HAAR DECOMPRESSION AND AMENABILITY OF ELLIS FLOWS

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Abstract. Let (X, G) be a tame flow and let K be an Ellis group of its enveloping semigroup $E(X, G)$. Although K is a compact Hausdorff topological group in its τ-topology, the inclusion $K \hookrightarrow E(X, G)$ need not be Borel. We show that normalized Haar measure on K nevertheless determines, via the Riesz–Markov theorem, a canonical regular Borel probability measure μ_K on $E(X, G)$, called its Haar decompression.

Our principal structural result states that, for every tame flow,

$$\big(E(X, G), G\big) \text{ is amenable} \quad \Longleftrightarrow \quad (X, G) \text{ is hereditarily amenable.}$$

For a tame hereditarily amenable flow, every Haar decompression is G-invariant whenever X is metrizable, or whenever G is countable. For metrizable minimal tame flows admitting an invariant measure, the evaluation pushforward of every Haar decompression, at every point of the flow, is the unique invariant measure. Moreover, every ergodic invariant measure on a metrizable tame flow has minimal support; consequently, every ergodic invariant measure on a metrizable tame ambit is obtained by evaluating a suitable Haar decompression.

1. Introduction

Haar decompression. Let G be a discrete group. Throughout, a G-flow is a compact Hausdorff space equipped with an action of G by homeomorphisms, and an ambit is a pointed flow (X, x_0, G) such that $\overline{Gx_0} = X$. The construction of Haar decompressions and the amenability results of Section 5 concern arbitrary flows. A distinguished transitive point is used only in the evaluation and completeness results of Section 4.

The enveloping semigroup $E(X, G)$ is the pointwise closure of G in X^X. Its minimal left ideals $\mathcal{M} \trianglelefteq_m E(X, G)$ and idempotents $u^2 = u \in \mathcal{M}$ give rise to the Ellis groups $K = u\mathcal{M}$, equipped with the τ-topology of Ellis and Glasner [Aus88; Gla76]. We work with tame flows, whose enveloping semigroups are characterized by the fragmentedness of their elements; see [GMU08; Gla06; GM18b; GM22; KL07; Hua06; CH23].

For a tame flow, every Ellis group (K, τ) is a compact Hausdorff topological group (Lemma 3.3, using [Gla18; CGK26; BZ24]), and hence carries normalized Haar measure $\mathfrak{h}_K$. The difficulty is that this measure does not directly define a Borel measure on $E(X, G)$: the τ-topology may be strictly coarser than the subspace topology; the inclusion $j : K \hookrightarrow E(X, G)$ need not be Borel, which we we

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illustrate by an example in Appendix A.1 (see also Remark 3.8 and Question 3.9). In the model-theoretic context of a group G definable in a structure M and acting on the space $S_{G,\mathrm{ext}}(M)$ of complete external types over M concentrated on G so that the resulting Ellis group is Hausdorff, it was proven in [CGK26, Theorem 1.7] that the function j is always Borel. This reveals an essential difference between this model-theoretic and the general abstract context. Here, we prove that, for every $F \in C(E(X,G),\mathbb{C})$, the restriction $F \circ j$ is $\mathfrak{h}_K$-measurable. Consequently,

$$F \longmapsto \int_K F(j(k))\, d\mathfrak{h}_K(k)$$

defines a state on $C(E(X,G),\mathbb{C})$ and hence, by the Riesz–Markov theorem, a unique regular Borel probability measure μ_K on $E(X,G)$. We call μ_K the *Haar decompression* of $\mathfrak{h}_K$ (Lemma 3.7). This construction requires neither metrizability nor amenability.

The paper studies two consequences of this construction: whether μ_K is invariant under the left action by G, and which ergodic measures on an ambit (X, x_0, G) arise as evaluations $(\mathrm{ev}_{x_0})_*\mu_K$. Here, *ergodic* measure is understood as an extreme point in the space of invariant regular Borel probability measures.

Main results. The principal result concerns amenability of the enveloping flow. A G-flow is *amenable* if it carries a G-invariant regular Borel probability measure. It is *hereditarily amenable* if every nonempty subflow of X is amenable.

Theorem (Corollary 5.6). Let (X,G) be tame. Then

$$\big(E(X,G),G\big) \text{ is amenable} \quad \Longleftrightarrow \quad (X,G) \text{ is hereditarily amenable.}$$

Theorem 5.3, which does not require tameness, first identifies amenability of the Ellis flow with hereditary amenability of all finite powers of X. Theorem 5.5 then shows that, for tame flows, ordinary hereditary amenability already implies this finite-power condition. Their combination yields Corollary 5.6. Under an additional separability hypothesis, the same finite-coordinate analysis gives the stronger conclusion that every Haar decompression is invariant.

The tameness hypothesis is essential: Appendix A.3 gives a metrizable hereditarily amenable flow (X,G) for which the Ellis flow $(E(X,G),G)$ is not amenable.

Theorem (Theorems 5.7 and 5.8). Let (X,G) be tame and hereditarily amenable. If X is metrizable, or G is countable, or, more generally, if every countable subset of $C(X,\mathbb{C})$ is contained in a separable unital G-invariant C^*-subalgebra, then every Haar decompression μ_K is G-invariant.

We also consider evaluation of decompressions. By the structure theorem for minimal tame systems [Gla07; Gla18] and the regularity theorem of [Fuh+21], a metrizable minimal tame flow admitting an invariant measure is uniquely ergodic and is an almost one-to-one extension of its maximal equicontinuous factor. We prove that the Haar decompression realizes this unique measure directly from every Ellis group.

Theorem (Theorem 4.3). Let (X,G) be a metrizable minimal tame flow admitting an invariant measure. Then, for every $x_0 \in X$ and every Ellis group K of $E(X,G)$, the measure

$$(\mathrm{ev}_{x_0})_*\mu_K$$

is the unique G-invariant regular Borel probability measure on X.

The following support theorem makes the localization argument complete in the metrizable tame setting.

Theorem (Proposition 4.4 and Corollary 4.6). Let (X, G) be a metrizable tame flow. Then every ergodic G-invariant regular Borel probability measure on X has minimal support (meaning that the support is a minimal subflow). Consequently, if (X, x_0, G) is an ambit, then every ergodic G-invariant regular Borel probability measure μ on X is of the form

$$\mu = (\mathrm{ev}_{x_0})_* \mu_K$$

for some Ellis group K of $E(X, G)$.

Thus, in the metrizable tame ambit setting, evaluation of Haar decompressions is surjective onto the set of ergodic invariant probability measures.

Examples in the Appendix. The appendix collects three examples clarifying the scope of the main results. Appendix A.1 shows that the Ellis-group inclusion $j : (K, \tau) \to E(X, G)$ need not be Borel, even for a minimal metrizable tame flow. Appendix A.2 computes a split-circle dihedral flow in which two Ellis groups are interchanged by the action but their Haar decompressions coincide and are invariant. Finally, Appendix A.3 shows that tameness is essential in the hereditary-amenability characterization of the Ellis flow.

Model-theoretic motivation. The construction is motivated by model-theoretic dynamics. Enveloping semigroups of type-space flows have played a central role since the work of Newelski [New09; New12]. Model-theoretic tameness (NIP) corresponds to dynamical tameness [Sim15; Iba16; KR20; CH23], and Ellis groups are closely related to canonical compact quotients such as G/G^{00} and the Kim–Pillay Galois group; see [KP17; CS18; KNS19; KP19; KPR18; HKP26]. In [HR25], jointly with Rzepecki, following ideas of [CS18], the first author described ergodic measures of $(S_{\bar{m}}(M), \mathrm{Aut}(M))$ for countable NIP theories in terms of Haar measure on the Kim–Pillay Galois group. The present paper isolates a similar, purely dynamical mechanism: Haar measure on an Ellis group is first regularized on the enveloping semigroup and then transferred by evaluation.

The idea of inducing a measure on the enveloping semigroup from the Haar measure on an Ellis group appeared already in [CGK26, Section 4] in the context of definable groups, but the situation there did not require regularization which we need in the abstract context.

Organization of the paper. Section 2 fixes the conventions and recalls the required facts about Ellis groups, the τ-topology, and tame flows. Section 3 constructs Haar decompressions and establishes their basic measurability and support properties. Section 4 studies ergodic transfer, proves that ergodic measures on metrizable tame flows have minimal support, and establishes complete Ellis transfer for every metrizable tame ambit. Section 5 proves the finite-power criterion for amenability of the Ellis flow, derives the hereditary-amenability characterization for tame flows, and establishes invariance of Haar decompressions under the stated metrizability of X / countability of G hypotheses.

Appendix A contains the non-Borel inclusion example, the split-circle-dihedral computation, and the counterexample showing that tameness in Theorem 5.5 / Corollary 5.6 is essential.

2. Preliminaries

Throughout, G denotes a discrete group. All compact spaces are assumed to be Hausdorff, all probability measures are regular Borel measures.

2.1. **Flows and invariant measures.** A *G-flow* is a compact space X equipped with an action of G by homeomorphisms. A *subflow* is a nonempty closed G-invariant subset, and a *factor map* is a continuous surjective G-map. A pointed flow (X, x_0, G) is an *ambit* if $\overline{Gx_0} = X$, and a flow is *minimal* if every orbit is dense.

For a compact space X, let $C(X) := C(X, \mathbb{C})$, endowed with the uniform norm and the involution $f^*(x) := \overline{f(x)}$. Its positive cone is

$$\begin{aligned} C(X)_+ &:= \{h^*h : h \in C(X)\} \\ &= \{f \in C(X) : f = f^* \text{ and } f(x) \geqslant 0 \text{ for every } x \in X\}. \end{aligned}$$

A *state* on a unital C^*-algebra A is a complex-linear functional $\Lambda : A \to \mathbb{C}$ such that $\Lambda(a^*a) \geqslant 0$ for every $a \in A$ and $\Lambda(1_A) = 1$.

We write $\mathrm{Prob}(X)$ for the compact convex space of regular Borel probability measures on X, equipped with the weak-$*$ topology inherited from $C(X)^*$. If $\pi : X \to Y$ is Borel and $\mu \in \mathrm{Prob}(X)$, then $\pi_*\mu$ denotes the pushforward of μ. For $g \in G$, we write $g_*\mu$ for the pushforward under $x \mapsto gx$, and put

$$\mathrm{Prob}_G(X) := \{\mu \in \mathrm{Prob}(X) : g_*\mu = \mu \text{ for every } g \in G\}.$$

An invariant measure is *ergodic* if it is an extreme point of $\mathrm{Prob}_G(X)$, and we write

$$\mathrm{Erg}_G(X) := \mathrm{ext}\big(\mathrm{Prob}_G(X)\big).$$

The flow (X, G) is *amenable* if $\mathrm{Prob}_G(X) \neq \varnothing$, and it is *uniquely ergodic* if $\mathrm{Prob}_G(X)$ is a singleton. It is *hereditarily amenable* if every subflow of X is amenable. The action of G on $C(X)$ is given by $(g \cdot f)(x) := f(g^{-1}x)$.

2.2. **Ellis semigroups, fragmented maps and tameness.** The *enveloping semigroup* of a flow (X, G) is

$$E(X, G) := \overline{\{g_X : g \in G\}}^{X^X}, \qquad g_X(x) := gx,$$

where X^X carries the product topology. Multiplication is composition: $(pq)(x) := p(q(x))$. For every fixed $q \in E(X, G)$, the map $p \mapsto pq$ is continuous. For $x \in X$, evaluation is the continuous map

$$\mathrm{ev}_x : E(X, G) \longrightarrow X, \qquad \mathrm{ev}_x(p) := p(x),$$

and $\mathrm{ev}_x[E(X, G)] = \overline{Gx}$. A factor map $\pi : X \to Y$ induces a continuous surjective G-equivariant semigroup homomorphism

$$\pi_E : E(X, G) \longrightarrow E(Y, G)$$

satisfying $\pi \circ p = \pi_E(p) \circ \pi$ for every $p \in E(X, G)$.

Put $E := E(X, G)$. We write $\mathcal{M} \trianglelefteq_m E$ for a minimal left ideal. If $u^2 = u \in \mathcal{M}$, then $K := u\mathcal{M}$ is the corresponding *Ellis group*. For $p \in E$ and $A \subseteq E$, define the Ellis circle operation by

$$\begin{aligned} p \circ A := \big\{q \in E : {} & \text{there are nets } (g_i)_i \subseteq G \text{ and } (a_i)_i \subseteq A \\ & \text{such that } g_i \to p \text{ and } g_i a_i \to q\big\}. \end{aligned}$$

For $A \subseteq K$, put

$$\mathrm{cl}_\tau(A) := K \cap (u \circ A) = u(u \circ A).$$

The operator cl_τ is a closure operator on K. The topology determined by this closure operator is called the *τ-topology.* The τ-topology is coarser than the topology on K inherited from E. Moreover, (K,τ) is always a quasi-compact T_1 semitopological group, that is, multiplication is separately continuous. No Hausdorffness is asserted here. See [Gla76] for the classical construction in the βG setting, and [Rze18, Appendix A] for the general formulation used here. Whenever (K,τ) is a compact Hausdorff topological group, $\mathfrak{h}_K$ denotes its normalized Haar measure.

Fact 2.1. *Let (X,G) and (Y,G) be flows, and let $\Phi : E(X,G) \longrightarrow E(Y,G)$ be a continuous surjective G-equivariant semigroup homomorphism. Let $\mathcal{M} \trianglelefteq_m E(X,G)$, $u^2 = u \in \mathcal{M}$, and put $\mathcal{N} := \Phi[\mathcal{M}]$, $v := \Phi(u)$, $K := u\mathcal{M}$, $L := v\mathcal{N} = \Phi[K]$. Then $\mathcal{N} \trianglelefteq_m E(Y,G)$, $v^2 = v \in \mathcal{N}$, and the restriction*

$$\Phi|_K : (K,\tau) \longrightarrow (L,\tau)$$

is a continuous group epimorphism and a quotient map for the τ-topologies. (In particular, the conclusion applies when $\Phi = \pi_E$ is induced by a factor map $\pi : X \to Y$, and when $Y \subseteq X$ is a subflow and $\Phi : E(X,G) \to E(Y,G)$ is the restriction map.) If (K,τ) and (L,τ) are compact Hausdorff topological groups, then $(\Phi|_K)_*\mathfrak{h}_K = \mathfrak{h}_L$.

Proof. The assertions concerning minimal left ideals, idempotents, Ellis groups, and the quotient property are [KLM22, Facts 2.3 and 2.5]. The last assertion follows from the uniqueness of normalized Haar measure on a compact Hausdorff topological group. □

For $\eta \in E(X,G)$ and $\bar{x} = (x_1, \dots, x_n) \in X^n$, we write

$$\eta(\bar{x}) := \big(\eta(x_1), \dots, \eta(x_n)\big).$$

Let X be a compact space and let Y be a uniform space. A map $f : X \to Y$ is *fragmented* if, for every nonempty closed set $A \subseteq X$ and every entourage V of Y, there is a nonempty relatively open set $U \subseteq A$ such that

$$(f(x), f(y)) \in V$$

for all $x, y \in U$. When Y is metrizable, this is equivalent to requiring that, for every $\varepsilon > 0$ and every nonempty closed $A \subseteq X$, there is a nonempty relatively open $U \subseteq A$ with

$$\operatorname{diam} f[U] < \varepsilon.$$

In particular, a bounded function $f : X \to \mathbb{C}$ is fragmented if and only if both $\operatorname{Re} f$ and $\operatorname{Im} f$ are fragmented. A flow (X,G) is *tame* if every element of $E(X,G)$ is a fragmented self-map of X. This is one of the standard equivalent definitions of tameness; see [GM18b, Definition 3.1]. We use this characterization throughout.

We equip every compact Hausdorff space with its unique compatible uniformity. A flow (X,G) is *equicontinuous* if the family

$$\{g_X : g \in G\} \subseteq X^X$$

is equicontinuous with respect to this uniformity.

Lemma 2.2. *Let (X,G) be a flow.*

1. *If (X,G) is equicontinuous, then, for every cardinal κ, the flow X^κ, equipped with the diagonal G-action, is equicontinuous.*

(2) If (X, G) is tame, then, for every cardinal κ, the flow X^κ, equipped with the diagonal G-action, is tame.

(3) If (X, G) is tame, then its enveloping semigroup $E(X, G)$, equipped with the natural left G-action, is tame.

Consequently, if X is metrizable and (X, G) is tame, then every subflow of every finite power X^n is metrizable and tame.

Proof. (1) Let V be an entourage of X^κ. By the definition of the product uniformity, there are a finite set $F \subseteq \kappa$ and entourages V_α of X, for $\alpha \in F$, such that

$$(x_\alpha, y_\alpha) \in V_\alpha \text{ for every } \alpha \in F \quad \Longrightarrow \quad (x, y) \in V.$$

For every $\alpha \in F$, equicontinuity of (X, G) gives an entourage U_α of X such that

$$(s, t) \in U_\alpha \quad \Longrightarrow \quad (gs, gt) \in V_\alpha \text{ for every } g \in G.$$

The basic entourage of X^κ determined by the U_α, $\alpha \in F$, witnesses equicontinuity of the diagonal action on X^κ.

For (2) and (3), see [GM12, Lemma 5.4 and the sentence immediately following its proof]. $\square$

For further background on tame flows and Ellis semigroups, see [Aus88; Gla76; Gla06; GM18b; CH23].

3. Haar decompression on the Ellis semigroup

We first establish the measurability input needed to define Haar decompression. The argument combines a descent property for fragmented functions with the measurability of bounded fragmented functions for the completion of every regular Borel probability measure.

Lemma 3.1. *Let $\pi : L \to K$ be a continuous surjection of compact Hausdorff spaces. Let $f : K \to \mathbb{C}$ be bounded. If $f \circ \pi : L \to \mathbb{C}$ is fragmented, then f is fragmented.*

Proof. This is a standard fact (see, for example, [Sim15, Lemma 2.4]). Suppose that f is not fragmented. Then there are $\varepsilon > 0$ and a nonempty closed set $F \subseteq K$ such that $\operatorname{diam} f[U] \geqslant \varepsilon$ for every nonempty relatively open subset $U \subseteq F$.

By Zorn's lemma choose a minimal closed set $L_0 \subseteq \pi^{-1}(F)$ such that $\pi[L_0] = F$: the existence follows because if $(L_i)_{i \in I}$ is a decreasing chain of closed subsets of $\pi^{-1}(F)$ mapping onto F, then for each $x \in F$ the compact sets $L_i \cap \pi^{-1}(\{x\})$ have the finite intersection property, so $\bigcap_{i \in I} L_i$ still maps onto F. Since $f \circ \pi$ is fragmented on L, and hence also on the closed subspace L_0, there is a nonempty relatively open set $V \subseteq L_0$ such that $\operatorname{diam}(f \circ \pi)[V] < \varepsilon$. If $L_0 \setminus V$ still mapped onto F, this would contradict the minimality of L_0. Hence,

$$U := F \setminus \pi[L_0 \setminus V]$$

is nonempty. Moreover, since $L_0 \setminus V$ is compact, $\pi[L_0 \setminus V]$ is closed in F, so U is relatively open in F.

For every $x \in U$, $L_0 \cap \pi^{-1}(\{x\}) \subseteq V$. Therefore, $f[U] \subseteq (f \circ \pi)[V]$. Consequently,

$$\operatorname{diam} f[U] \leqslant \operatorname{diam}(f \circ \pi)[V] < \varepsilon,$$

contradicting the choice of F. Thus, f is fragmented. $\square$

By [Luk+10, Theorem A.121], bounded fragmented functions on compact spaces are $\Sigma_1(H_s)$-measurable, where H_s denotes the family of resolvable sets. By [Luk+10, Proposition A.118], resolvable sets are universally measurable. Thus, the following fact is standard. For the convenience of the reader, we include a self-contained proof.

Fact 3.2. *Let K be a compact Hausdorff space and let $f : K \to \mathbb{C}$ be bounded and fragmented. Then, for every regular Borel probability measure λ on K, there are a Borel function $f_\lambda : K \to \mathbb{C}$ and a Borel set $N_\lambda \subseteq K$ with $\lambda(N_\lambda) = 0$ such that*

$$f = f_\lambda \quad \text{on } K \setminus N_\lambda.$$

In particular, f is λ-measurable (i.e., measurable for the λ-completion of the Borel σ-algebra on K). Consequently, if K is a compact Hausdorff group, then f is measurable for the completion of the Borel σ-algebra on K with respect to the normalized Haar measure on K.

Proof. For every $\varepsilon > 0$ and every nonempty closed subset $F \subseteq K$, there is a nonempty relatively open subset $V \subseteq F$ such that $\operatorname{diam} f[V] < \varepsilon$. Fix $\varepsilon > 0$. We construct, by transfinite recursion, a decreasing family of closed sets $(F_\gamma)_\gamma$ and disjoint Borel pieces $(P_\gamma)_\gamma$. Put $F_0 := K$. If $F_\gamma \neq \varnothing$, choose an open set $O_\gamma \subseteq K$ such that

$$P_\gamma := F_\gamma \cap O_\gamma \neq \varnothing \quad \text{and} \quad \operatorname{diam} f[P_\gamma] < \varepsilon,$$

and set

$$F_{\gamma+1} := F_\gamma \setminus P_\gamma = F_\gamma \cap (K \setminus O_\gamma).$$

At limit ordinals δ, put $F_\delta := \bigcap_{\gamma<\delta} F_\gamma$. Since the closed subsets of K form a set, this strictly decreasing process stops: there is an ordinal θ such that $F_\theta = \varnothing$. Hence,

$$K = \bigsqcup_{\gamma<\theta} P_\gamma.$$

Indeed, if $x \in K$, then the least ordinal γ such that $x \notin F_\gamma$ cannot be a limit ordinal, and therefore $\gamma = \delta + 1$ with $x \in P_\delta$. Each P_γ is Borel, being the intersection of a closed set and an open set.

Now, fix a regular Borel probability measure λ on K. We first recall the following continuity property for regular Borel probability measures on compact spaces: if $(C_i)_{i\in I}$ is a downward directed family of closed subsets of K, then

$$\lambda\Big(\bigcap_{i\in I} C_i\Big) = \inf_{i\in I} \lambda(C_i).$$

For $\gamma \leqslant \theta$, put

$$s_\gamma := \sum_{\delta<\gamma} \lambda(P_\delta),$$

where the sum means the supremum of all finite subsums. Since the sets P_δ are pairwise disjoint and $\lambda(K) = 1$, only countably many summands are positive, so this agrees with the usual countable sum over those positive summands. We claim that $\lambda(F_\gamma) = 1 - s_\gamma$, for all $\gamma \leqslant \theta$. This is proved by transfinite induction. The case $\gamma = 0$ is immediate. For the successor step, use $F_\gamma = F_{\gamma+1} \sqcup P_\gamma$, thus

$$\lambda(F_{\gamma+1}) = \lambda(F_\gamma) - \lambda(P_\gamma) = 1 - s_\gamma - \lambda(P_\gamma) = 1 - s_{\gamma+1}.$$

At a limit ordinal δ, the closed sets $(F_\gamma)_{\gamma<\delta}$ are downward directed and $F_\delta = \bigcap_{\gamma<\delta} F_\gamma$. By the continuity property given above,

$$\lambda(F_\delta) = \lambda\Big(\bigcap_{\gamma<\delta} F_\gamma\Big) = \inf_{\gamma<\delta} \lambda(F_\gamma) = \inf_{\gamma<\delta}(1 - s_\gamma) = 1 - \sup_{\gamma<\delta} s_\gamma = 1 - s_\delta.$$

Since $F_\theta = \varnothing$, we obtain $\sum_{\gamma<\theta} \lambda(P_\gamma) = 1$.

Let

$$\Gamma_\varepsilon = \{\gamma < \theta : \lambda(P_\gamma) > 0\}.$$

This set is countable: for each $n \geqslant 1$, there are only finitely many γ with $\lambda(P_\gamma) \geqslant 1/n$. Put

$$B_\varepsilon := \bigcup_{\gamma\in\Gamma_\varepsilon} P_\gamma.$$

Then B_ε is Borel and $\lambda(B_\varepsilon) = 1$. For each $\gamma \in \Gamma_\varepsilon$, choose a point $x_\gamma \in P_\gamma$, and set

$$c_\gamma := f(x_\gamma).$$

Define a Borel step function $g_\varepsilon : K \to \mathbb{C}$ by

$$g_\varepsilon = \sum_{\gamma\in\Gamma_\varepsilon} c_\gamma \, \mathbf{1}_{P_\gamma},$$

with value 0 off B_ε. Since the sets P_γ are pairwise disjoint and Γ_ε is countable, g_ε is Borel. For $x \in B_\varepsilon$, say $x \in P_\gamma$, we have

$$|f(x) - g_\varepsilon(x)| = |f(x) - c_\gamma| < \varepsilon,$$

because $\operatorname{diam} f[P_\gamma] < \varepsilon$.

Apply the construction with $\varepsilon = 1/n$. Put $N := \bigcup_{n=1}^{\infty}(K \setminus B_{1/n})$. Then N is Borel and $\lambda(N) = 0$. Let

$$h_n := g_{1/n} \cdot \mathbf{1}_{K\setminus N}.$$

The functions h_n are Borel, and for every $x \in K \setminus N$,

$$|h_n(x) - f(x)| < \frac{1}{n}.$$

Thus, $h_n \to f$ pointwise on $K \setminus N$, while $h_n = 0$ on N for all n. Consequently,

$$f_\lambda := \lim_{n\to\infty} h_n$$

is Borel and agrees with f on $K\setminus N$. Therefore, f is measurable for the λ-completion of the Borel σ-algebra. □

Let (X, G) be a tame flow, $u^2 = u \in \mathcal{M} \trianglelefteq E := E(X, G)$, $K := u\mathcal{M}$, let $j : K \to E$ be the inclusion map.

Lemma 3.3. *K is a compact Hausdorff group in the τ-topology.*

Proof. By Lemma 2.2(3), the flow $(E(X, G), G)$ is tame, and hence its minimal subflow $(\mathcal{M}, G)$ is tame. By [CGK26, Lemma 5.16], (K, τ) is isomorphic as a semitopological group to an Ellis group K' of $E(\mathcal{M}, G)$. Since $(\mathcal{M}, G)$ is minimal and tame, [BZ24, Theorem 11.7] implies that K' is Hausdorff, and therefore so is K.

By the basic properties of the τ-topology recalled in Section 2, (K, τ) is a quasi-compact T_1 semitopological group. By [BZ24, Theorem 11.7], it is Hausdorff. Hence, it is a compact Hausdorff semitopological group, and the Ellis joint continuity theorem implies that it is a topological group. □

Lemma 3.4. *If $F \in C(E, \mathbb{C})$, then $F \circ j : K \to \mathbb{C}$ is a fragmented map with respect to the τ-topology on K.*

Proof. Consider the map $\lambda_u : E \to E$ given by $\lambda_u(\eta) := u\eta$. By Lemma 2.2(3), the flow $(E(X,G), G)$ is tame. Moreover, $\lambda_u \in E(E(X,G), G)$. Therefore, $\lambda_u : E \to E$ is fragmented. In particular, its restriction $\lambda_u|_{\overline{K}} : \overline{K} \to \overline{K}$ is a fragmented map, and thus $F \circ \lambda_u|_{\overline{K}}$ is fragmented as well.

Because (X, G) is tame, we have that (K, τ) is a compact Hausdorff topological group by Lemma 3.3. By Lemma 4.1 from [KPR18], the map $\lambda_u : \overline{K} \to K$ is a continuous surjection (from the induced topology to τ-topology).

Now, the composition $(F \circ j) \circ \lambda_u|_{\overline{K}}$ is equal to $F \circ \lambda_u|_{\overline{K}}$, hence it is fragmented. We apply Lemma 3.1, to get the conclusion. □

Let $\mathfrak{h}_K$ be the normalized Haar measure on the compact Hausdorff group K.

Corollary 3.5. *If $F \in C(E, \mathbb{C})$ then $F \circ j : K \to \mathbb{C}$ is $\mathfrak{h}_K$-measurable.*

Proof. By Lemma 3.4, we have that $F \circ j : K \to \mathbb{C}$ is fragmented. The conclusion follows by Fact 3.2. □

Fact 3.6 (Riesz–Markov representation)**.** *Let Y be a compact Hausdorff space. If*

$$\Lambda : C(Y, \mathbb{C}) \to \mathbb{C}$$

is a positive unital complex-linear functional, then there is a unique regular Borel probability measure μ_Λ on Y such that

$$\Lambda(F) = \int_Y F \, d\mu_\Lambda \quad \text{for every } F \in C(Y, \mathbb{C}).$$

Lemma 3.7. *The formula*

$$\Lambda_K(F) := \int_K F(j(k)) \, d\mathfrak{h}_K(k) \qquad (F \in C(E, \mathbb{C}))$$

defines a positive unital complex-linear functional on $C(E, \mathbb{C})$. Consequently, by Fact 3.6, there is a unique regular Borel probability measure μ_K on E such that

$$\int_E F \, d\mu_K = \int_K F(j(k)) \, d\mathfrak{h}_K(k) \quad \text{for every } F \in C(E, \mathbb{C}).$$

We call μ_K the decompression of $\mathfrak{h}_K$.

Proof. The integral is well defined by Corollary 3.5. Complex linearity follows immediately from linearity of the integral. Moreover,

$$\Lambda_K(1_E) = \int_K 1 \, d\mathfrak{h}_K = 1.$$

If $F \in C(E)_+$, then $F(j(k)) \geqslant 0$ for every $k \in K$. Therefore, $F \circ j$ is real-valued and non-negative, and

$$\Lambda_K(F) = \int_K F(j(k)) \, d\mathfrak{h}_K(k) \geqslant 0.$$

Thus, Λ_K is positive. The Riesz–Markov representation theorem now yields the unique regular Borel probability measure μ_K with the displayed property. □

Remark 3.8. The measure μ_K is not defined as a raw pushforward. Although $F \circ j$ is $\mathfrak{h}_K$-measurable for every $F \in C(E)$, the inclusion $j : (K, \tau) \to E$ need not be Borel; see Appendix A.1. It is not known whether j is always measurable from the $\mathfrak{h}_K$-completion of the Borel σ-algebra on K to the Borel σ-algebra on E. Even when this measurability holds, it is not known whether the Borel pushforward $j_*\mathfrak{h}_K$ must be regular. The Riesz–Markov construction instead produces a canonical regular Borel probability measure on E. If j is measurable in the above sense and $j_*\mathfrak{h}_K$ is regular, then

$$\mu_K = j_*\mathfrak{h}_K,$$

since the two measures have the same integrals against every function in $C(E)$.

Question 3.9.
(1) Does there exist a tame flow (X, G) for which the inclusion $j : (K, \tau) \longrightarrow E(X, G)$ is not measurable with respect to the $\mathfrak{h}_K$-completion?
(2) Suppose that j is measurable for the $\mathfrak{h}_K$-completion. Must the Borel pushforward $j_*\mathfrak{h}_K$ be regular?

Remark 3.10.
(1) $\mathrm{supp}(\mu_K) \subseteq \overline{u\mathcal{M}} \subseteq \mathcal{M}$.
(2) If μ_K is G-invariant, then $\overline{u\mathcal{M}} = \mathcal{M}$.
(3) For every $x \in X$, $\mathrm{supp}((\mathrm{ev}_x)_*\mu_K) \subseteq \overline{u\mathcal{M}}\,x \subseteq \mathcal{M}x$.
(4) If $(\mathrm{ev}_x)_*\mu_K$ is G-invariant, then $\mathrm{supp}((\mathrm{ev}_x)_*\mu_K) = \mathcal{M}x$ is a minimal flow.

Proof. (1) If not, then there is $F \in C(E)_+$ and $\epsilon > 0$ such that $\mu_K(\{\eta \in E : F(\eta) > \epsilon\}) > 0$ and $F|_{\mathrm{cl}(u\mathcal{M})} = 0$. Then $0 < \int_E F d\mu_K = \int_K F(j(k)) d\mathfrak{h}_K(k) = 0$, a contradiction.

(2) If μ_K is G-invariant, then so is its support which is also closed, so the conclusion follows from (1). Also (3) follows from (1).

(4) By G-invariance, $\mathrm{supp}((\mathrm{ev}_x)_*\mu_K)$ is a subflow of (X, G) which, by (3), is contained in the minimal subflow $\mathcal{M}x$, and so it must be equal to $\mathcal{M}x$. □

For every $x \in X$, $\mathrm{supp}\big((\mathrm{ev}_x)_*\mu_K\big) \subseteq \mathcal{M}x$, and $\mathcal{M}x$ is a minimal subflow of X. Consequently, whenever $(\mathrm{ev}_x)_*\mu_K$ is G-invariant, its support is minimal. This already shows that evaluation of Haar decompressions cannot produce invariant measures with nonminimal support. Proposition 4.4 below shows that an ergodic invariant measure with nonminimal support cannot occur on a metrizable tame flow. For comparison, an ergodic Bernoulli measure on a full shift has nonminimal support; however, the full shift is not tame.

Question 3.11. Does there exist a nonmetrizable tame flow carrying an ergodic G-invariant regular Borel probability measure with nonminimal support?

An interesting feature of decompression is that μ_K may become G-invariant on $E(X, G)$ even when invariance is not visible on the original Ellis group. The split-circle dihedral example in Appendix A.2 exhibits this phenomenon.

4. Ergodic transfer

The transfer notions considered in this section depend on a distinguished point. Unless explicitly stated otherwise, (X, x_0, G) denotes an ambit, i.e. $\mathrm{ev}_{x_0}[E(X, G)] = \overline{Gx_0} = X$.

It is useful to distinguish two assertions. The *ergodic Ellis-transfer property* says that the pushforward under ev_{x_0} of every Haar decompression from an Ellis group is G-invariant and ergodic. The *complete Ellis-transfer property* says that every ergodic G-invariant regular Borel probability measure on X is obtained from at least one Ellis group in this way. For tame ambits the decompression itself is always defined by Lemma 3.7; the substantive issues are invariance, ergodicity and completeness. In the metrizable tame setting, the main result of this section shows that completeness is automatic.

4.1. **Basic positive cases.** Before turning to the general results, we record three situations in which the transfer mechanism is particularly transparent.

Equicontinuous ambits. If (X, x_0, G) is equicontinuous, then it is minimal and $E(X, G)$ is a compact topological group. Its normalized Haar measure evaluates to the unique G-invariant probability measure on X. Hence, the ergodic Ellis-transfer is complete.

WAP ambits. Suppose that (X, x_0, G) is WAP. Let $\mathcal{M} \trianglelefteq_m E(X, G)$, let $u^2 = u \in \mathcal{M}$, and put $K := u\mathcal{M}$ and $Y := \mathcal{M}x_0$. By [EN89, Proposition II.5], one has $K = \mathcal{M}$. Moreover, the τ-topology on K coincides with the topology induced from $E(X, G)$: the latter is compact Hausdorff, the former is Hausdorff by Lemma 3.3, and the identity map from the induced topology to the weaker τ-topology is therefore a homeomorphism. Hence, the inclusion $j : (K, \tau) \to E(X, G)$ is continuous, and Remark 3.8 gives $\mu_K = j_*\mathfrak{h}_K$. For $g \in G$ and $\eta \in K = \mathcal{M}$, $g\eta = (gu)\eta$, so the left action of G on K is given by group translations. Therefore, μ_K is G-invariant. Since Y is minimal and WAP, it is equicontinuous, and hence $(\mathrm{ev}_{x_0})_*\mu_K$ is the unique invariant probability measure on Y; in particular, it is ergodic.

Thus, every Ellis group gives an ergodic transfer. If X is metrizable, the transfer is complete by Corollary 4.6. More generally, the proof of Lemma 4.5 shows that completeness follows whenever every ergodic invariant measure on X has minimal support.

Abelian tame ambits. Suppose that G is abelian and (X, x_0, G) is tame. For every $g \in G$ and $\eta \in E(X, G)$ one has $g\eta = \eta g$. If $K = u\mathcal{M}$ is an Ellis group, then $gu \in K$ and $gk = (gu)k$ for every $k \in K$. Thus, the action of G on K is given by translations by the subgroup Gu. This subgroup is τ-dense in K: indeed, $Gu \subseteq u\mathcal{M}$ and, by [Rze18, Fact A.35], $u\overline{Gu} \subseteq \mathrm{cl}_\tau(Gu)$, while $K = u\mathcal{M} = u\overline{Gu}$.

The normalized Haar measure $\mathfrak{h}_K$ is the unique G-invariant regular Borel probability measure on K. Indeed, if λ is G-invariant, then it is invariant under translations by Gu. For every $f \in C(K)$, the map

$$a \longmapsto \int_K f(ak)\, d\lambda(k)$$

is continuous on K, so τ-density of Gu implies that λ is invariant under every left translation by an element of K. Therefore, $\lambda = \mathfrak{h}_K$. In particular, $\mathfrak{h}_K$ is ergodic under the action of G.

We claim that μ_K is G-invariant and ergodic. Invariance follows, for $F \in C(E(X, G))$ and $g \in G$, from

$$\int_E F(g\eta)\, d\mu_K(\eta) = \int_K F(gk)\, d\mathfrak{h}_K(k) = \int_K F((gu)k)\, d\mathfrak{h}_K(k) = \int_E F\, d\mu_K.$$

To prove ergodicity, suppose that $\mu_K = t\nu_1 + (1-t)\nu_2$, where $0 < t < 1$ and $\nu_1, \nu_2 \in \mathrm{Prob}_G(E)$. Put $r_i := d\nu_i/d\mu_K$. The functions r_i are G-invariant μ_K-almost everywhere. The map $T : C(E) \longrightarrow L^1(K, \mathfrak{h}_K)$, given by $T(F) := F \circ j$, is an isometry for the L^1-norms, by the definition of μ_K, and hence extends to an injective positive isometry $T : L^1(E, \mu_K) \to L^1(K, \mathfrak{h}_K)$. Since $gk = (gu)k$ for $g \in G$ and $k \in K$, the function $T(r_i)$ is invariant under translations by the τ-dense subgroup Gu, and hence under all left translations by elements of K. Therefore, $T(r_i)\mathfrak{h}_K$ is a left-invariant probability measure on K, so it equals $\mathfrak{h}_K$. Thus, $T(r_i) = 1$ almost everywhere, and injectivity of T yields $r_i = 1$ almost everywhere. Hence, $\nu_1 = \nu_2 = \mu_K$, proving that μ_K is ergodic. Since $\mathrm{ev}_{x_0} : E(X,G) \to X$ is continuous and G-equivariant, the pushforward $(\mathrm{ev}_{x_0})_*\mu_K$ of the ergodic measure μ_K is ergodic.

Consequently, every Ellis group gives an invariant ergodic transfer. If X is metrizable, completeness follows from Corollary 4.6. More generally, by the proof of Lemma 4.5, a prescribed ergodic invariant measure μ is obtained from an Ellis group whenever the restricted flow on $\mathrm{supp}(\mu)$ is uniquely ergodic: one chooses a minimal left ideal contained in $\mathrm{ev}_{x_0}^{-1}[\mathrm{supp}(\mu)]$ and uses unique ergodicity to identify the resulting transfer with μ.

One should not confuse failure of a standard invariance argument with failure of the transfer itself. In the split-circle dihedral example the left action interchanges the two ordinary Ellis groups, so no single Ellis group is invariant; nevertheless the two decompressions coincide and both evaluate to the unique ergodic measure.

Fact 4.1. *Let G be an amenable discrete group, and let $\pi : X \longrightarrow Y$ be a factor map of compact G-flows. Then every $\mu \in \mathrm{Erg}_G(Y)$ admits a lift $\widetilde{\mu} \in \mathrm{Erg}_G(X)$ such that $\pi_*\widetilde{\mu} = \mu$.*

Proof. The proof is standard; for the countable metrizable case, see [BF19, Lemma 3.2(ii)]. Put $\mathrm{Lift}(\mu) := \{\nu \in \mathrm{Prob}(X) : \pi_*\nu = \mu\}$. This set is nonempty. Indeed, the functional

$$L_0 : \pi^*C(Y) \longrightarrow \mathbb{C}, \qquad L_0(f \circ \pi) := \int_Y f \, d\mu,$$

is a state on the unital C^*-subalgebra $\pi^*C(Y) = \{f \circ \pi : f \in C(Y)\} \subseteq C(X)$. It extends to a state on $C(X)$, and the Riesz–Markov theorem provides a regular Borel probability measure $\nu \in \mathrm{Prob}(X)$ satisfying $\pi_*\nu = \mu$.

The set $\mathrm{Lift}(\mu)$ is convex and weak-$*$ compact. It is preserved by the affine weak-$*$ continuous action $g \cdot \nu := g_*\nu$ (since $\pi_*(g_*\nu) = g_*(\pi_*\nu) = g_*\mu = \mu$). Hence, by Day's fixed-point theorem [Day61], this affine action has a fixed point. Thus, the G-invariant part of $\mathrm{Lift}(\mu)$ is a nonempty compact convex set. By the Krein–Milman theorem, it has an extreme point, and it is not difficult to show that this is the desired ergodic lift $\tilde{\mu}$ of μ. □

Proposition 4.2. *Assume that the acting group G is amenable. Let*

$$\pi : (X, x_0, G) \longrightarrow (Y, y_0, G)$$

be a factor map of tame ambits. If X has complete Ellis-transfer property, then so does Y.

Proof. Let $\mu \in \mathrm{Erg}_G(Y)$. By Fact 4.1, amenability gives an invariant ergodic lift $\widetilde{\mu} \in \mathrm{Erg}_G(X)$ with $\pi_*\widetilde{\mu} = \mu$. By completeness of Ellis-transfer on X, choose an Ellis group $K = u\mathcal{M}$ with $\widetilde{\mu} = (\mathrm{ev}_{x_0})_*\mu_K$. The factor map induces a continuous semigroup epimorphism $\pi_E : E(X,G) \longrightarrow E(Y,G)$. Set

$$\mathcal{N} := \pi_E[\mathcal{M}], \qquad v := \pi_E(u), \qquad L := v\mathcal{N} = \pi_E[K].$$

By Fact 2.1, $\mathcal{N}$ is a minimal left ideal, v is idempotent, and

$$\pi_E|_K : (K, \tau) \longrightarrow (L, \tau)$$

is a continuous group epimorphism. Moreover, $(\pi_E|_K)_* \mathfrak{h}_K = \mathfrak{h}_L$.

Consequently, for every $F \in C(E(Y, G), \mathbb{C})$,

$$\int_{E(Y,G)} F \, d(\pi_E)_* \mu_K = \int_K F(\pi_E(k)) \, d\mathfrak{h}_K(k) = \int_L F(\ell) \, d\mathfrak{h}_L(\ell),$$

so $(\pi_E)_* \mu_K = \mu_L$. Using

$$\pi \circ \mathrm{ev}_{x_0} = \mathrm{ev}_{y_0} \circ \pi_E,$$

we obtain the required Ellis-transfer representation of μ on Y. $\square$

4.2. **Ergodic transfer in the metrizable case.** We now turn to completeness of Ellis transfer in the metrizable tame setting. The proof has two ingredients: the almost-automorphic structure of minimal tame flows and the fact that the support of every ergodic invariant measure is minimal.

Let $\pi : X \to Z$ be a factor map of compact metrizable flows, and put

$$Z_0 := \{z \in Z : |\pi^{-1}(z)| = 1\}.$$

The extension π is *almost one-to-one* if Z_0 is dense in Z. If (Z, G) is uniquely ergodic, with invariant probability measure λ_Z, then π is called *regular* if $\lambda_Z(Z_0) = 1$.

Theorem 4.3. *Let (X, G) be a metrizable minimal tame amenable flow, let $x_0 \in X$, and let $\mathcal{M} \trianglelefteq_m E := E(X, G)$, $u^2 = u \in \mathcal{M}$ and $K := u\mathcal{M}$. Then $\nu_K := (\mathrm{ev}_{x_0})_* \mu_K$ is a unique G-invariant regular Borel probability measure on X.*

Proof. Let

$$\pi : X \longrightarrow Z, \qquad z_0 := \pi(x_0),$$

be the maximal equicontinuous factor of (X, G). By [Gla18, Corollary 5.4], the map π is almost one-to-one and X is uniquely ergodic. Moreover, by [Fuh+21, Theorem 1.2], the extension is regular. The minimal equicontinuous flow (Z, G) is uniquely ergodic. Thus, if λ_Z denotes the unique invariant probability measure on Z, then

$$\lambda_Z(Z_0) = 1, \qquad Z_0 := \{z \in Z : |\pi^{-1}(z)| = 1\}. \tag{1}$$

Put $L := E(Z, G)$. Since (Z, G) is minimal and equicontinuous, L is a compact Hausdorff topological group, and

$$\lambda_Z = (\mathrm{ev}_{z_0})_* \mathfrak{h}_L, \tag{2}$$

where $\mathfrak{h}_L$ is the normalized Haar measure on L. The factor map π induces a continuous surjective semigroup homomorphism $\pi_E : E(X, G) \longrightarrow E(Z, G) = L$ such that $\pi \circ \eta = \pi_E(\eta) \circ \pi$ for every $\eta \in E$. We have $\pi_E[\mathcal{M}] = L$ and $\pi_E(u)$ is equal to the identity of L, and the restriction $\pi_E|_K : K \longrightarrow L$ is a continuous epimorphism of compact Hausdorff groups. Hence,

$$(\pi_E|_K)_* \mathfrak{h}_K = \mathfrak{h}_L. \tag{3}$$

We now compute the projection of ν_K to Z. For every $\varphi \in C(Z, \mathbb{C})$, the function

$$E \ni \eta \longmapsto \varphi\big(\pi(\eta(x_0))\big) \in \mathbb{C},$$

is continuous. Using decompression, (3), and (2), we obtain

$$\begin{aligned}\int_Z \varphi\, d(\pi_*\nu_K) &= \int_E \varphi\big(\pi(\eta(x_0))\big)\, d\mu_K(\eta) = \int_K \varphi\big(\pi(k(x_0))\big)\, d\mathfrak{h}_K(k)\\ &= \int_K \varphi\big(\pi_E|_K(k) z_0\big)\, d\mathfrak{h}_K(k) = \int_L \varphi(\ell z_0)\, d\mathfrak{h}_L(\ell)\\ &= \int_Z \varphi\, d\lambda_Z.\end{aligned}$$

Hence,

$$\pi_*\nu_K = \lambda_Z. \tag{4}$$

It remains to use regularity of the almost one-to-one extension. By a standard argument the set Z_0 is a G_δ subset of Z, and hence is Polish. The set $X_0 := \pi^{-1}(Z_0)$ is likewise a G_δ subset of X, and hence is Polish. The restriction

$$\pi|_{X_0} : X_0 \longrightarrow Z_0$$

is a continuous bijection. By the Lusin–Souslin theorem, its inverse

$$s : Z_0 \longrightarrow X_0$$

is Borel. Now, let ρ be a Borel probability measure on X such that $\pi_*\rho = \lambda_Z$. Since $\lambda_Z(Z_0) = 1$, we have $\rho(X_0) = 1$. Therefore, for every Borel set $B \subseteq X$,

$$\rho(B) = \lambda_Z\big(s^{-1}(B \cap X_0)\big).$$

Thus, ρ is uniquely determined by λ_Z, and consequently there is exactly one Borel probability measure on X which projects to λ_Z.

By (4), the measure ν_K is such a lift. The unique invariant measure, say μ_X, is also such a lift, because $\pi_*\mu_X$ is G-invariant on the uniquely ergodic flow (Z, G) and hence equals λ_Z. □

The following proposition strengthens conditional minimal-support [Rom16, Theorem 3.6] given for tame semicascades:

Proposition 4.4. *Let (X, G) be a metrizable tame flow, and let μ be an ergodic G-invariant regular Borel probability measure on X. Then* $\operatorname{supp}(\mu)$ *is a minimal subflow of X.*

Proof. Set $Y := \operatorname{supp}(\mu)$, and regard μ as a probability measure on Y. Then (Y, G) is metrizable and tame, while μ is ergodic and has full support on Y.

Let $E_Y := E(Y, G)$ and choose a minimal left ideal $\mathcal{M}_Y \trianglelefteq_m E_Y$ and an idempotent $u \in \mathcal{M}_Y$, then $E_Y u = \mathcal{M}_Y$.

By the metrizable characterization of tameness coming from the dynamical BFT dichotomy [Gla06, Theorem 1.2], the space E_Y is a Rosenthal compactum. Every Rosenthal compactum is angelic, and hence Fréchet–Urysohn [BFT78, Theorem 3F]. Therefore, since G is dense in E_Y, there is a sequence $(g_n)_{n<\omega}$ in G converging pointwise on Y to u. In particular, $u : Y \to Y$ is Borel.

For every $f \in C(Y, \mathbb{C})$, dominated convergence and G-invariance of μ give

$$\int_Y f(uy)\, d\mu(y) = \lim_{n\to\infty} \int_Y f(g_n y)\, d\mu(y) = \int_Y f(y)\, d\mu(y).$$

Consequently, $u_*\mu = \mu$.

The set $F := \{y \in Y : uy = y\}$ is Borel. Since $u^2 = u$, one has $u(Y) \subseteq F$, and hence

$$\mu(F) = (u_*\mu)(F) = \mu\big(u^{-1}[F]\big) = 1.$$

Let $(U_m)_{m<\omega}$ be a countable basis consisting of nonempty open subsets of Y, and set

$$A_m := \bigcup_{g \in G} g^{-1} U_m.$$

Each A_m is a G-invariant open subset of Y. Since μ has full support, $\mu(A_m) \geqslant \mu(U_m) > 0$. Ergodicity therefore gives $\mu(A_m) = 1$ for every $m < \omega$ (by Proposition 12.4 from [Phe01]). Thus,

$$T := \bigcap_{m<\omega} A_m$$

has full measure. By the definition of T, every point of T has dense G-orbit in Y.

Choose $y \in F \cap T$. Since $uy = y$ and $\overline{Gy} = Y$, we obtain

$$Y = E_Y y = E_Y uy = (E_Y u)y = \mathcal{M}_Y y.$$

Thus, Y is minimal. □

Lemma 4.5. *Let (X, G) be a metrizable tame flow, fix $x_0 \in X$, and let μ be an ergodic G-invariant regular Borel probability measure on X. Assume that $Y :=$ $\operatorname{supp}(\mu)$ is contained in $\overline{Gx_0}$. Put $I_Y := \operatorname{ev}_{x_0}^{-1}[Y] \subseteq E(X, G)$. Then, for every choice of*

$$\mathcal{M} \trianglelefteq_m E(X, G), \qquad \mathcal{M} \subseteq I_Y, \qquad u^2 = u \in \mathcal{M},$$

and $K := u\mathcal{M}$, one has $(\operatorname{ev}_{x_0})_* \mu_K = \mu$.

Proof. Let $y_0 := u(x_0) \in Y$. By Proposition 4.4, the flow (Y, G) is minimal; it is also metrizable and tame, and it carries the invariant probability measure $\mu|_Y$.

Restriction to Y defines a continuous semigroup epimorphism

$$r : E(X, G) \longrightarrow E(Y, G), \qquad r(\eta) := \eta|_Y.$$

Put

$$\mathcal{M}_Y := r[\mathcal{M}], \qquad v := r(u), \qquad K_Y := v\mathcal{M}_Y = r[K].$$

By Fact 2.1, $\mathcal{M}_Y$ is a minimal left ideal, v is idempotent, and

$$r|_K : (K, \tau) \longrightarrow (K_Y, \tau)$$

is a continuous group epimorphism satisfying $(r|_K)_* \mathfrak{h}_K = \mathfrak{h}_{K_Y}$.

Let $\nu_K := (\operatorname{ev}_{x_0})_* \mu_K$. By Remark 3.10(3), $\operatorname{supp}(\nu_K) \subseteq \mathcal{M}x_0 \subseteq I_Y x_0 \subseteq Y$. Since also $\operatorname{supp}(\mu) = Y$, the measures ν_K and μ are equal if and only if their restrictions to Y are equal. By Theorem 4.3, $(\operatorname{ev}_{y_0})_* \mu_{K_Y}$ is the unique G-invariant probability measure on Y. Therefore, it remains to prove that

$$\nu_K|_Y = (\operatorname{ev}_{y_0})_* \mu_{K_Y}.$$

Fix $f \in C(Y, \mathbb{C})$ and choose an extension $\widetilde{f} \in C(X, \mathbb{C})$. Using $ku = k$ for $k \in K$ and $y_0 = u(x_0)$, we obtain

$$\begin{aligned} \int_Y f \, d(\nu_K|_Y) = & \int_X \widetilde{f} \, d\nu_K = \int_K \widetilde{f}\big(k(x_0)\big) \, d\mathfrak{h}_K(k) \\ &= \int_K \widetilde{f}\big(k(u(x_0))\big) \, d\mathfrak{h}_K(k) = \int_K f\big(r(k)(y_0)\big) \, d\mathfrak{h}_K(k) \end{aligned}$$

$$= \int_{K_Y} f\big(\ell(y_0)\big)\, d\mathfrak{h}_{K_Y}(\ell) \;= \int_{E(Y,G)} f\big(\mathrm{ev}_{y_0}(\eta)\big)\, d\mu_{K_Y}(\eta)$$
$$= \int_Y f\, d\big((\mathrm{ev}_{y_0})_*\mu_{K_Y}\big).$$

This proves $\nu_K|_Y = (\mathrm{ev}_{y_0})_*\mu_{K_Y}$, and therefore $\nu_K = \mu$. □

Corollary 4.6. *Let (X, x_0, G) be a metrizable tame amenable ambit. Consider an ergodic G-invariant regular Borel probability measure μ on X. Then μ is the pushforward under ev_{x_0} of the decompression of Haar measure from an Ellis group of $E(X,G)$.*

Proof. By Lemma 4.5. □

For comparison, in the more restrictive one-map setting of tame $\mathbb{N}_0$-semicascades, Romanov [Rom19] realizes ergodic measures through generalized ergodic averages in the Köhler operator semigroup, a mechanism different from Haar decompression from Ellis groups.

4.3. **Summary and the Choquet theorem.** In this subsection, assume that (X, x_0, G) is a metrizable tame hereditarily amenable ambit. Then the Ellis realization of $\mathrm{Erg}_G(X)$ is particularly uniform.

Indeed, let $\mathcal{M} \trianglelefteq_m E(X,G)$, $u^2 = u \in \mathcal{M}$, $K := u\mathcal{M}$. Then $Y_K := \mathcal{M}x_0$ is a minimal subflow of X. By hereditary amenability, Y_K admits an invariant probability measure μ. Since Y_K is metrizable, minimal and tame, this measure is unique and so ergodic. So μ is also ergodic as a measure on X, and $\mathrm{supp}(\mu) = Y_K = \mathcal{M}x_0$ which implies that $\mathcal{M} \subseteq \mathrm{ev}_{x_0}^{-1}[Y_K]$. Thus, the assumptions of Lemma 4.5 are satisfied, and so we conclude that

$$(\mathrm{ev}_{x_0})_*\mu_K = \mu \in \mathrm{Erg}_G(X), \quad \mathrm{supp}\,\big((\mathrm{ev}_{x_0})_*\mu_K\big) = Y_K.$$

Conversely, every $\nu \in \mathrm{Erg}_G(X)$ has minimal support by Proposition 4.4, and every Ellis group K obtained from a minimal left ideal contained in $\mathrm{ev}_{x_0}^{-1}[\mathrm{supp}(\nu)]$ evaluates to ν, that is $(\mathrm{ev}_{x_0})_*\mu_K = \nu$. Thus, the pushforwards of the Haar decompressions of all Ellis groups are precisely the ergodic invariant measures, possibly with several Ellis groups producing the same measure.

We briefly recall the relevant Choquet-theoretic background (cf. [Phe01]). Since X is compact metrizable, the weak-$*$ compact convex set $\mathrm{Prob}_G(X)$ is a metrizable Choquet simplex, and its extreme boundary is precisely

$$\mathrm{ext}\,\mathrm{Prob}_G(X) = \mathrm{Erg}_G(X).$$

Moreover, $\mathrm{Erg}_G(X)$ is a G_δ-subset of $\mathrm{Prob}_G(X)$ (cf. [Alf66]). Thus, $\mathrm{Prob}_G(X)$ is a Bauer simplex precisely when $\mathrm{Erg}_G(X)$ is closed, whereas, in the nontrivial case, it is a Poulsen simplex precisely when $\mathrm{Erg}_G(X)$ is dense (cf. [LOS78]). In particular, every invariant measure admits a unique barycentric decomposition over the ergodic invariant measures.

Let $\mathcal{E}(X,G) := \big\{u\mathcal{M} : \mathcal{M} \trianglelefteq_m E(X,G),\ u^2 = u \in \mathcal{M}\big\}$ be the collection of all Ellis groups of $E(X,G)$, and define

$$\Pi\colon \mathcal{E}(X,G) \longrightarrow \mathrm{Erg}_G(X), \qquad \Pi(K) := (\mathrm{ev}_{x_0})_*\mu_K.$$

The preceding discussion shows that Π is surjective.

Equip $\mathcal{E}(X,G)$ with the σ-algebra $\big\{\Pi^{-1}[B] : B \in \mathcal{B}(\mathrm{Erg}_G(X))\big\}$. For every $\mu \in \mathrm{Prob}_G(X)$, let λ_μ denote its unique Choquet measure on $\mathrm{Erg}_G(X)$. Since Π is surjective, there is a unique probability measure $\widehat{\mu}$ on $\mathcal{E}(X,G)$ determined by $\Pi_*\widehat{\mu} = \lambda_\mu$.

Corollary 4.7. *Let (X, x_0, G) be a metrizable tame hereditarily amenable ambit. Then, for every $\mu \in \mathrm{Prob}_G(X)$,*

$$\mu = \int_{\mathcal{E}(X,G)} (\mathrm{ev}_{x_0})_*\mu_K \, d\widehat{\mu}(K)$$

in the weak-* sense.

Proof. By the Choquet representation theorem and its application to invariant measures [Phe01, Proposition 3.1 and Chapters 10 and 12], μ is the barycenter of its unique representing probability measure λ_μ on $\mathrm{Erg}_G(X)$. □

5. Invariant measures on the Ellis semigroup

We first characterize amenability of the Ellis flow in terms of hereditary amenability of all finite powers of X; this criterion does not require tameness (Theorem 5.3). We then prove that, for tame flows, ordinary hereditary amenability implies the above finite-power condition (Theorem 5.5), which yields Corollary 5.6. Finally, the same finite-coordinate analysis is used to prove G-invariance of every Haar decompression in the metrizable case, the countable-group case, and, more generally, under the separable-generation hypothesis of Theorem 5.8. For the rest of this section, let (X,G) denote a flow. (We do not assume that (X,G) has a dense orbit.)

5.1. Finite-power hereditary amenability.

Theorem 5.3 below shows, without any tameness assumption, that amenability of the Ellis flow is equivalent to hereditary amenability of all finite powers of X. Its proof uses the dense algebra of finite-coordinate functions introduced below. This motivates the following definition.

Definition 5.1. The flow X is *finitely power-hereditarily amenable* if, for every $n \geqslant 1$, every subflow of X^n, equipped with the diagonal action

$$g(x_1,\ldots,x_n) := (gx_1,\ldots,gx_n),$$

carries a G-invariant regular Borel probability measure.

We shall use the algebra of finite-coordinate functions

$$\mathcal{A}_{\mathrm{fin}} := \left\{ F \in C(E(X,G),\mathbb{C}) : \begin{array}{l} \text{for some } n \geqslant 1,\ \bar{x} \in X^n, \text{ and } \varphi \in C(X^n,\mathbb{C}), \\ F(\eta) = \varphi(\eta\bar{x}) \text{ for every } \eta \in E(X,G) \end{array} \right\}.$$

By concatenating the tuples on which two functions depend, one sees that $\mathcal{A}_{\mathrm{fin}}$ is closed under sums and products; it is also closed under complex conjugation and contains the constants. It separates points of $E(X,G)$: if $\eta \neq \xi$, choose $x \in X$ with $\eta(x) \neq \xi(x)$ and then choose $f \in C(X,\mathbb{C})$ separating these two points. The function

$$\zeta \longmapsto f(\zeta(x))$$

belongs to $\mathcal{A}_{\mathrm{fin}}$ and separates η and ξ. Hence, by the complex Stone–Weierstrass theorem,

$$\overline{\mathcal{A}_{\mathrm{fin}}}^{\|\cdot\|_\infty} = C(E(X,G),\mathbb{C}).$$

Remark 5.2. $(E(X,G),G)$ is amenable if and only if $(E(X,G),G)$ is hereditarily amenable.

Proof. Only the left-to-right implication requires proof. Put $E := E(X,G)$, and let ν be a G-invariant regular Borel probability measure on E.

Let $Y \subseteq E$ be a nonempty subflow and fix $q \in Y$. By the left-continuity in E, the right translation

$$R_q : E \longrightarrow E, \qquad R_q(p) := pq,$$

is continuous and G-equivariant (i.e. $R_q(gp) = gR_q(p)$). Moreover, $R_q[E] = Eq = \overline{Gq}$. Since Y is closed and G-invariant and $q \in Y$, one has $\overline{Gq} \subseteq Y$. Thus, $(R_q)_*\nu$ is a probability measure on Y. Because R_q is G-equivariant, $(R_q)_*\nu$ is G-invariant. $\square$

Theorem 5.3. *$\big(E(X,G),G\big)$ is amenable if and only if (X,G) is finitely power-hereditarily amenable.*

Proof. Assume first that $\big(E(X,G),G\big)$ is amenable. Fix $n \geqslant 1$, and let $W \subseteq X^n$ be a nonempty subflow. Choose a minimal subflow $V \subseteq W$ and a point $\bar{x} = (x_1,\dots,x_n) \in V$. Consider the evaluation map

$$e_{\bar{x}} : E(X,G) \longrightarrow V, \qquad e_{\bar{x}}(\eta) := \eta(\bar{x}).$$

It is continuous and G-equivariant. Moreover, the image of $e_{\bar{x}}$ is compact and contains $G\bar{x}$, so minimality of V gives $e_{\bar{x}}[E(X,G)] = V$.

The pushforward of a G-invariant probability measure on $E(X,G)$ is therefore a G-invariant regular Borel probability measure on V, and hence, can be viewed as a measure on W which is supported on V.

Now, we prove that if (X,G) is finitely power-hereditarily amenable then the flow $\big(E(X,G),G\big)$ is amenable. Let $E := E(X,G)$. For $g \in G$, $F \in C(E,\mathbb{C})$, and $\varepsilon > 0$, consider

$$D(g,F,\varepsilon) := \left\{\rho \in \mathrm{Prob}(E) : \left|\int_E F(g\eta)\,d\rho(\eta) - \int_E F(\eta)\,d\rho(\eta)\right| \leqslant \varepsilon\right\}.$$

Each $D(g,F,\varepsilon)$ is weak-$*$ closed in $\mathrm{Prob}(E)$. As $\mathrm{Prob}(E)$ is weak-$*$ compact, it is enough to show that these sets have the finite-intersection property (see the end of the proof).

Fix $g_1,\dots,g_r \in G$, $F_1,\dots,F_r \in C(E,\mathbb{C})$ and $\epsilon_1,\dots,\epsilon_r > 0$. By the density of the algebra $\mathcal{A}_{\mathrm{fin}}$ in $C(E,\mathbb{C})$ (see the paragraph after Definition 5.1), for every $j \leqslant r$ choose $A_j \in \mathcal{A}_{\mathrm{fin}}$ such that

$$\|F_j - A_j\|_\infty < \frac{\varepsilon_j}{3}. \tag{5}$$

After concatenating the finitely many tuples on which the functions A_j depend, there are a single finite tuple $\bar{x} = (x_1,\dots,x_n) \in X^n$ and functions $\varphi_j \in C(X^n,\mathbb{C})$ such that

$$A_j(\eta) = \varphi_j(\eta\bar{x}) \qquad (\eta \in E,\ 1 \leqslant j \leqslant r). \tag{6}$$

Consider $e_{\bar{x}} : E \longrightarrow X^n$ given by $e_{\bar{x}}(\eta) := \eta(\bar{x})$, and set $Y_{\bar{x}} := e_{\bar{x}}[E]$. The set $Y_{\bar{x}}$ is a subflow of X^n, and $Y_{\bar{x}} = \overline{G\bar{x}}$.

Since X is finitely power-hereditarily amenable, there is a G-invariant regular Borel probability measure λ on $Y_{\bar{x}}$.

Claim 1. λ has a lift ρ to E.

Proof. Since $e_{\bar{x}} : E \to Y_{\bar{x}}$ is a continuous surjection, the formula

$$L_0(f \circ e_{\bar{x}}) := \int_{Y_{\bar{x}}} f \, d\lambda$$

defines a state on the unital C^*-subalgebra $e_{\bar{x}}^* C(Y_{\bar{x}}, \mathbb{C}) \subseteq C(E, \mathbb{C})$. By Krein extension theorem, we can extend L_0 to a state on $C(E, \mathbb{C})$. By the Riesz–Markov theorem, the extension is integration against some regular Borel probability measure ρ on E, and, by construction,

$$(e_{\bar{x}})_* \rho = \lambda. \tag{7}$$

□(claim)

Using (6), (7), and the G-invariance of λ, we obtain, for every $1 \leqslant j \leqslant r$,

$$\begin{aligned}\int_E A_j(g_j\eta) \, d\rho(\eta) &= \int_{Y_{\bar{x}}} \varphi_j(g_j y) \, d\lambda(y) \\ &= \int_{Y_{\bar{x}}} \varphi_j(y) \, d\lambda(y) \\ &= \int_E A_j(\eta) \, d\rho(\eta).\end{aligned}$$

Consequently, by (5),

$$\begin{aligned}&\left| \int_E F_j(g_j\eta) \, d\rho(\eta) - \int_E F_j(\eta) \, d\rho(\eta) \right| \\ &\quad \leqslant \left| \int_E (F_j - A_j)(g_j\eta) \, d\rho(\eta) \right| + \left| \int_E (F_j - A_j)(\eta) \, d\rho(\eta) \right| \\ &\quad \leqslant 2\|F_j - A_j\|_\infty < \frac{2\varepsilon_j}{3} < \varepsilon_j.\end{aligned}$$

Thus,

$$\rho \in \bigcap_{j=1}^{r} D(g_j, F_j, \varepsilon_j),$$

which proves the finite-intersection property.

By compactness of $\mathrm{Prob}(E)$, there is

$$\rho_0 \in \bigcap_{\substack{g \in G,\ F \in C(E,\mathbb{C}) \\ m \geqslant 1}} D\left(g, F, \frac{1}{m}\right).$$

Therefore, for every $g \in G$ and every $F \in C(E, \mathbb{C})$,

$$\int_E F(g\eta) \, d\rho_0(\eta) = \int_E F(\eta) \, d\rho_0(\eta).$$

Equivalently, $g_* \rho_0 = \rho_0$ for every $g \in G$. Hence, the Ellis flow $(E(X, G), G)$ carries a G-invariant regular Borel probability measure. □

5.2. **Hereditary amenability for tame flows.** Tameness will be needed in order to pass from hereditary amenability of X to hereditary amenability of its finite powers. We first prove a joining lemma for minimal metrizable tame flows and then use a separable reduction to treat arbitrary compact tame flows.

Lemma 5.4. *Let H be a group, let $Y_1, \dots, Y_n$ be minimal metrizable tame H-flows, each carrying an H-invariant regular Borel probability measure, and let*

$$V \subseteq Y_1 \times \cdots \times Y_n$$

be a minimal H-subflow. Then V carries an H-invariant regular Borel probability measure.

Proof. For each i, let

$$\pi_i : Y_i \longrightarrow Z_i$$

be the maximal equicontinuous factor. As in the proof of Theorem 4.3, by [Gla18, Corollary 5.4(2)] and [Fuh+21, Theorem 1.2], if $\mathfrak{m}_i$ denotes the unique invariant probability measure on Z_i, then

$$\mathfrak{m}_i(Z_i^0) = 1, \qquad Z_i^0 := \left\{ z \in Z_i : |\pi_i^{-1}(z)| = 1 \right\}.$$

Define

$$\pi_V : V \longrightarrow Z_1 \times \cdots \times Z_n$$

as $\pi_V := \pi_1 \times \ldots \times \pi_n$, and put $Z := \pi_V[V]$. By Lemma 2.2(1), the product $Z_1 \times \cdots \times Z_n$ is equicontinuous. Hence, its subflow Z is equicontinuous. Since Z is minimal, it has a unique invariant probability measure $\mathfrak{m}_Z$.

For each i, the image of the coordinate projection $V \longrightarrow Y_i$ is a nonempty closed H-invariant subset of the minimal flow Y_i, thus equal to Y_i. It follows that the induced projection $p_i : Z \longrightarrow Z_i$ is also onto. Therefore,

$$(p_i)_* \mathfrak{m}_Z = \mathfrak{m}_i.$$

Consequently, $\mathfrak{m}_Z(Z^0) = 1$ for

$$Z^0 := Z \cap \left(Z_1^0 \times \cdots \times Z_n^0 \right) = \bigcap_{i=1}^{n} p_i^{-1}[Z_i^0].$$

Now, if $z = (z_1, \dots, z_n) \in Z^0$, then

$$\pi_V^{-1}(z) \subseteq \pi_1^{-1}(z_1) \times \cdots \times \pi_n^{-1}(z_n).$$

The set on the right is a singleton. Since $z \in \pi_V[V]$, the fibre $\pi_V^{-1}(z)$ is nonempty and hence is itself a singleton.

A probability lift of $\mathfrak{m}_Z$ to V exists by applying the Krein extension theorem to extend the state

$$f \circ \pi_V \mapsto \int_Z f \, d\mathfrak{m}_Z$$

from the unital C^*-subalgebra $\pi_V^* C(Z)$ to a state on $C(V)$, and subsequently applying the Riesz–Markov representation theorem.

This lift is unique. Indeed, each Z_i^0 is a G_δ subset of Z_i, and hence Z^0 is a G_δ subset of the compact metrizable space Z. Put $V^0 := \pi_V^{-1}[Z^0]$. Then V^0 and Z^0 are Polish spaces, and $\pi_V|_{V^0} : V^0 \longrightarrow Z^0$ is a continuous bijection. By the Lusin–Souslin theorem, its inverse is Borel. Every probability measure on V which projects to $\mathfrak{m}_Z$ is concentrated on V^0, and is therefore the pushforward of $\mathfrak{m}_Z|_{Z^0}$ under this inverse. Thus, the lift is unique.

For every $h \in H$, the translate of this lift is again a lift of $\mathfrak{m}_Z$. By uniqueness, the lift is H-invariant. □

We briefly recall the form of Gelfand duality used below (see, for example, [Mur90]). For a unital commutative C^*-algebra A, its spectrum $\mathrm{Spec}(A)$ is the set of characters, i.e. unital $*$-homomorphisms $\chi : A \to \mathbb{C}$, equipped with the weak-$*$ topology. It is compact Hausdorff, and the Gelfand transform identifies A isometrically with $C(\mathrm{Spec}(A), \mathbb{C})$. If $A \subseteq C(Y, \mathbb{C})$ is a unital C^*-subalgebra, then

$$q_A : Y \longrightarrow \mathrm{Spec}(A), \qquad q_A(y)(a) := a(y),$$

is a continuous surjection; it is equivariant whenever A is invariant under the acting group. An inclusion $A \subseteq B$ induces the continuous surjection $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ given by restriction of characters. Moreover, $\mathrm{Spec}(A)$ is metrizable when A is separable.

Theorem 5.5. *Let (X, G) be tame. If X is hereditarily amenable, then X is finitely power-hereditarily amenable.*

Proof. Fix $n \geqslant 1$, and let $W \subseteq X^n$ be a nonempty subflow. It is enough to construct an invariant probability measure on a minimal subflow of W. Replacing W by such a subflow, we assume that W is minimal.

For $1 \leqslant i \leqslant n$, put

$$Y_i := \mathrm{pr}_i[W].$$

Each Y_i is a minimal subflow of X. By hereditary amenability, choose a G-invariant regular Borel probability measure μ_i on Y_i. For $g \in G$ and $f \in C(W, \mathbb{C})$, we consider

$$C(g, f) := \left\{ \nu \in \mathrm{Prob}(W) : \int_W f(gw)\, d\nu(w) = \int_W f(w)\, d\nu(w) \right\}.$$

Each $C(g, f)$ is weak-$*$ closed, and

$$\mathrm{Prob}_G(W) = \bigcap_{\substack{g \in G \\ f \in C(W, \mathbb{C})}} C(g, f).$$

We verify that the above family has the finite-intersection property.

Fix $g_1, \dots, g_r \in G$ and $f_1, \dots, f_r \in C(W, \mathbb{C})$ and let $H_0 \leqslant G$ be the subgroup generated by $g_1, \dots, g_r$. Since W is a closed subset of the compact Hausdorff, and hence normal, space

$$Y_1 \times \cdots \times Y_n,$$

the Tietze extension theorem, applied separately to the real and imaginary parts, gives, for every $j \leqslant r$, an extension $\widetilde{f_j} \in C(Y_1 \times \cdots \times Y_n, \mathbb{C})$ of f_j.

By the complex Stone–Weierstrass theorem, each $\widetilde{f_j}$ is a uniform limit of finite sums of products of coordinate functions. Choose a sequence of such approximations for every j, and, for $1 \leqslant i \leqslant n$, let $\mathcal{D}_i \subseteq C(Y_i, \mathbb{C})$ be the countable family of all i-th coordinate functions which occur in these approximations.

We now construct recursively an increasing sequence of countable subgroups

$$H_0 \leqslant H_1 \leqslant H_2 \leqslant \cdots \leqslant G$$

and, simultaneously for every $1 \leqslant i \leqslant n$, an increasing sequence of separable unital H_m-invariant C^*-subalgebras

$$A_{i,m} \subseteq C(Y_i, \mathbb{C}).$$

The action on functions is given by $(h \cdot a)(y) := a(h^{-1}y)$. We already have H_0. For each i, let $A_{i,0}$ be the unital C^*-algebra generated by $\{h \cdot a : h \in H_0,\ a \in \mathcal{D}_i\}$. Now, assume that H_m and all the algebras $A_{i,m}$ have been constructed. Let

$$q_{i,m} : Y_i \longrightarrow Y_{i,m} := \mathrm{Spec}(A_{i,m})$$

be the associated factor maps of H_m-flows, with metrizable targets $Y_{i,m}$, and put

$$q_m := (q_{1,m}, \ldots, q_{n,m})|_W, \qquad W_m := q_m[W].$$

Choose a countable basis $\mathcal{B}_m$ consisting of nonempty open subsets of W_m. For $B \in \mathcal{B}_m$, the set $U_B := q_m^{-1}[B]$ is a nonempty open subset of the minimal G-flow W, and so the family $\{gU_B : g \in G\}$ covers W. By compactness, choose a finite set $F_B \subseteq G$ such that

$$W = \bigcup_{g \in F_B} gU_B.$$

Let H_{m+1} be the subgroup generated by

$$H_m \cup \bigcup_{B \in \mathcal{B}_m} F_B.$$

This group is countable. For every i, let $A_{i,m+1}$ be the unital C^*-algebra generated by all H_{m+1}-translates of $A_{i,m}$. Then $A_{i,m+1}$ is separable and H_{m+1}-invariant. This completes the recursive construction.

Now, we introduce

$$H := \bigcup_{m<\omega} H_m$$

and

$$A_i := \overline{\bigcup_{m<\omega} A_{i,m}}.$$

Then H is countable, and each A_i is a separable unital H-invariant C^*-subalgebra of $C(Y_i, \mathbb{C})$. Let

$$q_i : Y_i \longrightarrow Y_i' := \mathrm{Spec}(A_i)$$

be the associated factor map of H-flows, with metrizable target Y_i', and put

$$q := (q_1, \ldots, q_n)|_W, \qquad W' := q[W].$$

Claim 1. Every f_j factors through q, i.e. $f_j = \hat{f}_j \circ q$ for a unique function $\hat{f}_j \in C(W', \mathbb{C})$.

Proof. Suppose that $q(w) = q(w')$. Then, for every i and every $a \in A_i$, the function a has the same value on the i-th coordinates of w and w'. In particular, this holds for every member of $\mathcal{D}_i$. Thus, all the chosen finite sums of products of coordinate functions have the same value at w and w'. Passing to the uniform limit gives $f_j(w) = f_j(w')$. Since $q : W \to W'$ is a continuous surjection from a compact space onto a Hausdorff space, it is a quotient map. Hence, there is a unique function $\widehat{f_j} \in C(W', \mathbb{C})$ such that $f_j = \widehat{f_j} \circ q$. $\square$(claim)

Claim 2. W' is minimal as an H-flow.

Proof. By Gelfand duality, the inclusions $A_{i,m} \subseteq A_i$ induce factor maps $r_{i,m} : Y_i' \longrightarrow Y_{i,m}$ and hence continuous maps $r_m : W' \longrightarrow W_m$ such that

$$q_m = r_m \circ q.$$

The algebra

$$\bigcup_{m<\omega} r_m^* C(W_m, \mathbb{C})$$

is uniformly dense in $C(W', \mathbb{C})$. Indeed, the coordinate functions coming from the algebras A_i separate points of W', so the self-adjoint unital algebra generated by them is dense in $C(W', \mathbb{C})$ by the complex Stone–Weierstrass theorem. Every member of A_i is a uniform limit of members of the algebras $A_{i,m}$. By the isometry of the Gelfand transform, such approximation in A_i yields uniform approximation of the corresponding coordinate functions on Y_i'. Finally, it is easy to see that the coordinate functions coming from the algebras $A_{i,m}$ belong to

$$\bigcup_{m<\omega} r_m^* C(W_m, \mathbb{C}).$$

It follows that the sets $r_m^{-1}[B]$, with $m < \omega$ and $B \in \mathcal{B}_m$, form a basis of the topology of W'. Explicitly, let $z \in O \subseteq W'$, where O is open. Choose

$$\phi \in C(W', [0,1])$$

such that $\phi(z) = 1$ and $\phi = 0$ on $W' \setminus O$. By the preceding density, choose m and

$$\chi_m \in C(W_m, \mathbb{C})$$

such that $\|\phi - \chi_m \circ r_m\|_\infty < \frac{1}{3}$. Let $\psi_m := \mathrm{Re}(\chi_m) \in C(W_m, \mathbb{R})$. Since ϕ is real-valued,

$$\|\phi - \psi_m \circ r_m\|_\infty < \frac{1}{3}.$$

Therefore,

$$z \in r_m^{-1}\left[\left\{y \in W_m : \psi_m(y) > \frac{2}{3}\right\}\right] \subseteq O.$$

Choosing a basis element $B \in \mathcal{B}_m$ contained in the open set $\{y \in W_m : \psi_m(y) > \frac{2}{3}\}$ and containing $r_m(z)$, we get $z \in r_m^{-1}[B] \subseteq O$.

For such $B \in \mathcal{B}_m$, we already fixed a finite set $F_B \subseteq H$ such that

$$W = \bigcup_{g \in F_B} g q_m^{-1}[B].$$

Since q is H-equivariant and $q_m = r_m \circ q$, applying q yields

$$W' = \bigcup_{g \in F_B} g\, r_m^{-1}[B] \subseteq \bigcup_{g \in F_B} gO.$$

Thus, finitely many H-translates of every nonempty open subset of W' cover W'. Hence, every H-orbit in W' is dense, and W' is minimal. $\square$(claim)

For each i, the coordinate map

$$\mathrm{pr}_i|_{W'} : W' \longrightarrow Y_i'$$

is continuous and H-equivariant. It is also onto as $W' = q[W] = (q_1, \ldots, q_n)[W]$ and we have $\mathrm{pr}_i[W'] = q_i[\mathrm{pr}_i[W]] = q_i[Y_i] = Y_i'$. Therefore, by Claim 2, Y_i' is an H-factor of the minimal flow W', and hence Y_i' is minimal.

Since Y_i is a subflow of the tame G-flow X, the restricted H-flow Y_i is tame. Tameness passes to factors, so Y_i' is tame. Moreover,

$$\mu_i' := (q_i)_* \mu_i$$

is an H-invariant regular Borel probability measure on Y_i'. Lemma 5.4 applied to $W' \subseteq Y_1' \times \cdots \times Y_n'$, gives an H-invariant regular Borel probability measure λ on W'.

Define a state on the unital C^*-subalgebra $q^*C(W', \mathbb{C}) \subseteq C(W, \mathbb{C})$ by

$$L(\varphi \circ q) := \int_{W'} \varphi \, d\lambda.$$

Extend L to a state on $C(W, \mathbb{C})$, and let ν be its Riesz measure on W. Then $q_*\nu = \lambda$.

Using the factorization $f_j = \widehat{f}_j \circ q$ from Claim 1, the H-equivariance of q, the inclusion $g_j \in H$, and the H-invariance of λ, we obtain

$$\begin{aligned} \int_W f_j(g_j w) \, d\nu(w) &= \int_{W'} \widehat{f}_j(g_j z) \, d\lambda(z) \\ &= \int_{W'} \widehat{f}_j(z) \, d\lambda(z) \\ &= \int_W f_j(w) \, d\nu(w). \end{aligned}$$

Thus, $\nu \in C(g_1, f_1) \cap \cdots \cap C(g_r, f_r)$, and so we have proved that the family

$$\{C(g, f) : g \in G, \ f \in C(W, \mathbb{C})\}$$

has nonempty finite intersections. By compactness,

$$\operatorname{Prob}_G(W) = \bigcap_{\substack{g \in G \\ f \in C(W, \mathbb{C})}} C(g, f) \neq \varnothing.$$

Thus, W carries a G-invariant regular Borel probability measure. $\square$

Corollary 5.6. *Let (X, G) be tame. The following conditions are equivalent:*

(i) $\big(E(X, G), G\big)$ *is hereditarily amenable;*
(ii) $\big(E(X, G), G\big)$ *is amenable;*
(iii) X *is hereditarily amenable.*

Proof. Equivalence between (i) and (ii) is the content of Remark 5.2.

Assume (ii). By Theorem 5.3, the flow X is finitely power-hereditarily amenable, and hence is hereditarily amenable. Thus, (iii) holds.

Conversely, (iii) implies (ii) by Theorem 5.5 combined with Theorem 5.3. $\square$

The tameness assumption in Corollary 5.6 cannot be omitted: Appendix A.3 constructs a metrizable hereditarily amenable flow whose Ellis flow is not amenable.

5.3. **Invariance of Haar decompressions.** The preceding results guarantee the existence of some invariant measure on the Ellis flow. We now prove the stronger statement that the canonical Haar decompression associated with each Ellis group is invariant, first in the metrizable case and then under the general separable-generation hypothesis.

Theorem 5.7. *Let (X, G) be tame, metrizable and hereditarily amenable. Let $\mathcal{M} \trianglelefteq_m E := E(X, G)$, $u^2 = u \in \mathcal{M}$ and set $K := u\mathcal{M}$. Then the Haar decompression μ_K is a G-invariant regular Borel probability measure on $E(X, G)$.*

Proof. By Theorem 5.5, (X, G) is finitely power-hereditarily amenable. Fix a finite tuple $\bar{x} = (x_1, \ldots, x_n) \in X^n$ and set $W_{\bar{x}} := \mathcal{M}\bar{x}$. The flow $W_{\bar{x}}$ is minimal. By finite-power hereditary amenability, $W_{\bar{x}}$ carries a G-invariant regular Borel probability measure. By Lemma 2.2(2), the flow $W_{\bar{x}}$ is tame. It is metrizable because it is a subflow of the finite power X^n. Let $w_{\bar{x}} := u\bar{x}$.

For $\eta \in E(X, G)$, let $\Delta_n(\eta) : X^n \longrightarrow X^n$, be given by

$$\Delta_n(\eta)(y_1, \ldots, y_n) := \big(\eta(y_1), \ldots, \eta(y_n)\big).$$

Then

$$r_{\bar{x}} : E(X, G) \longrightarrow E(W_{\bar{x}}, G), \qquad r_{\bar{x}}(\eta) := \Delta_n(\eta)|_{W_{\bar{x}}},$$

is a continuous semigroup epimorphism. Set

$$\mathcal{N}_{\bar{x}} := r_{\bar{x}}[\mathcal{M}], \qquad v_{\bar{x}} := r_{\bar{x}}(u), \qquad L_{\bar{x}} := v_{\bar{x}}\mathcal{N}_{\bar{x}} = r_{\bar{x}}[K].$$

By Fact 2.1, $\mathcal{N}_{\bar{x}}$ is a minimal left ideal, $v_{\bar{x}}$ is idempotent, and

$$r_{\bar{x}}|_K : (K, \tau) \longrightarrow (L_{\bar{x}}, \tau)$$

is a continuous group epimorphism and a quotient map. Moreover,

$$(r_{\bar{x}}|_K)_*\mathfrak{h}_K = \mathfrak{h}_{L_{\bar{x}}}.$$

Let $e_{\bar{x}} : E(X, G) \longrightarrow X^n$ be given by $e_{\bar{x}}(\eta) := \eta(\bar{x})$, and set $\nu_{\bar{x}} := (e_{\bar{x}})_*\mu_K$. Since $\operatorname{supp}(\mu_K) \subseteq \mathcal{M}$, the measure $\nu_{\bar{x}}$ is supported on $W_{\bar{x}}$, and we regard it as a probability measure on that flow.

Let $\varphi \in C(W_{\bar{x}}, \mathbb{C})$, and choose an extension $\widetilde{\varphi} \in C(X^n, \mathbb{C})$. Using the definition of Haar decompression, $ku = k$ for $k \in K$, we obtain

$$\begin{aligned}
\int_{W_{\bar{x}}} \varphi \, d\nu_{\bar{x}} &= \int_K \widetilde{\varphi}(k\bar{x}) \, d\mathfrak{h}_K(k) \\
&= \int_K \widetilde{\varphi}\big(k(u\bar{x})\big) \, d\mathfrak{h}_K(k) \\
&= \int_K \varphi\big(r_{\bar{x}}(k)(w_{\bar{x}})\big) \, d\mathfrak{h}_K(k) \\
&= \int_{L_{\bar{x}}} \varphi\big(\ell(w_{\bar{x}})\big) \, d\mathfrak{h}_{L_{\bar{x}}}(\ell).
\end{aligned}$$

The last expression is the transfer associated with the Ellis group $L_{\bar{x}}$ of the minimal metrizable tame flow $W_{\bar{x}}$. By Theorem 4.3, $\nu_{\bar{x}}$ is the unique G-invariant regular Borel probability measure on $W_{\bar{x}}$.

By the finite-coordinate Stone–Weierstrass argument following Definition 5.1, $\mathcal{A}_{\mathrm{fin}}$ is uniformly dense in $C(E, \mathbb{C})$.

Fix $g \in G$ and $F \in \mathcal{A}_{\mathrm{fin}}$. Choose a representation $F(\eta) = \psi(\eta\bar{x})$, where $\eta \in E$, with $\bar{x} \in X^n$ and $\psi \in C(X^n, \mathbb{C})$. Since $\nu_{\bar{x}}$ is G-invariant,

$$\begin{aligned}
\int_E F(g\eta) \, d\mu_K(\eta) &= \int_E \psi\big(g(\eta\bar{x})\big) \, d\mu_K(\eta) \\
&= \int_{W_{\bar{x}}} \psi(gw) \, d\nu_{\bar{x}}(w) \\
&= \int_{W_{\bar{x}}} \psi(w) \, d\nu_{\bar{x}}(w) \\
&= \int_E F(\eta) \, d\mu_K(\eta).
\end{aligned}$$

Uniform density extends this identity to every $F \in C(E,\mathbb{C})$. Hence,

$$g_*\mu_K = \mu_K$$

for every $g \in G$. □

Theorem 5.8. *Let (X,G) be tame and hereditarily amenable, and assume that every countable subset of $C(X,\mathbb{C})$ is contained in a separable unital G-invariant C^*-subalgebra of $C(X,\mathbb{C})$, where $(g\cdot f)(x) := f(g^{-1}x)$. Let*

$$\mathcal{M} \trianglelefteq_m E(X,G), \qquad u^2 = u \in \mathcal{M}, \qquad K := u\mathcal{M}.$$

Then the Haar decompression μ_K is a G-invariant regular Borel probability measure on $E(X,G)$.

Theorem 5.8 extends Theorem 5.7. Indeed, if X is metrizable, then $C(X,\mathbb{C})$ is separable, so the separable-generation hypothesis is automatic. It is also automatic when G is countable: for every countable set $\mathcal{D} \subseteq C(X,\mathbb{C})$, the algebra

$$A := C^*(\{1\} \cup \{g\cdot f : g \in G,\ f \in \mathcal{D}\})$$

is a separable unital G-invariant C^*-subalgebra of $C(X,\mathbb{C})$ containing $\mathcal{D}$. Thus, the theorem applies, in particular, to every tame hereditarily amenable flow with a countable acting group (without any metrizability assumption on X).

The proof reduces the general case to the metrizable case of Theorem 5.7.

Proof. Let $E := E(X,G)$. We shall use the functoriality of decompression for factor maps (already appearing in the proofs of Proposition 4.2 and Lemma 4.5). Namely, if $\pi : X \longrightarrow Y$ is a factor map, $\pi_E : E(X,G) \to E(Y,G)$ is the induced semigroup epimorphism, then

$$\pi_E|_K : (K,\tau) \longrightarrow (L,\tau)$$

is a continuous group epimorphism between Ellis groups and so

$$(\pi_E)_*\mu_K = \mu_L. \tag{1}$$

We reduce the assertion to Theorem 5.7. Fix $g \in G$, $F \in C(E,\mathbb{C})$, and let $\varepsilon > 0$. By the density of $\mathcal{A}_{\mathrm{fin}}$, there are $n \geqslant 1$, a tuple $\bar{x} = (x_1,\ldots,x_n) \in X^n$, and a function $\varphi \in C(X^n,\mathbb{C})$ such that, putting $F_0 : \eta \mapsto \varphi(\eta\bar{x})$, one has

$$\|F - F_0\|_\infty < \varepsilon. \tag{2}$$

By the complex Stone–Weierstrass theorem, finite sums of products of coordinate functions are uniformly dense in $C(X^n,\mathbb{C})$. Hence, there are functions $f_{r,i} \in C(X,\mathbb{C})$, with $1 \leqslant r \leqslant m$ and $1 \leqslant i \leqslant n$, such that

$$\psi(y_1,\ldots,y_n) := \sum_{r=1}^{m}\prod_{i=1}^{n} f_{r,i}(y_i)$$

satisfies $\|\varphi - \psi\|_\infty < \varepsilon$. By the separable-generation hypothesis, the finite family

$$\{f_{r,i} : 1 \leqslant r \leqslant m,\ 1 \leqslant i \leqslant n\}$$

is contained in a separable unital G-invariant C^*-subalgebra $A \subseteq C(X,\mathbb{C})$. Let

$$\pi : X \longrightarrow Y := \mathrm{Spec}(A)$$

be the associated factor map. Since A is separable, Y is metrizable; since tameness passes to factors, (Y,G) is tame.

The flow Y is hereditarily amenable. Indeed, if $Z \subseteq Y$ is a subflow, then $\pi^{-1}[Z]$ is a subflow of X. An invariant regular Borel probability measure on $\pi^{-1}[Z]$ pushes forward under π to an invariant regular Borel probability measure on Z.

Let $\pi_E : E(X,G) \longrightarrow E(Y,G)$ be the induced semigroup epimorphism, and let

$$\mathcal{N} := \pi_E[\mathcal{M}], \qquad v := \pi_E(u), \qquad L := v\mathcal{N}.$$

By (1),

$$(\pi_E)_*\mu_K = \mu_L.$$

The flow Y is metrizable, tame and hereditarily amenable, so Theorem 5.7 implies that μ_L is G-invariant.

For every r, i, let $\widehat{f}_{r,i} \in C(Y,\mathbb{C})$ be the function determined by

$$f_{r,i} = \widehat{f}_{r,i} \circ \pi.$$

Define

$$\widehat{\psi}(z_1,\ldots,z_n) := \sum_{r=1}^{m}\prod_{i=1}^{n} \widehat{f}_{r,i}(z_i)$$

and

$$\widehat{F}(\theta) := \widehat{\psi}\big(\theta(\pi(x_1)),\ldots,\theta(\pi(x_n))\big) \qquad (\theta \in E(Y,G)).$$

Then $\widehat{F} \in C(E(Y,G),\mathbb{C})$ and, for every $\eta \in E$,

$$\begin{aligned}
\widehat{F}\big(\pi_E(\eta)\big) &= \widehat{\psi}\big(\pi_E(\eta)(\pi(x_1)),\ldots,\pi_E(\eta)(\pi(x_n))\big)\\
&= \widehat{\psi}\big(\pi(\eta(x_1)),\ldots,\pi(\eta(x_n))\big)\\
&= \psi\big(\eta(x_1),\ldots,\eta(x_n)\big).
\end{aligned}$$

It follows from (2) and $\|\varphi - \psi\|_\infty < \varepsilon$ that

$$\|F - \widehat{F}\circ\pi_E\|_\infty < 2\varepsilon. \tag{3}$$

Using $(\pi_E)_*\mu_K = \mu_L$, equivariance of π_E, and G-invariance of μ_L, we obtain

$$\begin{aligned}
\int_E \widehat{F}\big(\pi_E(g\eta)\big)\,d\mu_K(\eta) &= \int_{E(Y,G)} \widehat{F}(g\theta)\,d\mu_L(\theta)\\
&= \int_{E(Y,G)} \widehat{F}(\theta)\,d\mu_L(\theta)\\
&= \int_E \widehat{F}\big(\pi_E(\eta)\big)\,d\mu_K(\eta).
\end{aligned}$$

Therefore, by (3),

$$\begin{aligned}
\left|\int_E F(g\eta)\,d\mu_K(\eta) - \int_E F(\eta)\,d\mu_K(\eta)\right| &\leqslant 2\|F - \widehat{F}\circ\pi_E\|_\infty\\
&< 4\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary,

$$\int_E F(g\eta)\,d\mu_K(\eta) = \int_E F(\eta)\,d\mu_K(\eta).$$

As $g \in G$ and $F \in C(E,\mathbb{C})$ were arbitrary, μ_K is G-invariant. $\square$

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Appendix A. Examples

We collect three examples. The first shows that the inclusion of an Ellis group into the enveloping semigroup need not be Borel. The second computes a split-circle dihedral flow in which Haar decompression restores invariance not present on either Ellis group separately. The third shows that tameness is essential in the characterization of amenability of the Ellis flow.

A.1. **Non-Borel inclusion of Ellis group.** Put $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, equipped with its usual circular order. For $A \subseteq \mathbb{T}$, let $\mathrm{Split}(\mathbb{T}; A)$ be the circularly ordered space obtained by replacing each $t \in A$ by two adjacent points $t^- < t^+$ and leaving every $t \notin A$ unsplit. We use the convention $t^- = t^+ = t$ at unsplit points, and equip $\mathrm{Split}(\mathbb{T}; A)$ with the cyclic-order topology, that is, the topology generated by the open cyclic intervals.

Choose $\alpha, \beta \in \mathbb{R}$ such that $1, \alpha, \beta$ are linearly independent over $\mathbb{Q}$. For $\xi \in \{\alpha, \beta\}$, put $A_\xi := \{n\xi : n \in \mathbb{Z}\} \subseteq \mathbb{T}$ and let $X_\xi := \mathrm{Split}(\mathbb{T}; A_\xi)$. Let $T_\xi(t^\rho) := (t+\xi)^\rho$, and equip $X := X_\alpha \times X_\beta$ with the diagonal $\mathbb{Z}$-action generated by $T_\alpha \times T_\beta$.

By [GM22, Proposition 4.1], this is a minimal metrizable tame system, and

$$\mathcal{M} := \left\{ \left(p^\sigma_{\alpha,\gamma}, p^\rho_{\beta,\eta}\right) : \sigma, \rho \in \{+,-\},\ \gamma, \eta \in \mathbb{T} \right\}$$

is a minimal $\mathbb{Z}$-subflow of $E(X, \mathbb{Z})$, hence a minimal left ideal. Here $p^\sigma_{\xi,\gamma}(t^\rho) := (t+\gamma)^\sigma$. The composition rule

$$p^\sigma_{\xi,\gamma} \circ p^\rho_{\xi,\eta} = p^\sigma_{\xi,\gamma+\eta}$$

shows that $u := \left(p^+_{\alpha,0}, p^+_{\beta,0}\right)$ is an idempotent and that

$$K := u\mathcal{M} = \left\{ \left(p^+_{\alpha,\gamma}, p^+_{\beta,\eta}\right) : (\gamma, \eta) \in \mathbb{T}^2 \right\}.$$

The natural factor map $X \to \mathbb{T}^2$ induces a semigroup epimorphism

$$\pi_E : E(X, \mathbb{Z}) \longrightarrow \mathbb{T}^2.$$

By Fact 2.1, its restriction $\pi_E|_K : (K, \tau) \longrightarrow \mathbb{T}^2$, given by $\left(p^+_{\alpha,\gamma}, p^+_{\beta,\eta}\right) \longmapsto (\gamma, \eta)$, is a continuous quotient group homomorphism. It is bijective, and therefore it is a homeomorphism.

On the other hand, [GM22, Proposition 4.1] shows that

$$\Delta := \left\{ \left(p^+_{\alpha,\gamma}, p^+_{\beta,-\gamma}\right) : \gamma \in \mathbb{T} \right\}$$

is discrete in the topology inherited from $E(X,\mathbb{Z})$. In the τ-topology it is a closed ordinary circle. Choose a non-Borel set $B \subsetneq \mathbb{T}$ and put

$$\Delta_B := \left\{ \left(p^+_{\alpha,\gamma}, p^+_{\beta,-\gamma}\right) : \gamma \in B \right\}.$$

Since Δ is discrete in the inherited topology, there is an open set $O \subseteq E(X,\mathbb{Z})$ such that $O \cap \Delta = \Delta_B$. If the inclusion $j : (K,\tau) \to E(X,\mathbb{Z})$ were Borel, then $j^{-1}[O] \cap \Delta = \Delta_B$ would be Borel in the ordinary circle Δ, a contradiction.

A.2. **Haar decompression restores invariance.** This example exhibits two Ellis groups which are interchanged by the reflection, although their Haar decompressions coincide and are G-invariant.

The *fully split circle* is $X = \mathbb{T}^{\bowtie} := \mathrm{Split}(\mathbb{T};\mathbb{T})$ (cf. Subsection A.1). Thus every $t \in \mathbb{T}$ is replaced by two adjacent points $t^- < t^+$. The arcs $[a^+, b^-]$, $a \neq b$, form a clopen basis, and the map $\pi : X \longrightarrow \mathbb{T}$, $\pi(t^\pm) := t$, is continuous and onto. For $\rho \in \{+,-\}$, set $\rho^1 := \rho$ and let ρ^{-1} be the opposite sign. For $\varepsilon \in \{\pm 1\}$ and $a \in \mathbb{T}$, define

$$P_{\varepsilon,a}(t^\rho) := (\varepsilon t + a)^{\rho^\varepsilon}.$$

Then $P_{\sigma,d} \circ P_{\varepsilon,a} = P_{\sigma\varepsilon, d+\sigma a}$. Fix an irrational $\alpha \in \mathbb{T}$, put $D := \langle \alpha \rangle$, and let

$$R := P_{1,\alpha}, \qquad J := P_{-1,0}, \qquad G := \langle R, J \rangle = \{P_{\varepsilon,d} : \varepsilon \in \{\pm 1\},\ d \in D\}.$$

Thus, $G \cong D_\infty$ is amenable, and (X, x_0, G) is an ambit for $x_0 := 0^+$. The flow is tame: the index-two subgroup $\langle R \rangle$ acts tamely by [GM18a, Theorem 4.5], and the G-orbit of every $f \in C(X)$ is the union of two translates of its $\langle R \rangle$-orbit.

For $\delta \in \{+,-\}$, $\varepsilon \in \{\pm 1\}$, and $a \in \mathbb{T}$, define the collapsed map

$$P^\delta_{\varepsilon,a}(t^\rho) := (\varepsilon t + a)^\delta, \qquad M_\delta := \{P^\delta_{\varepsilon,a} : \varepsilon \in \{\pm 1\},\ a \in \mathbb{T}\}.$$

The superscript records the side onto which every two-point fibre is collapsed. The product formulas are

$$P^\delta_{\varepsilon,a} \circ P^{\delta'}_{\varepsilon',b} = P^\delta_{\varepsilon\varepsilon', a+\varepsilon b}, \quad P_{\sigma,d} \circ P^\delta_{\varepsilon,a} = P^{\delta^\sigma}_{\sigma\varepsilon, d+\sigma a}, \quad P^\delta_{\varepsilon,a} \circ P_{\sigma,d} = P^\delta_{\varepsilon\sigma, a+\varepsilon d}.$$

Lemma A.1. *One has*

$$E(X,G) = G \mathbin{\dot\cup} M_+ \mathbin{\dot\cup} M_-.$$

Moreover, $M := M_+ \cup M_-$ is the unique minimal left ideal. For $\delta \in \{+,-\}$,

$$u_\delta := P^\delta_{1,0} \qquad \text{and} \qquad K_\delta := u_\delta M = M_\delta.$$

Then u_δ is idempotent, and

$$(K_\delta, \tau) \longrightarrow O(2) \cong \{\pm 1\} \ltimes \mathbb{T}, \qquad P^\delta_{\varepsilon,a} \longmapsto (\varepsilon, a),$$

is an isomorphism of compact Hausdorff topological groups.

Proof. If $d_i \in D$ approaches a strictly from side δ, then

$$P_{\varepsilon,d_i} \longrightarrow P^\delta_{\varepsilon,a}$$

pointwise on X. Conversely, after passing to a subnet, every net of principal maps has constant orientation and its translation parameters are either eventually constant or approach their limit strictly from one side. This gives the classification and shows that, for each fixed $\varepsilon \in \{\pm 1\}$, the map

$$\mathbb{T}^{\bowtie} \longrightarrow E(X,G), \qquad a^\delta \longmapsto P^\delta_{\varepsilon,a},$$

is a homeomorphism onto its image.

The product formulas imply that M is a left ideal and that every nonempty left ideal contains M, and they give the asserted idempotents and Ellis groups. The factor map π induces a continuous semigroup epimorphism

$$\pi_E : E(X,G) \longrightarrow O(2)$$

which sends both $P_{\varepsilon,a}$ and $P^{\delta}_{\varepsilon,a}$ to (ε, a). Its restriction to K_δ is a continuous bijective homomorphism of compact Hausdorff groups, and hence is an isomorphism. $\square$

Lemma A.2. *For every $f \in C(X,\mathbb{C})$, the set*

$$J_f := \{a \in \mathbb{T} : f(a^+) \neq f(a^-)\}$$

is countable. The two sections

$$a \longmapsto a^+ \qquad \text{and} \qquad a \longmapsto a^-$$

are Borel.

Proof. For $n \geqslant 1$, put

$$J_{f,n} := \left\{ a \in \mathbb{T} : |f(a^+) - f(a^-)| \geqslant \frac{1}{n} \right\}.$$

If $J_{f,n}$ were infinite, it would contain a sequence (a_i) of distinct points converging to some $a \in \mathbb{T}$. Passing to a subsequence, all a_i approach a from the same side. Then the pairs a_i^-, a_i^+ converge to the same one of a^-, a^+, contradicting continuity of f. Thus, each $J_{f,n}$ is finite, and $J_f = \bigcup_{n\geqslant 1} J_{f,n}$ is countable. The Borel assertion follows directly by taking preimages of the clopen basis arcs $[a^+, b^-]$. $\square$

Let $\mathfrak{h}_\delta$ be the normalized Haar measure on K_δ, and let μ_δ be its decompression. Let da denote normalized Haar measure on $\mathbb{T}$. Under the identification

$$K_\delta \cong \{\pm 1\} \ltimes \mathbb{T}, \qquad P^{\delta}_{\varepsilon,a} \longmapsto (\varepsilon, a),$$

the measure $\mathfrak{h}_\delta$ corresponds to $\frac{1}{2}\sum_{\varepsilon\in\{\pm 1\}} \delta_\varepsilon \otimes da$.

Proposition A.3. *For every $F \in C(E(X,G),\mathbb{C})$,*

$$\int_{E(X,G)} F\, d\mu_\delta = \frac{1}{2} \sum_{\varepsilon\in\{\pm 1\}} \int_{\mathbb{T}} F(P^{\delta}_{\varepsilon,a})\, da. \tag{A.1}$$

The two decompressions coincide, $\mu_+ = \mu_- =: \mu_E$, and μ_E is G-invariant.

The ambit (X, x_0, G) is uniquely ergodic. Its unique invariant probability measure ν_X satisfies

$$\int_X \varphi\, d\nu_X = \frac{1}{2} \int_{\mathbb{T}} \big(\varphi(a^+) + \varphi(a^-)\big)\, da \tag{A.2}$$

where $\varphi \in C(X,\mathbb{C})$, and we have $(\mathrm{ev}_{x_0})_*\mu_+ = (\mathrm{ev}_{x_0})_*\mu_- = \nu_X$.

Proof. The identification $K_\delta \cong O(2)$ gives (A.1). For fixed ε, the restriction of F to the double circle

$$\{P^{\delta}_{\varepsilon,a} : \delta \in \{+,-\},\ a \in \mathbb{T}\}$$

is continuous. Lemma A.2 therefore shows that $F(P^{+}_{\varepsilon,a}) = F(P^{-}_{\varepsilon,a})$ for Lebesgue-almost every a, and hence $\mu_+ = \mu_-$. The product formulas imply

$$(P_{\sigma,d} \circ -)_*\mu_\delta = \mu_{\delta^\sigma},$$

so the common decompression is G-invariant.

Since $P^{\delta}_{\varepsilon,a}(x_0) = a^{\delta}$, formula (A.1) and Lemma A.2 give the evaluation identity and (A.2).

Finally, let ν be any G-invariant probability measure on X. It is atomless because all R-orbits are infinite, and $\pi_*\nu$ is Lebesgue measure by unique ergodicity of the irrational rotation. For $\varphi \in C(X)$, put $\bar{\varphi}(a) := \varphi(a^+)$. The function $\bar{\varphi}$ is Borel, and

$$\varphi(x) = \bar{\varphi}(\pi(x))$$

outside the countable union of fibres over J_φ. Since ν is atomless, this exceptional set is ν-null. Thus,

$$\int_X \varphi \, d\nu = \int_{\mathbb{T}} \bar{\varphi}(a) \, da,$$

which is independent of ν. Hence, $\nu = \nu_X$. □

We emphasize that the individual Ellis groups are not G-invariant: the rotation preserves each of them, whereas the reflection interchanges them,

$$R \circ K_\delta = K_\delta, \qquad J \circ K_\delta = K_{\delta^{-1}}.$$

Nevertheless, their common decompression $\mu_E = (j_+)_*\mathfrak{h}_+ = (j_-)_*\mathfrak{h}_-$ is G-invariant.

A.3. **Necessity of tameness.** The following example shows that hereditary amenability alone does not imply amenability of the Ellis flow, so the tameness hypothesis in Theorem 5.5 and Corollary 5.6 is indispensable.

Proposition A.4. *There is a metrizable hereditarily amenable flow (X, G) such that $(E(X, G), G)$ is not amenable.*

Proof. Let $H := F_2 = \langle a, b\rangle$ and $S := \{a, a^{-1}, b, b^{-1}\}$. We identify the boundary ∂H with the compact space of infinite reduced words in the alphabet S, equipped with the cylinder topology and the natural left H-action. For $s \in S$, let $[s] \subseteq \partial H$ denote the cylinder of words beginning with s.

Put $C_2 := \mathbb{Z}/2\mathbb{Z}$, $X := C_2^{H\times S}$, and let H act by

$$(h \cdot x)(q, s) := x(h^{-1}q, s).$$

Let

$$D := \bigoplus_{(q,s)\in H\times S} C_2 = \{d \in X : \mathrm{supp}(d) \text{ is finite}\}$$

be the dense subgroup of finitely supported functions, and put $G := D \rtimes H$. The affine action is

$$(d, h) \cdot x := d + h \cdot x.$$

Every D-orbit is dense in X, so the G-flow X is minimal. The normalized Haar measure on the compact group X is invariant under translations by D and under the coordinate permutations induced by H. Thus the G-flow X is amenable. Since it is minimal, its only nonempty subflow is X itself, and hence (X, G) is hereditarily amenable.

Define

$$\iota : \partial H \longrightarrow X, \qquad \iota(\xi)(q, s) := \mathbf{1}_{[s]}(q^{-1}\xi).$$

This map is continuous and H-equivariant. It is easy too see that ι is injective; since ∂H is compact and X is Hausdorff, it is an embedding. Put $Y := \iota[\partial H]$. The H-flow $Y \cong \partial H$ has no invariant probability measure. Indeed, for every $s \in S$,

$$s^{-1}[s] = \partial H \setminus [s^{-1}].$$

If λ were an invariant Borel probability measure, then $\lambda([s])+\lambda([s^{-1}])=1$. Taking $s=a$ and $s=b$ would make the sum of the measures of the four first-letter cylinders equal to 2, although these cylinders partition ∂H.

Now, we give Y the G-action induced by the quotient $G \twoheadrightarrow H$, so that D acts trivially. Define

$$\delta : X^2 \longrightarrow X, \qquad \delta(x,y) := y - x,$$

and put $W := \delta^{-1}[Y]$. For $(d,h) \in G$,

$$\delta\big((d,h)\cdot x, (d,h)\cdot y\big) = h \cdot \delta(x,y).$$

Therefore, W is a G-subflow and $\delta|_W : W \longrightarrow Y$ is a continuous G-map. It is onto, since $\delta(0,y) = y$ for every $y \in Y$. Thus it is a factor map.

If W carried a G-invariant probability measure, its pushforward under $\delta|_W$ would be a G-invariant probability measure on Y, a contradiction. Hence, the second power X^2 contains a nonamenable subflow, so (X,G) is not finitely power-hereditarily amenable. By Theorem 5.3, the Ellis flow $(E(X,G),G)$ is not amenable. In particular, (X,G) is not tame by Corollary 5.6. $\square$

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