m+1}$ be the unital C^*-algebra generated by all H_{m+1}-translates of $A_{i,m}$. Then $A_{i,m+1}$ is separable and H_{m+1}-invariant. This completes the recursive construction.

Now, we introduce

$$H := \bigcup_{m<\omega} H_m$$

and

$$A_i := \overline{\bigcup_{m<\omega} A_{i,m}}.$$

Then H is countable, and each A_i is a separable unital H-invariant C^*-subalgebra of $C(Y_i, \mathbb{C})$. Let

$$q_i : Y_i \longrightarrow Y_i' := \operatorname{Spec}(A_i)$$

be the associated factor map of H-flows, with metrizable target Y_i', and put

$$q := (q_1, \ldots, q_n)|_W, \qquad W' := q[W].$$

Claim 1. Every f_j factors through q, i.e. $f_j = \hat{f}_j \circ q$ for a unique function $\hat{f}_j \in C(W', \mathbb{C})$.

Proof. Suppose that $q(w) = q(w')$. Then, for every i and every $a \in A_i$, the function a has the same value on the i-th coordinates of w and w'. In particular, this holds for every member of $\mathcal{D}_i$. Thus, all the chosen finite sums of products of coordinate functions have the same value at w and w'. Passing to the uniform limit gives $f_j(w) = f_j(w')$. Since $q : W \to W'$ is a continuous surjection from a compact space onto a Hausdorff space, it is a quotient map. Hence, there is a unique function $\widehat{f}_j \in C(W', \mathbb{C})$ such that $f_j = \widehat{f}_j \circ q$. □(claim)

Claim 2. W' is minimal as an H-flow.

Proof. By Gelfand duality, the inclusions $A_{i,m} \subseteq A_i$ induce factor maps $r_{i,m} : Y_i' \longrightarrow Y_{i,m}$ and hence continuous maps $r_m : W' \longrightarrow W_m$ such that

$$q_m = r_m \circ q.$$

The algebra

$$\bigcup_{m<\omega} r_m^* C(W_m, \mathbb{C})$$

is uniformly dense in $C(W', \mathbb{C})$. Indeed, the coordinate functions coming from the algebras A_i separate points of W', so the self-adjoint unital algebra generated by them is dense in $C(W', \mathbb{C})$ by the complex Stone–Weierstrass theorem. Every member of A_i is a uniform limit of members of the algebras $A_{i,m}$. By the isometry of the Gelfand transform, such approximation in A_i yields uniform approximation of the corresponding coordinate functions on Y_i'. Finally, it is easy to see that the coordinate functions coming from the algebras $A_{i,m}$ belong to

$$\bigcup_{m<\omega} r_m^* C(W_m, \mathbb{C}).$$

It follows that the sets $r_m^{-1}[B]$, with $m < \omega$ and $B \in \mathcal{B}_m$, form a basis of the topology of W'. Explicitly, let $z \in O \subseteq W'$, where O is open. Choose

$$\phi \in C(W', [0,1])$$

such that $\phi(z) = 1$ and $\phi = 0$ on $W' \setminus O$. By the preceding density, choose m and

$$\chi_m \in C(W_m, \mathbb{C})$$

such that $\|\phi - \chi_m \circ r_m\|_\infty < \frac{1}{3}$. Let $\psi_m := \mathrm{Re}(\chi_m) \in C(W_m, \mathbb{R})$. Since ϕ is real-valued,

$$\|\phi - \psi_m \circ r_m\|_\infty < \frac{1}{3}.$$

Therefore,

$$z \in r_m^{-1}\left[\left\{y \in W_m : \psi_m(y) > \frac{2}{3}\right\}\right] \subseteq O.$$

Choosing a basis element $B \in \mathcal{B}_m$ contained in the open set $\{y \in W_m : \psi_m(y) > \frac{2}{3}\}$ and containing $r_m(z)$, we get $z \in r_m^{-1}[B] \subseteq O$.

For such $B \in \mathcal{B}_m$, we already fixed a finite set $F_B \subseteq H$ such that

$$W = \bigcup_{g \in F_B} g q_m^{-1}[B].$$

Since q is H-equivariant and $q_m = r_m \circ q$, applying q yields

$$W' = \bigcup_{g \in F_B} g\, r_m^{-1}[B] \subseteq \bigcup_{g \in F_B} gO.$$

Thus, finitely many H-translates of every nonempty open subset of W' cover W'. Hence, every H-orbit in W' is dense, and W' is minimal. □(claim)

For each i, the coordinate map

$$\mathrm{pr}_i|_{W'} : W' \longrightarrow Y_i'$$

is continuous and H-equivariant. It is also onto as $W' = q[W] = (q_1, \dots, q_n)[W]$ and we have $\mathrm{pr}_i[W'] = q_i[\mathrm{pr}_i[W]] = q_i[Y_i] = Y_i'$. Therefore, by Claim 2, Y_i' is an H-factor of the minimal flow W', and hence Y_i' is minimal.

Since Y_i is a subflow of the tame G-flow X, the restricted H-flow Y_i is tame. Tameness passes to factors, so Y_i' is tame. Moreover,

$$\mu_i' := (q_i)_* \mu_i$$

is an H-invariant regular Borel probability measure on Y_i'. Lemma 5.4 applied to $W' \subseteq Y_1' \times \cdots \times Y_n'$, gives an H-invariant regular Borel probability measure λ on W'.

Define a state on the unital C^*-subalgebra $q^* C(W', \mathbb{C}) \subseteq C(W, \mathbb{C})$ by

$$L(\varphi \circ q) := \int_{W'} \varphi \, d\lambda.$$

Extend L to a state on $C(W, \mathbb{C})$, and let ν be its Riesz measure on W. Then $q_* \nu = \lambda$.

Using the factorization $f_j = \widehat{f_j} \circ q$ from Claim 1, the H-equivariance of q, the inclusion $g_j \in H$, and the H-invariance of λ, we obtain

$$\begin{aligned} \int_W f_j(g_j w) \, d\nu(w) &= \int_{W'} \widehat{f_j}(g_j z) \, d\lambda(z) = \int_{W'} \widehat{f_j}(z) \, d\lambda(z) \\ &= \int_W f_j(w) \, d\nu(w). \end{aligned}$$

Thus, $\nu \in C(g_1, f_1) \cap \cdots \cap C(g_r, f_r)$, and so we have proved that the family

$$\{C(g, f) : g \in G, \ f \in C(W, \mathbb{C})\}$$

has nonempty finite intersections. By compactness,

$$\mathrm{Prob}_G(W) = \bigcap_{\substack{g \in G \\ f \in C(W, \mathbb{C})}} C(g, f) \neq \varnothing.$$

Thus, W carries a G-invariant regular Borel probability measure. □

Corollary 5.6. *Let (X, G) be tame. The following conditions are equivalent:*

(i) $\big(E(X,G), G\big)$ *is hereditarily amenable;*
(ii) $\big(E(X,G), G\big)$ *is amenable;*
(iii) X *is hereditarily amenable.*

Proof. Equivalence between (i) and (ii) is the content of Remark 5.2.

Assume (ii). By Theorem 5.3, the flow X is finitely power-hereditarily amenable, and hence is hereditarily amenable. Thus, (iii) holds.

Conversely, (iii) implies (ii) by Theorem 5.5 combined with Theorem 5.3. □

The tameness assumption in Corollary 5.6 cannot be omitted: Appendix A.3 constructs a metrizable hereditarily amenable flow whose Ellis flow is not amenable.

5.3. **Invariance of Haar decompressions.** The preceding results guarantee the existence of some invariant measure on the Ellis flow. We now prove the stronger statement that the canonical Haar decompression associated with each Ellis group is invariant. In the first version of this paper, we obtained the result under an additional hypothesis that the flow (X, G) is an inverse limit of metrizable G-flows. Then, during our work in a subsequent project [HK26], we were able to use a conclusion from *Generalized Newelski Conjecture* (GNC) established there to prove results of this subsection without any metrizability related hypothesis. Finally, we obtained arguments not requiring theorems from [HK26] to prove what follows below. On the other hand, Lemma 5.7(1) will be used in [HK26] to give a short proof of GNC.

Now, let (X,G) be a flow, $\mathcal{M} \trianglelefteq_m E(X,G)$, $u^2 = u \in \mathcal{M}$ and set $K := u\mathcal{M}$.

Lemma 5.7. *Assume that (X,G) is tame and $(\mathcal{M},G)$ is amenable. Then*

(1) $\overline{K} = \mathcal{M}$,

(2) $\zeta : \mathcal{M} \to K$, given by $\zeta(\eta) = u\eta$, is continuous with respect to the τ-topology on K.

Proof. Let $\mu \in \mathrm{Prob}_G(\mathcal{M})$. Suppose that $m_0 \in \mathcal{M} \setminus \overline{K}$. There exists $f \in C(\mathcal{M},[0,1])$ such that $f(m_0) = 1$ and $f|_{\overline{K}} \equiv 0$. In particular, $f(um) = 0$ for every $m \in \mathcal{M}$. On one hand, we have that $\int_{\mathcal{M}} f\, d\mu > 0$. On the other hand, by G-invariance of μ and Lemma 4.3, we see that

$$0 < \int_{\mathcal{M}} f(m)\, d\mu(m) = \int_{\mathcal{M}} f(m)\, d(u^{\#}\mu)(m) = \int_{\mathcal{M}} f(um)\, d\mu(m) = 0,$$

a contradiction. Hence (1). Then we obtain (2) by [KPR18, Lemma 3.1]. □

Remark 5.8. The amenability assumption in Lemma 5.7 is essential. For the tame flow studied in [GPP15], the corresponding minimal left ideal $\mathcal{M}$ and Ellis group $K = u\mathcal{M}$ satisfy $\overline{K} \subsetneq \mathcal{M}$. Consequently, $(\mathcal{M},G)$ is not amenable, since otherwise Lemma 5.7 would give $\overline{K} = \mathcal{M}$. By Theorem 5.9 below, the associated Haar decompression μ_K is therefore not G-invariant.

Theorem 5.9. *Assume that (X,G) is tame. The following are equivalent:*

(1) The Haar decompression μ_K is G-invariant,

(2) $(\mathcal{M},G)$ is amenable,

In particular, if $(\mathcal{M},G)$ is amenable, then μ_K is the unique element of $\mathrm{Prob}_G(\mathcal{M})$.

Proof. Assume (1). By Remark 3.10, we have that $\mathrm{supp}(\mu_K) \subseteq \mathcal{M}$. Hence (2).

Now, assume (2), and consider the function $\zeta : \mathcal{M} \to K$ given by $\zeta(\eta) = u\eta$. By Lemma 5.7 we have that $\overline{K} = \mathcal{M}$ and that ζ is continuous with respect to the τ-topology on K.

Let $\mu \in \mathrm{Prob}_G(\mathcal{M})$ and note that $\zeta_*\mu = \mathfrak{h}_K$. Indeed, $\mathcal{M} = E(X,G)u$ and so Gu is dense in $\mathcal{M}$, thus, we have that $\zeta[Gu] = \{ugu : g \in G\}$ is dense in K (with respect to the τ-topology). If $f \in C(K)$ then $f \circ \zeta \in C(\mathcal{M})$ and we apply Lemma 4.3 and G-invariance of μ to show that $\zeta_*\mu$ is $\zeta[Gu]$-invariant:

$$\begin{aligned}\int\limits_{k\in K} f(ugu\,k)\, d(\zeta_*\mu)(k) &= \int\limits_{m\in\mathcal{M}} f(ugu\,um)\, d\mu(m) = \int\limits_{m\in\mathcal{M}} (f\circ\zeta)(gu\,m)\, d\mu(m)\\ &= \int\limits_{m\in\mathcal{M}} (f\circ\zeta)(m)\, d\big((gu)^{\#}\mu\big)(m) = \int\limits_{m\in\mathcal{M}} (f\circ\zeta)(m)\, d\mu(m)\\ &= \int\limits_{k\in K} f(k)\, d(\zeta_*\mu)(k).\end{aligned}$$

Therefore, $\zeta_*\mu$ is a $\zeta[Gu]$-invariant regular Borel probability measure on the compact group K, and as such must be equal to its Haar measure $\mathfrak{h}_K$.

Now, we will obtain G-invariance of μ_K by showing that $\mu_K = \mu$. Let $f \in C(E)$. By Corollary 3.5, $f \circ j$ is measurable for the $\mathfrak{h}_K$-completion of the Borel σ-algebra on K. Hence, there are a bounded Borel function $\widetilde{f} : K \to \mathbb{C}$ and a Borel set $N \subseteq K$ such that $\mathfrak{h}_K(N) = 0$ and $\widetilde{f} = f \circ j$ on $K \setminus N$. Since $\zeta_* \mu = \mathfrak{h}_K$, we have $\mu(\zeta^{-1}[N]) = 0$, and therefore $\widetilde{f} \circ \zeta = f \circ j \circ \zeta$ μ-almost everywhere. Moreover, $u^{\#}\mu = \mu$, since the G-invariant measure μ is fixed by every element of $E(\mathrm{Prob}(\mathcal{M}), G)$. Consequently, using Lemma 4.3,

$$\begin{aligned}
\int_E f \, d\mu_K &= \int_K (f \circ j)(k) \, d\mathfrak{h}_K(k) = \int_K \widetilde{f}(k) \, d\mathfrak{h}_K(k) \\
&= \int_K \widetilde{f}(k) \, d(\zeta_* \mu)(k) \;\; = \int_{\mathcal{M}} \widetilde{f}(\zeta(m)) \, d\mu(m) \\
&= \int_{\mathcal{M}} f(um) \, d\mu(m) \;\; = \int_{\mathcal{M}} f(m) \, d(u^{\#}\mu)(m) \\
&= \int_{\mathcal{M}} f(m) \, d\mu(m).
\end{aligned}$$

Thus, regarding μ as a probability measure on E concentrated on $\mathcal{M}$, we obtain $\mu_K = \mu$. This ends the proof of (1).

If $\mu' \in \mathrm{Prob}_G(\mathcal{M})$ then, by repeating the proof above, we obtain that $\mu' = \mu_K = \mu$. Thus μ_K is the unique elment of $\mathrm{Prob}_G(\mathcal{M})$. Alternatively, uniqueness follows from Theorem 4.2. □

Remark 5.10.

(1) Under the equivalent conditions of Theorem 5.9, one has

$$\mu_K \in \mathrm{Erg}_G\big(E(X,G)\big) \qquad \text{and} \qquad \mathrm{supp}(\mu_K) = \mathcal{M} = \overline{K}.$$

Indeed, Lemma 5.7 gives $\overline{K} = \mathcal{M}$. The latter thing follows from Remark 3.10. Theorem 5.9 shows that μ_K is the unique invariant probability measure on $\mathcal{M}$. Any convex decomposition of μ_K into invariant probabilities on $E(X,G)$ is necessarily concentrated on $\mathcal{M}$, and uniqueness on $\mathcal{M}$ therefore makes the decomposition trivial. Thus, invariance of an individual Haar decompression already implies its ergodicity.

(2) More generally, for tame (X,G), Haar decompression leads to all ergodic measures on its Ellis flow:

$$\mathrm{Erg}_G(E) = \left\{ \mu_{u\mathcal{M}} : \mathcal{M} \trianglelefteq_m E, \; u^2 = u \in \mathcal{M}, \; (\mathcal{M}, G) \text{ is amenable} \right\}.$$

The right-to-left inclusion follows by the first point. Now, if we take $\mu \in \mathrm{Erg}_G(E(X,G))$ then Corollary 4.6 implies that $\mathrm{supp}(\mu) = \mathcal{M}$ for some minimal left ideal $\mathcal{M}$. Because $(\mathcal{M}, G)$ is amenable, we may use Theorem 5.9 and obtain that $\mu = \mu_K$ for some Ellis group K of $\mathcal{M}$.

(3) Let (X,G) be a tame hereditarily amenable flow and let K and K' be two Ellis groups in $E(X,G)$. Then $\mu_K = \mu_{K'}$ if and only if K and K' are contained in the same minimal left ideal in $E(X,G)$. This follows by the first point and the uniqueness from Theorem 5.9.

Corollary 5.11. *Let (X,G) be tame. The following conditions are equivalent:*

(i) (X,G) is hereditarily amenable;
(ii) $(E(X,G),G)$ is amenable;
(iii) every Haar decompression on $E(X,G)$ is G-invariant;
(iv) some Haar decompression on $E(X,G)$ is G-invariant.

Proof. By combining Corollary 5.6 with Theorem 5.9. □

6. Ergodic Ellis transfer and realization

The transfer and realization properties considered in this section depend on a distinguished point.

It is useful to distinguish two assertions. Let (X, x_0, G) be an ambit. The *ergodic Ellis-transfer property* says that the pushforward under ev_{x_0} of every Haar decompression from an Ellis group is G-invariant and ergodic (e.g.: Lemma 4.1, Corollary 6.1). The *ergodic Ellis-realization property* says that every ergodic G-invariant regular Borel probability measure on X is obtained from at least one Ellis group in this way (e.g.: Corollary 6.3). For tame ambits the decompression itself is always defined by Lemma 3.7. The main result of the present section establishes the realization property in the tame setting.

6.1. **Ergodic Ellis-transfer property.** We begin with the transfer problem. The invariance criterion from Subsection 5.3, together with preservation of ergodicity under equivariant factors, yields the desired conclusion. Let (X, G) be a flow, put $E := E(X, G)$, let $\mathcal{M} \trianglelefteq_m E$, choose an idempotent $u \in \mathcal{M}$, and put $K := u\mathcal{M}$.

Corollary 6.1. *Let (X, G) be tame and hereditarily amenable. Then $\mu_K \in \mathrm{Erg}_G(E)$ and, for every $x \in X$, $(\mathrm{ev}_x)_*\mu_K \in \mathrm{Erg}_G(X)$. In particular, every tame hereditarily amenable ambit has the ergodic Ellis-transfer property.*

Proof. Put $E := E(X, G)$. By Corollary 5.11, μ_K is G-invariant. Remark 5.10 therefore gives $\mu_K \in \mathrm{Erg}_G(E)$.

Now, fix $x \in X$. Since ev_x is G-equivariant, $(\mathrm{ev}_x)_*\mu_K$ is G-invariant. Moreover, the inverse image under ev_x of every G-invariant measurable subset of X is a G-invariant measurable subset of $E(X, G)$. Since μ_K is ergodic, such an inverse image has μ_K-measure either 0 or 1 (by [Phe01, Proposition 12.4]). Hence, the $(\mathrm{ev}_x)_*\mu_K$-measure of the original subset of X is either 0 or 1 as well, and so $(\mathrm{ev}_x)_*\mu_K$ is ergodic by Remark 4.5. □

6.2. **Ergodic Ellis-realization property.** We now address the complementary realization problem.

Lemma 6.2. *Let (X, G) be a tame flow, fix $x_0 \in X$, and let $\mu \in \mathrm{Erg}_G(X)$. Assume that $Y := \mathrm{supp}(\mu)$ is contained in $\overline{Gx_0}$. Put $I_Y := \mathrm{ev}_{x_0}^{-1}[Y] \subseteq E(X, G)$. Then, for every choice of*

$$\mathcal{M} \trianglelefteq_m E(X, G), \qquad \mathcal{M} \subseteq I_Y, \qquad u^2 = u \in \mathcal{M},$$

and $K := u\mathcal{M}$, one has $(\mathrm{ev}_{x_0})_*\mu_K = \mu$.

Proof. Let $y_0 := u(x_0) \in Y$. By Proposition 4.4, the flow (Y, G) is minimal; it is also tame, and it carries the invariant probability measure $\mu|_Y$. Restriction to Y defines a continuous semigroup epimorphism

$$r : E(X, G) \longrightarrow E(Y, G), \qquad r(\eta) := \eta|_Y.$$

Put

$$\mathcal{M}_Y := r[\mathcal{M}], \qquad v := r(u), \qquad K_Y := v\mathcal{M}_Y = r[K].$$

By Fact 2.1, $\mathcal{M}_Y$ is a minimal left ideal, v is idempotent, and

$$r|_K : (K, \tau) \longrightarrow (K_Y, \tau)$$

is a continuous group epimorphism satisfying $(r|_K)_*\mathfrak{h}_K = \mathfrak{h}_{K_Y}$.

By Corollary 6.1, the measure $(\mathrm{ev}_{y_0})_*\mu_{K_Y}$ is G-invariant. On the other hand, Theorem 4.2 shows that (Y,G) is uniquely ergodic. Hence, $(\mathrm{ev}_{y_0})_*\mu_{K_Y} = \mu|_Y$.

Fix $f \in C(X,\mathbb{C})$. Using $ku = k$ for $k \in K$, $y_0 = u(x_0)$, $(\mathrm{ev}_{y_0})_*\mu_{K_Y} = \mathfrak{h}_{K_Y}$ and $(r|_K)_*\mathfrak{h}_K = \mathfrak{h}_{K_Y}$, we obtain

$$\begin{aligned}\int_X f\, d\big((\mathrm{ev}_{x_0})_*\mu_K\big) &= \int_K f\big(k(x_0)\big)\, d\mathfrak{h}_K(k) = \int_K f\big(k(u(x_0))\big)\, d\mathfrak{h}_K(k)\\ &= \int_K f\big(r(k)(y_0)\big)\, d\mathfrak{h}_K(k) = \int_{K_Y} f\big(\ell(y_0)\big)\, d\mathfrak{h}_{K_Y}(\ell)\\ &= \int_Y f|_Y\, d(\mathrm{ev}_{y_0})_*\mu_{K_Y} = \int_X f\, d\mu.\end{aligned}$$

Since $f \in C(X,\mathbb{C})$ was arbitrary, $(\mathrm{ev}_{x_0})_*\mu_K = \mu$. ☐

Corollary 6.3. *Every tame ambit has the ergodic Ellis-realization property.*

Proof. Let (X, x_0, G) be a tame ambit, fix $\mu \in \mathrm{Erg}_G(X)$, and put

$$Y := \mathrm{supp}(\mu), \qquad I_Y := \mathrm{ev}_{x_0}^{-1}[Y].$$

Note that I_Y is a left ideal of $E(X,G)$.

Choose a minimal left ideal $\mathcal{M} \trianglelefteq_m E(X,G)$ contained in I_Y, an idempotent $u \in \mathcal{M}$, and put $K := u\mathcal{M}$. Since $Y \subseteq X = \overline{Gx_0}$, Lemma 6.2 gives $(\mathrm{ev}_{x_0})_*\mu_K = \mu$. ☐

For comparison, in the more restrictive one-map setting of tame $\mathbb{N}_0$-semicascades, Romanov [Rom19] realizes ergodic measures through generalized ergodic averages in the Köhler operator semigroup, a mechanism different from Haar decompression from Ellis groups.

6.3. **Summary and the Choquet theorem.** Let (X, x_0, G) be a tame hereditarily amenable ambit. Put $E := E(X,G)$ and

$$\mathcal{E}(X,G) := \{u\mathcal{M} : \mathcal{M} \trianglelefteq_m E,\ u^2 = u \in \mathcal{M}\}.$$

By Corollary 6.1, the map $\Pi : \mathcal{E}(X,G) \longrightarrow \mathrm{Erg}_G(X)$, given by $\Pi(K) := (\mathrm{ev}_{x_0})_*\mu_K$, is well defined, and Corollary 6.3 shows that it is surjective. Thus,

$$\{(\mathrm{ev}_{x_0})_*\mu_K : K \in \mathcal{E}(X,G)\} = \mathrm{Erg}_G(X).$$

Here hereditary amenability is used only to ensure that every Ellis group produces an invariant ergodic measure; the ergodic Ellis-realization property itself holds for every tame ambit.

We now assume additionally that X is metrizable and briefly recall the relevant Choquet-theoretic background (cf. [Phe01]). Since X is compact metrizable, the weak-$*$ compact convex set $\mathrm{Prob}_G(X)$ is a metrizable Choquet simplex, and its extreme boundary is precisely

$$\operatorname{ext} \mathrm{Prob}_G(X) = \mathrm{Erg}_G(X).$$

Moreover, $\mathrm{Erg}_G(X)$ is a G_δ-subset of $\mathrm{Prob}_G(X)$ (cf. [Alf66]). Thus, $\mathrm{Prob}_G(X)$ is a Bauer simplex precisely when $\mathrm{Erg}_G(X)$ is closed, whereas, in the nontrivial case, it is a Poulsen simplex precisely when $\mathrm{Erg}_G(X)$ is dense (cf. [LOS78]). In particular, every $\mu \in \mathrm{Prob}_G(X)$ has a unique representing probability measure λ_μ on $\mathrm{Erg}_G(X)$ [Phe01, Proposition 3.1 and Chapters 10 and 12].

Equip $\mathcal{E}(X,G)$ with the σ-algebra $\big\{\Pi^{-1}[B] : B \in \mathcal{B}(\mathrm{Erg}_G(X))\big\}$. Since Π is surjective, there is a unique probability measure $\widehat{\mu}$ on $\mathcal{E}(X,G)$ determined by $\Pi_*\widehat{\mu} = \lambda_\mu$.

Corollary 6.4. *Let (X, x_0, G) be a metrizable tame hereditarily amenable ambit. Then, for every $\mu \in \mathrm{Prob}_G(X)$,*

$$\mu = \int_{\mathcal{E}(X,G)} (\mathrm{ev}_{x_0})_* \mu_K \, d\widehat{\mu}(K)$$

in the weak-* sense.

Proof. By the Choquet representation theorem and its application to invariant measures [Phe01, Proposition 3.1 and Chapters 10 and 12], μ is the barycenter of its unique representing probability measure λ_μ on $\mathrm{Erg}_G(X)$. □

6.4. **A link to model theory.** We conclude by showing that the abstract transfer and realization mechanisms recover two model-theoretic classifications of ergodic measures. The first concerns definable groups in NIP theories and the measures of Chernikov and Simon [CS18]; the second concerns automorphism flows in amenable NIP theories and the measures of the first author and Rzepecki [HR25].

Remark 6.5. Evaluation of Haar decompressions recovers the description of ergodic measures by Chernikov and Simon in the context of definable groups in NIP theories from [CS18, Theorem 4.5].

Proof. Consider a definably amenable group G definable in an NIP theory T. Let $\mathfrak{C} \models T$ be a monster model, and consider the ambit $(X, x_0, G) := (S_G(\mathfrak{C}), \mathrm{tp}(e/\mathfrak{C}), G)$ (where $G = G(\mathfrak{C})$). Since by [CPS14] amenability transfers to the Shelah expansion, the flow $(S_G(\mathfrak{C}^{\mathrm{ext}}), G)$ is amenable, and hence $(E(S_G(\mathfrak{C}), G), G)$ is amenable as a factor of $(S_G(\mathfrak{C}^{\mathrm{ext}}), G)$. Thus, $(S_G(\mathfrak{C}), \mathrm{tp}(e/\mathfrak{C}), G)$ is hereditarily amenable by Corollary 5.6. This flow is also tame by the NIP assumption (see the introduction in [CS18]).

As observed above, Corollaries 6.1 and 6.3 yield the following description of all ergodic measures on X:

$$\mathrm{Erg}_G(X) = \{(\mathrm{ev}_{x_0})_* \mu_K : K \text{ is an Ellis group in } E(X, G)\}.$$

For an f-generic type $p \in S_G(\mathfrak{C})$, Definition 3.16 in [CS18] yields a G-invariant measure μ_p on X which is concentrated on $\overline{Gp}$ by [CS18, Remark 3.17]. Theorem 4.5 in [CS18] classifies the ergodic measures on X as all the measures μ_p for f-generic p. By the comment after [CS18, Proposition 3.31], instead of f-generics we can use here almost periodic types. Our goal is to show that this classification follows from our results. For that it remains to show that

$$\{\mu_p : p \in X \text{ almost periodic}\} = \{(\mathrm{ev}_{x_0})_* \mu_K : K \text{ an Ellis group in } E(X, G)\}.$$

For $(\subseteq)$ consider any almost periodic $p \in X$. Then there is a minimal left ideal $\mathcal{M}$ in $E(X, G)$ such that $p \in \mathrm{ev}_{x_0}[\mathcal{M}] = \mathcal{M}x_0 = \mathcal{M}p = \overline{Gp}$. Consider an arbitrary Ellis group K contained in $\mathcal{M}$. By Theorem 5.9, Remark 5.10(1), and Theorem 4.2, $(\mathrm{ev}_{x_0})_* \mu_K$ is the unique invariant measure concentrated on the tame minimal flow $\mathcal{M}x_0$. Since μ_p is an invariant measure concentrated on $\overline{Gp} = \mathcal{M}x_0$, we get $\mu_p = (\mathrm{ev}_{x_0})_* \mu_K$.

For $(\supseteq)$ start from an Ellis group K contained in some minimal left ideal $\mathcal{M}$. Take any $p \in \mathrm{ev}_{x_0}[\mathcal{M}]$. Then p is almost periodic, so the above argument yields $\mu_p = (\mathrm{ev}_{x_0})_* \mu_K$. □

Remark 6.6. The same mechanism recovers the conclusions of [HR25], i.e. the description of ergodic measures in the context of countable amenable NIP theory, where the acting group is the group of automorphisms.

Proof. Recall the notation of [HR25, Section 5]: let T be an amenable NIP theory, $M \preceq N \preceq \mathfrak{C}$, let $\bar{n}$ enumerate N, and put $X := S_{\bar{n}}(\mathfrak{C})$, $x_0 := \mathrm{tp}(\bar{n}/\mathfrak{C})$, $G := \mathrm{Aut}(\mathfrak{C})$, and let $I := \mathrm{fGen} \subseteq S^{\mathrm{inv}}_{\bar{n}}(\mathfrak{C}, M)$ be the collection of f-generic types. Because T is NIP, we have that the flow (X, G) is tame (e.g. by [KR20] or [CH23]). Because T is amenable, we have that the flow (X, G) is amenable (by [HKP26]). Every minimal G-subflow Z of X is uniquely ergodic with the unique G-invariant Keisler measure μ_p defined in [HR25, Definition 5.2] for $p \in Z \subseteq I$ (by [HR25, Theorem 5.4] and Theorem 4.2). This was established in [HR25] after assuming that T and M are countable (cf. [HR25, Corollary 5.14]). Moreover, if T and M are countable, then $\{\mu_p : p \in I\} = \mathrm{Erg}_G(X)$ by [HR25, Theorem 5.17]. Again, using our more general machinery, we can remove the aforementioned countability assumption of [HR25, Theorem 5.17] in exchange for working with almost periodic types: by Corollary 4.6 and [HR25, Theorem 5.4] we get

$$\mathrm{Erg}_G(X) = \{\mu_p : p \in X \text{ is almost periodic}\}.$$

Now, we realize measures μ_p of almost periodic types as Haar decompressions. Indeed, because flow (X, G) is tame and hereditarily amenable, we may use Corollaries 6.1 and 6.3 and conclude

$$\{\mu_p : p \in X \text{ almost periodic}\} = \{(\mathrm{ev}_{x_0})_* \mu_K : K \text{ an Ellis group in } E(X, G)\},$$

for every amenable NIP theory T (not necessarily countable). □

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In the final stages of manuscript preparation, the authors used Aristotle, developed by Harmonic, to mechanically verify selected statements in Lean 4. The authors take full responsibility for the mathematical content of the paper.

Appendix A. Examples

We collect three examples. The first shows that the inclusion of an Ellis group into the enveloping semigroup need not be Borel. The second gives an explicit split-circle dihedral illustration of Theorem 5.9: neither Ellis group is G-invariant as a subset of the enveloping semigroup, but their Haar decompressions coincide and are G-invariant. The third shows that tameness is essential in the characterization of amenability of the Ellis flow.

A.1. **Non-Borel inclusion of Ellis group.** Put $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, equipped with its usual circular order. For $A \subseteq \mathbb{T}$, let $\mathrm{Split}(\mathbb{T}; A)$ be the circularly ordered space obtained by replacing each $t \in A$ by two adjacent points $t^- < t^+$ and leaving every $t \notin A$ unsplit. We use the convention $t^- = t^+ = t$ at unsplit points, and equip $\mathrm{Split}(\mathbb{T}; A)$ with the cyclic-order topology, that is, the topology generated by the open cyclic intervals.

Choose $\alpha, \beta \in \mathbb{R}$ such that $1, \alpha, \beta$ are linearly independent over $\mathbb{Q}$. For $\xi \in \{\alpha, \beta\}$, put $A_\xi := \{n\xi : n \in \mathbb{Z}\} \subseteq \mathbb{T}$ and let $X_\xi := \mathrm{Split}(\mathbb{T}; A_\xi)$. Let $T_\xi(t^\rho) := (t+\xi)^\rho$, and equip $X := X_\alpha \times X_\beta$ with the diagonal $\mathbb{Z}$-action generated by $T_\alpha \times T_\beta$.

By [GM22, Proposition 4.1], this is a minimal metrizable tame system, and

$$\mathcal{M} := \left\{ \left(p^\sigma_{\alpha,\gamma}, p^\rho_{\beta,\eta}\right) : \sigma, \rho \in \{+,-\},\ \gamma, \eta \in \mathbb{T} \right\}$$

is a minimal $\mathbb{Z}$-subflow of $E(X,\mathbb{Z})$, hence a minimal left ideal. Here $p^\sigma_{\xi,\gamma}(t^\rho) := (t+\gamma)^\sigma$. The composition rule

$$p^\sigma_{\xi,\gamma} \circ p^\rho_{\xi,\eta} = p^\sigma_{\xi,\gamma+\eta}$$

shows that $u := \left(p^+_{\alpha,0}, p^+_{\beta,0}\right)$ is an idempotent and that

$$K := u\mathcal{M} = \left\{ \left(p^+_{\alpha,\gamma}, p^+_{\beta,\eta}\right) : (\gamma, \eta) \in \mathbb{T}^2 \right\}.$$

The natural factor map $X \to \mathbb{T}^2$ induces a semigroup epimorphism

$$\pi_E : E(X,\mathbb{Z}) \longrightarrow \mathbb{T}^2.$$

By Fact 2.1, its restriction $\pi_E|_K : (K, \tau) \longrightarrow \mathbb{T}^2$, given by $\left(p^+_{\alpha,\gamma}, p^+_{\beta,\eta}\right) \longmapsto (\gamma, \eta)$, is a continuous quotient group homomorphism. It is bijective, and therefore it is a homeomorphism.

On the other hand, [GM22, Proposition 4.1] shows that

$$\Delta := \left\{ \left(p^+_{\alpha,\gamma}, p^+_{\beta,-\gamma}\right) : \gamma \in \mathbb{T} \right\}$$

is discrete in the topology inherited from $E(X,\mathbb{Z})$. In the τ-topology it is a closed ordinary circle. Choose a non-Borel set $B \subseteq \mathbb{T}$ and put

$$\Delta_B := \left\{ \left(p^+_{\alpha,\gamma}, p^+_{\beta,-\gamma}\right) : \gamma \in B \right\}.$$

Since Δ is discrete in the inherited topology, there is an open set $O \subseteq E(X,\mathbb{Z})$ such that $O \cap \Delta = \Delta_B$. If the inclusion $j : (K,\tau) \to E(X,\mathbb{Z})$ were Borel, then $j^{-1}[O] \cap \Delta = \Delta_B$ would be Borel in the ordinary circle Δ, a contradiction.

A.2. **Haar decompression restores invariance.** This example exhibits two Ellis groups which are interchanged by the reflection, although their Haar decompressions coincide and are G-invariant.

The *fully split circle* is $X = \mathbb{T}^{\bowtie} := \mathrm{Split}(\mathbb{T}; \mathbb{T})$ (cf. Subsection A.1). Thus every $t \in \mathbb{T}$ is replaced by two adjacent points $t^- < t^+$. The arcs $[a^+, b^-]$, $a \neq b$, form a clopen basis, and the map $\pi : X \longrightarrow \mathbb{T}$, $\pi(t^\pm) := t$, is continuous and onto. For $\rho \in \{+,-\}$, set $\rho^1 := \rho$ and let ρ^{-1} be the opposite sign. For $\varepsilon \in \{\pm 1\}$ and $a \in \mathbb{T}$, define

$$P_{\varepsilon,a}(t^\rho) := (\varepsilon t + a)^{\rho^\varepsilon}.$$

Then $P_{\sigma,d} \circ P_{\varepsilon,a} = P_{\sigma\varepsilon, d+\sigma a}$. Fix an irrational $\alpha \in \mathbb{T}$, put $D := \langle \alpha \rangle$, and let

$$R := P_{1,\alpha}, \qquad J := P_{-1,0}, \qquad G := \langle R, J \rangle = \{P_{\varepsilon,d} : \varepsilon \in \{\pm 1\},\ d \in D\}.$$

Thus, $G \cong D_\infty$ is amenable, and (X, x_0, G) is an ambit for $x_0 := 0^+$. The flow is tame: the index-two subgroup $\langle R\rangle$ acts tamely by [GM18a, Theorem 4.5], and the G-orbit of every $f \in C(X)$ is the union of two translates of its $\langle R\rangle$-orbit.

For $\delta \in \{+,-\}$, $\varepsilon \in \{\pm 1\}$, and $a \in \mathbb{T}$, define the collapsed map

$$P^\delta_{\varepsilon,a}(t^\rho) := (\varepsilon t + a)^\delta, \qquad M_\delta := \{P^\delta_{\varepsilon,a} : \varepsilon \in \{\pm 1\},\ a \in \mathbb{T}\}.$$

The superscript records the side onto which every two-point fibre is collapsed. The product formulas are

$$P^\delta_{\varepsilon,a} \circ P^{\delta'}_{\varepsilon',b} = P^\delta_{\varepsilon\varepsilon',a+\varepsilon b}, \quad P_{\sigma,d} \circ P^\delta_{\varepsilon,a} = P^{\delta^\sigma}_{\sigma\varepsilon,d+\sigma a}, \quad P^\delta_{\varepsilon,a} \circ P_{\sigma,d} = P^\delta_{\varepsilon\sigma,a+\varepsilon d}.$$

Lemma A.1. *One has*

$$E(X,G) = G \,\dot\cup\, M_+ \,\dot\cup\, M_-.$$

Moreover, $M := M_+ \cup M_-$ is the unique minimal left ideal. For $\delta \in \{+,-\}$,

$$u_\delta := P^\delta_{1,0} \qquad \textit{and} \qquad K_\delta := u_\delta M = M_\delta.$$

Then u_δ is idempotent, and

$$(K_\delta, \tau) \longrightarrow O(2) \cong \{\pm 1\} \ltimes \mathbb{T}, \qquad P^\delta_{\varepsilon,a} \longmapsto (\varepsilon, a),$$

is an isomorphism of compact Hausdorff topological groups.

Proof. If $d_i \in D$ approaches a strictly from side δ, then

$$P_{\varepsilon,d_i} \longrightarrow P^\delta_{\varepsilon,a}$$

pointwise on X. Conversely, after passing to a subnet, every net of principal maps has constant orientation and its translation parameters are either eventually constant or approach their limit strictly from one side. This gives the classification and shows that, for each fixed $\varepsilon \in \{\pm 1\}$, the map

$$\mathbb{T}^{\bowtie} \longrightarrow E(X,G), \qquad a^\delta \longmapsto P^\delta_{\varepsilon,a},$$

is a homeomorphism onto its image.

The product formulas imply that M is a left ideal and that every nonempty left ideal contains M, and they give the asserted idempotents and Ellis groups. The factor map π induces a continuous semigroup epimorphism

$$\pi_E : E(X,G) \longrightarrow O(2)$$

which sends both $P_{\varepsilon,a}$ and $P^\delta_{\varepsilon,a}$ to (ε, a). Its restriction to K_δ is a continuous bijective homomorphism of compact Hausdorff groups, and hence is an isomorphism. $\square$

Lemma A.2. *For every $f \in C(X,\mathbb{C})$, the set*

$$J_f := \{a \in \mathbb{T} : f(a^+) \neq f(a^-)\}$$

is countable. The two sections

$$a \longmapsto a^+ \qquad \textit{and} \qquad a \longmapsto a^-$$

are Borel.

Proof. For $n \geqslant 1$, put

$$J_{f,n} := \left\{ a \in \mathbb{T} : |f(a^+) - f(a^-)| \geqslant \frac{1}{n} \right\}.$$

If $J_{f,n}$ were infinite, it would contain a sequence (a_i) of distinct points converging to some $a \in \mathbb{T}$. Passing to a subsequence, all a_i approach a from the same side. Then the pairs a_i^-, a_i^+ converge to the same one of a^-, a^+, contradicting continuity of f. Thus, each $J_{f,n}$ is finite, and $J_f = \bigcup_{n\geqslant 1} J_{f,n}$ is countable. The Borel assertion follows directly by taking preimages of the clopen basis arcs $[a^+, b^-]$. $\square$

Let $\mathfrak{h}_\delta$ be the normalized Haar measure on K_δ, and let μ_δ be its decompression. Let da denote normalized Haar measure on $\mathbb{T}$. Under the identification

$$K_\delta \cong \{\pm 1\} \ltimes \mathbb{T}, \qquad P^\delta_{\varepsilon,a} \longmapsto (\varepsilon, a),$$

the measure $\mathfrak{h}_\delta$ corresponds to $\frac{1}{2}\sum_{\varepsilon\in\{\pm1\}} \delta_\varepsilon \otimes da$.

Proposition A.3. *For every $F \in C(E(X,G),\mathbb{C})$,*

$$\int_{E(X,G)} F\, d\mu_\delta = \frac{1}{2}\sum_{\varepsilon\in\{\pm1\}} \int_{\mathbb{T}} F(P^\delta_{\varepsilon,a})\, da. \tag{A.1}$$

The two decompressions coincide, $\mu_+ = \mu_- =: \mu_E$, and μ_E is G-invariant.

The ambit (X, x_0, G) is uniquely ergodic. Its unique invariant probability measure ν_X satisfies

$$\int_X \varphi\, d\nu_X = \frac{1}{2}\int_{\mathbb{T}} \big(\varphi(a^+) + \varphi(a^-)\big)\, da \tag{A.2}$$

where $\varphi \in C(X,\mathbb{C})$, and we have $(\mathrm{ev}_{x_0})_*\mu_+ = (\mathrm{ev}_{x_0})_*\mu_- = \nu_X$.

Proof. The identification $K_\delta \cong O(2)$ gives (A.1). For fixed ε, the restriction of F to the double circle

$$\{P^\delta_{\varepsilon,a} : \delta \in \{+,-\},\ a \in \mathbb{T}\}$$

is continuous. Lemma A.2 therefore shows that $F(P^+_{\varepsilon,a}) = F(P^-_{\varepsilon,a})$ for Lebesgue-almost every a, and hence $\mu_+ = \mu_-$. The product formulas imply

$$(P_{\sigma,d} \circ -)_*\mu_\delta = \mu_{\delta^\sigma},$$

so the common decompression is G-invariant.

Since $P^\delta_{\varepsilon,a}(x_0) = a^\delta$, formula (A.1) and Lemma A.2 give the evaluation identity and (A.2).

Finally, let ν be any G-invariant probability measure on X. It is atomless because all R-orbits are infinite, and $\pi_*\nu$ is Lebesgue measure by unique ergodicity of the irrational rotation. For $\varphi \in C(X)$, put $\bar\varphi(a) := \varphi(a^+)$. The function $\bar\varphi$ is Borel, and

$$\varphi(x) = \bar\varphi(\pi(x))$$

outside the countable union of fibres over J_φ. Since ν is atomless, this exceptional set is ν-null. Thus,

$$\int_X \varphi\, d\nu = \int_{\mathbb{T}} \bar\varphi(a)\, da,$$

which is independent of ν. Hence, $\nu = \nu_X$. $\square$

We emphasize that the individual Ellis groups are not G-invariant: the rotation preserves each of them, whereas the reflection interchanges them,

$$R \circ K_\delta = K_\delta, \qquad J \circ K_\delta = K_{\delta^{-1}}.$$

Nevertheless, their common decompression $\mu_E = (j_+)_*\mathfrak{h}_+ = (j_-)_*\mathfrak{h}_-$ is G-invariant.

A.3. **Necessity of tameness.** The following example shows that hereditary amenability alone does not imply amenability of the Ellis flow, so the tameness hypothesis in Theorem 5.5 and Corollary 5.6 is indispensable.

Proposition A.4. *There is a metrizable hereditarily amenable flow (X, G) such that $(E(X, G), G)$ is not amenable.*

Proof. Let $H := F_2 = \langle a, b\rangle$ and $S := \{a, a^{-1}, b, b^{-1}\}$. We identify the boundary ∂H with the compact space of infinite reduced words in the alphabet S, equipped with the cylinder topology and the natural left H-action. For $s \in S$, let $[s] \subseteq \partial H$ denote the cylinder of words beginning with s.

Put $C_2 := \mathbb{Z}/2\mathbb{Z}$, $X := C_2^{H\times S}$, and let H act by

$$(h \cdot x)(q, s) := x(h^{-1}q, s).$$

Let

$$D := \bigoplus_{(q,s)\in H\times S} C_2 = \{d \in X : \operatorname{supp}(d) \text{ is finite}\}$$

be the dense subgroup of finitely supported functions, and put $G := D \rtimes H$. The affine action is

$$(d, h) \cdot x := d + h \cdot x.$$

Every D-orbit is dense in X, so the G-flow X is minimal. The normalized Haar measure on the compact group X is invariant under translations by D and under the coordinate permutations induced by H. Thus the G-flow X is amenable. Since it is minimal, its only nonempty subflow is X itself, and hence (X, G) is hereditarily amenable.

Define

$$\iota : \partial H \longrightarrow X, \qquad \iota(\xi)(q, s) := \mathbf{1}_{[s]}(q^{-1}\xi).$$

This map is continuous and H-equivariant. It is easy too see that ι is injective; since ∂H is compact and X is Hausdorff, it is an embedding. Put $Y := \iota[\partial H]$. The H-flow $Y \cong \partial H$ has no invariant probability measure. Indeed, for every $s \in S$,

$$s^{-1}[s] = \partial H \setminus [s^{-1}].$$

If λ were an invariant Borel probability measure, then $\lambda([s]) + \lambda([s^{-1}]) = 1$. Taking $s = a$ and $s = b$ would make the sum of the measures of the four first-letter cylinders equal to 2, although these cylinders partition ∂H.

Now, we give Y the G-action induced by the quotient $G \twoheadrightarrow H$, so that D acts trivially. Define

$$\delta : X^2 \longrightarrow X, \qquad \delta(x, y) := y - x,$$

and put $W := \delta^{-1}[Y]$. For $(d, h) \in G$,

$$\delta\big((d, h) \cdot x, (d, h) \cdot y\big) = h \cdot \delta(x, y).$$

Therefore, W is a G-subflow and $\delta|_W : W \longrightarrow Y$ is a continuous G-map. It is onto, since $\delta(0, y) = y$ for every $y \in Y$. Thus it is a factor map.

If W carried a G-invariant probability measure, its pushforward under $\delta|_W$ would be a G-invariant probability measure on Y, a contradiction. Hence, the second power X^2 contains a nonamenable subflow, so (X, G) is not finitely power-hereditarily amenable. By Theorem 5.3, the Ellis flow $(E(X, G), G)$ is not amenable. In particular, (X, G) is not tame by Corollary 5.6. $\square$

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