INVARIANT MEASURES IN SIMPLE AND IN SMALL THEORIES

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Abstract. We give examples of (i) a simple theory with a formula (with parameters) which does not fork over $\emptyset$ but has μ-measure 0 for every automorphism invariant Keisler measure μ, and (ii) a definable group G in a simple theory such that G is not definably amenable, i.e. there is no translation invariant Keisler measure on G.

We also discuss paradoxical decompositions both in the setting of discrete groups and of definable groups, and prove some positive results about small theories, including the definable amenability of definable groups.

1. Introduction and preliminaries

We begin with an introduction for a general audience. The paper is about *amenability* in model-theoretic environments, with both nonexistence and existence theorems. The expression "amenability" often refers to the existence of a finitely additive probability measure μ on some suitable collection $\mathcal{B}$ of subsets of a given set X, which is invariant under a certain action of a certain group G. When $X = G$, $\mathcal{B}$ is the collection of *all* subsets of G, and the action is the action of G on $\mathcal{B}$ by left translation, then we obtain precisely the "classical" notion of amenability of G as a discrete group. Remaining in this context, one could replace the Boolean algebra of *all* subsets of G by some other Boolean algebra of subsets of G invariant under left translation, and ask for amenability with respect to the new Boolean algebra. In some interesting examples one obtains strikingly different behaviour when passing to natural and reasonably rich Boolean algebras. For example the free group F_2 on two generators is not amenable as a discrete group, but if we choose instead the Boolean algebra $\mathcal{B}$ to be the collection of subsets of F_2 which are *definable* (with parameters) in the structure $(F_2, \times)$, then not only do we get amenability, but "unique ergodicity": there is a unique invariant measure which is moreover $\{0,1\}$-valued. This is a consequence of the fact that the first order theory $Th((F_2, \times))$ of the structure $(F_2, \times)$ has a property called *stability*, which can be summed up by the statement that "any stable group is (uniquely) definably amenable". In addition to the free group, all commutative groups and all algebraic groups over algebraically closed fields are stable. A more general class of first order theories, the class of so-called *simple theories* was defined and studied beginning in the 1980's, often in the context of specific examples of independent interest such as *pseudofinite fields*

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(logical limits of finite fields). Early applications were to algebraic groups over finite fields [12]. Groups definable in pseudofinite fields are definably amenable witnessed by a "nonstandard counting measure". It was asked around ten years ago whether groups definable in any simple theory are definably amenable. One of our main theorems appearing in Section 3 (as in (ii) of the abstract) is a counterexample.

We now give some background for (i) in the abstract, which on the face of it, may seem less accessible to the general reader. The context, implicit in the paragraph above, is a structure M in the sense of model theory, namely an underlying set which we also call M, equipped with a collection D of distinguished subsets of various Cartesian powers M^n of M, including the diagonal $\subset M^2$. The automorphism group $Aut(M)$ is the group of permutations of M which fix setwise each of the distinguished sets. Closing under the operations of finite Boolean combination, and projection (from M^{n+1} to M^n), we obtain the class D_1 of $\emptyset$-definable sets. For $X \subseteq M^{n+k}$ in D_1, and $\bar{a} \in M^k$, let $X_{\bar{a}} = \{\bar{b} \in M^n : (\bar{b}, \bar{a}) \in X\}$. These various $X_{\bar{a}}$ (as X and $\bar{a}$ vary), are called the definable (with parameters) sets in the structure M. $Aut(M)$ acts on the collection of definable sets. We fix some ambient Cartesian power M^n of M, and consider the Boolean algebra $\mathcal{B}$ of definable subsets of M^n, again acted on by $Aut(M)$. We make an additional assumption on M (saturation) ensuring that $Aut(M)$ is "large" in a suitable sense. One of the recent waves of connections between model theory and combinatorics, specifically [10], was largely based on an analogy between two kinds of ($Aut(M)$-)invariant ideals of $\mathcal{B}$: the "forking ideal" I_f in the case that the first order theory $Th(M)$ is simple (see below for details and definitions) and for any invariant finitely additive probability measure μ on $\mathcal{B}$, such as the nonstandard counting measure when M is pseudofinite, the μ-measure 0 ideal I_μ. We always have that $I_f \subseteq I_\mu$, and it was an open question whether for simple theories I_f is precisely the intersection of the I_μ as μ varies over all invariant measures. We answer this question negatively in this paper. The main example is constructed in Section 2, producing a theory with many invariant measures, and a formula which is in I_μ for all μ but not in I_f. On the other hand, a corollary of the main theorem in Section 3, is the existence of a simple theory and a "sort" on which there are *no* invariant measures, giving another route to a negative answer to the question.

Another aspect of the paper, which is made explicit in Section 4, concerns the "paradoxical decomposition" obstructions to amenability in the various senses. We are interested in definable versions of paradoxical decompositions, and which model theoretic properties of theories T are incompatible with definable paradoxical decompositions. Various results are obtained including the definable amenability of definable groups in "small" theories (where the Boolean algebras of $\emptyset$-definable sets admit a Cantor-Bendixon analysis).

We now pass on to a more technical introduction, for readers familiar with model theory.

In stable theories, Keisler measures are very well understood, originating in [18]. There was a comprehensive study of Keisler measures in NIP theories, starting with [14], [15], [16]. It is very natural to ask what happens in simple theories. The main thrust of the current paper is to give counterexamples to some of these questions, as in the abstract. Another aspect of the paper is to give some positive results in the case of countable small theories.

Partly as motivation we will, in this introduction, discuss and recall what is known about Keisler measures and forking in general, as well as in stable and NIP theories and then state the questions which are answered in the body of this paper.

Our model-theoretic notation is standard. Models will be denoted by $M, N, \ldots$ and subsets (sets of parameters) by $A, B, \ldots$. If and when we work with a complete theory T then we often work in a sufficiently saturated model, called $\mathfrak{C}$ or $\bar{M}$; $a, b, \ldots$ refer to tuples in models of T unless we say otherwise or clear from the context.

The study of *stable theories* is connected to *categoricity* and is largely due to Shelah [27]. There are many other reference books, including [25]. In the middle 1990's the machinery of stability theory was extended or generalized to the class of *simple theories* which had been defined earlier by Shelah in [28]. This development was closely connected to and went in parallel with the concrete analysis of several kinds of structures and theories, including Lie coordinatizable and smoothly approximable structures ([17], [3]), and bounded PAC fields ([13] and the later published [9]), using tools with a stability theoretic flavour. In fact Hrushovski's S_1-*theories* already provided a certain abstract finite rank environment for adapting stability to the more general situations. The technical breakthroughs came with Byunghan Kim's thesis [19], [20] followed by [22]. Kim showed that all the machinery of nonforking independence extended word-for-word from stable theories to simple theories, except for stationarity of types over models (or more generally algebraically closed sets), and [22] found the appropriate weak version of the stationarity theory: the Independence Theorem over a model, or more generally for Lascar strong types. The latter, improved to so-called Kim-Pillay strong types, migrated and became essential in all of model theory, and also made connections to combinatorics and Lie groups possible, although we still do not know, whether this level of generality, versus the strong types of Shelah, is really needed in simple theories. The expression "Independence Theorem" already appears in the earlier work on S_1-theories, and was borrowed from there. In addition to the original papers, there are several good texts on simple theories [29], [21], [1]. The original definition of simplicity was in terms of not having the "tree property". We will define it here in terms of "dividing" as it is an opportunity to introduce dividing and forking.

Definition 1.1. (i) A formula $\phi(x, b)$ *divides* over A if there exists an A-indiscernible sequence $(b_i : i < \omega)$ with $b = b_0$ such that $\{\phi(x, b_i) : i < \omega\}$ is inconsistent.

(ii) If $\Sigma(x)$ is a partial type over a set B closed under conjunctions and $A \subseteq B$, then $\Sigma(x)$ *divides* over A if some formula $\phi(x, b) \in \Sigma(x)$ divides over A.

(iii) A formula *forks* over A if it implies a finite disjunction of formulas each of which divides over A.

(iv) For $\Sigma(x)$, $A \subseteq B$ as in (ii), $\Sigma(x)$ *forks* over A if some formula in $\Sigma(x)$ forks over A.

(v) The complete theory T is said to be *simple* if for any complete type $p(x) \in S(B)$ there is a subset $A \subseteq B$ of cardinality at most $|T|$ such that $p(x)$ does not divide over A.

In simple theories, dividing and forking coincide. Stable theories can be characterized as simple theories such that for any model M, $p(x) \in S(M)$, and $M \prec N$, $p(x)$ has a *unique* extension to a complete type $q(x) \in S(N)$ which does not fork over M.

The *stable forking conjecture* says that in a simple theory T, forking is explained by the "stable part" of T (in a sense that we will not describe in detail). There are many simple theories T which have a stable reduct T_0 (with quantifier elimination) such that T is the model companion of T_0 together with the new relations (possibly modulo some mild universal theory). Typically in such a situation forking in T is witnessed by forking in T_0 so the stable forking conjecture holds. Our two main examples of simple theories will have this feature.

In a simple theory T we will say that a and b are *independent over* A (in the sense of nonforking) if $tp(a/A, b)$ does not fork over A. This satisfies a number of properties: invariance, finite character, local character, existence of nonforking extensions, symmetry, transitivity, and the "Independence Theorem over a model". Moreover the existence of an "abstract independence relation" satisfying these properties implies simplicity of T as well as that this relation coincides with nonforking. This will be used in Sections 2 and 3 and we will give a few more details there. Among the "simplest" simple theories are the theories of SU-rank 1, where every complete nonalgebraic 1-type has only algebraic forking extensions.

Although NIP theories are *not* really objects of study in the current paper, they form part of the motivation. A theory T is NIP if there is no formula $\phi(x, y)$ and a_i for $i \in \omega$ and b_S for $S \subseteq \omega$ in some model M of T such that for all i, S, $M \models \phi(a_i, b_S)$ iff $i \in S$. NIP theories are generalization of stable theories in an orthogonal direction from simple theories, and in fact T is stable if and only if T is both simple and NIP. Although forking is not so well-behaved in NIP unstable theories, it still plays a big role. In particular, forking coincides with dividing over models [5], and global nonforking extensions of types over a model M are precisely extensions which are invariant under automorphisms fixing M pointwise. For a type $p(x)$ over a set A its global nonforking extensions (if they exist) are rather invariant over the bounded closure "$bdd(A)$".

The other main ingredients in this paper are *Keisler measures.* Given a structure M (or model M of T), and variable x, a Keisler measure μ_x over M is a finitely additive probability measure on the Boolean algebra of definable (with parameters) subsets of the x-sort in M. Keisler measures generalize complete types $p(x)$ over M which are the special case where the measure is $\{0, 1\}$-valued (0 for false, 1 for true). It took a long time for Keisler measures to become part of everyday model theory (see [4] for a quick survey). They were studied by Keisler in [18] which is, on the face of it, about NIP theories, but where, among the main points, is that for stable theories, *locally* (formula-by-formula) Keisler measures are weighted, possibly infinite, sums of types. (See also [26] where this is used to give a pseudofinite account of the stable regularity lemma.) In the NIP environment, Keisler measures were a very useful tool in solving some conjectures about definable groups in o-minimal structures [14]. In [15], [16], the ubiquity of automorphism (translation) invariant Keisler measures in NIP theories (groups) was pointed out. In [11] a first-order theory was defined to be *amenable* if every complete type over $\emptyset$ extends to a global automorphism invariant Keisler measure.

For pseudofinite fields, the nonstandard counting measure provides both automorphism invariant measures on definable sets, as well as translation invariant measures on definable groups (with very good definability properties). The examples given in Sections 2 and 3 of the current paper show in particular that such behaviour does not extend to simple theories in general.

We will now describe the main results of the paper, with motivations coming from what is known in the stable context.

We will talk about (non-) forking over $\emptyset$, but $\emptyset$ can be systematically replaced by any small set A of parameters.

The following is well-known ([14], [24]) but we recall the proof anyway.

Fact 1.2. *(No assumption on T.) Suppose $\phi(x,b)$ forks over $\emptyset$. Then $\mu(\phi(x,b)) = 0$ for any automorphism invariant global Keisler measure $\mu(x)$.*

Proof. Working in the saturated model $\bar{M}$ we may assume that $\phi(x,b)$ divides over $\emptyset$, witnessed by indiscernible sequence $(b_0, b_1, \ldots)$ with $b_0 = b$ such that $\{\phi(x,b_i) : i < \omega\}$ is inconsistent. So $\phi(x,b_0)\wedge\phi(x,b_1)\wedge\ldots\wedge\phi(x,b_k)$ is inconsistent for some $k \geq 1$. Assume for a contradiction that $\mu(\phi(x,b)) > 0$ for some automorphism invariant global Keisler measure μ. Choose $0 \leq r < k$ maximum such that $\mu(\phi(x,b_0) \wedge \ldots \wedge \phi(x,b_r)) = t$ for some $t > 0$. Let $\psi_j(x) = \phi(x,b_0) \wedge \ldots \wedge \phi(x,b_{r-1}) \wedge \phi(x,b_j)$ for $j = r, r+1, r+2, \ldots$. Then by indiscernibility, invariance of μ and choice of r, we have that $\mu(\psi_j(x))) = t$ for all $j \geq r$, but $\mu(\psi_j(x) \wedge \psi_{j'}(x)) = 0$ for $r \leq j < j'$ — a contradiction as $\mu(x = x) = 1$. □

Remark 1.3. Suppose T is stable (and complete in language L), and $p(x)$ is a complete type over $\emptyset$. Then there is a global Keisler measure $\mu(x)$ (i.e. over a saturated model $\bar{M}$) which extends $p(x)$ and is $Aut(\bar{M})$-invariant. Moreover μ is the *unique* $Aut(\bar{M})$-invariant global Keisler measure extending p.

Proof. Again we give a proof, for completeness. The reader is referred to Section 2 of Chapter 1 of [25] for notation and facts that we use. Fix a finite set Δ of L-formulas of the form $\phi(x,y)$, and consider the collection of $p'(x)|\Delta$ where p' is a global nonforking extension of p. We know that there are only finitely many such, say $p_1, \ldots, p_n$. Let μ_Δ be the average of $\{p_1, \ldots, p_n\}$, namely for each $\phi(x,y) \in \Delta$ and $b \in M$, $\mu_\Delta(\phi(x,b)) = (1/n)(\sum p_i(\phi(x,b)))$ (where $p_i(\phi(x,b)) = 1$ if $\phi(x,b) \in p_i$ and 0 otherwise).

One has to check that $\Delta \subseteq \Delta'$ implies that $\mu_{\Delta'}$ agrees with μ_Δ on Δ-formulas, so that the directed union of the μ_Δ gives a global Keisler measure μ. For this we use transitivity of the action of $Aut(\bar{M})$ on the set of global nonforking extensions of p. From the definition of μ and invariance of non-forking, we deduce that μ is $Aut(\bar{M})$-invariant.

Uniqueness of μ follows from Fact 1.2. □

Corollary 1.4. *Suppose that T is stable and $\phi(x,b)$ is a formula which does not fork over $\emptyset$. Then there is an $Aut(\bar{M})$-invariant global Keisler measure giving $\phi(x,b)$ positive measure.*

Proof. Let p' be a global type which contains $\phi(x,b)$ and does not fork over $\emptyset$, and let p be the restriction of p' to $\emptyset$. The $Aut(\bar{M})$-invariant Keisler measure extending p constructed in Remark 1.3 gives $\phi(x,b)$ positive measure. □

A weak version of the corollary above holds in NIP theories using Proposition 4.7 of [15].

The issue for the current paper is what happens in simple theories, where the role, if any, of Keisler measures was not well understood. We will expand on some earlier comments. We fix a complete theory T, saturated model $\bar{M}$, sort S, and the Boolean algebra $\mathcal{B}$ of definable (with parameters) in $\bar{M}$ subsets of the sort S. The ideal I_f is the collection of such definable sets which fork over $\emptyset$. For any $Aut(\bar{M})$-invariant Keisler measure μ on S, let I_μ be the ideal of definable sets with μ-measure 0. Fact 1.2 says that $I_f \subseteq I_\mu$ for all such μ. In [10], an $Aut(\bar{M})$-invariant ideal I of $\mathcal{B}$ was defined to be an S_1-ideal if for any L-formula $\phi(x,y)$ (where x is of sort S) and indiscernible sequence $(b_n : n < \omega)$, if $\phi(x,b_1) \wedge \phi(x,b_2) \in I$, then $\phi(x,b_1) \in I$. Such $S1$ ideals appeared in the "Stabilizer Theorem" from [10]. Among the analogies between the forking ideal I_f and the ideals I_μ is that (i) I_μ is an S_1 ideal, and (ii) for simple T, I_f is an S_1 ideal [20]. The open problem (raised also by both the first author and Leo Harrington in personal communications) is whether, in a simple theory T (and working in a fixed sort), I_f is the intersection of the I_μ for μ ranging over invariant global Keisler measures. In the light of Fact 1.2, this reduces to the question whether (in a simple theory) any formula (with parameters) which does not fork over $\emptyset$, has μ-measure > 0 for some invariant measure μ. Of course if there are no invariant Keisler measures on sort S, then the question has a negative answer, and Corollary 1.9 below gives such an example. However we are also interested in the situation where there do exist (many) invariant measures, namely where T is also amenable in the sense described earlier. So we prove:

Theorem 1.5. *There is a simple theory T (of SU-rank 1) which is amenable, together with a formula $\phi(x,b)$ which does not fork over $\emptyset$ but has measure 0 for all automorphism invariant global Keisler measures.*

We now turn to the case of definable groups. Recall:

Definition 1.6. Let G be a group definable (say without parameters) in a structure M. Then G is said to be *definably amenable* if there is a Keisler measure on G over M which is invariant under left translation by G.

So definable amenability is a function not just of $(G,\cdot)$ but of the ambient structure M.

Recall from Section 5 of [14] that definable amenability of G depends only on $Th(M)$, not the particular model chosen. The relation with paradoxical decompositions will be discussed in detail in Section 4. The group version of Remark 1.3 is:

Fact 1.7. *Stable groups are definably amenable. More precisely if $Th(M)$ is stable and G a group definable in M, then G is definably amenable. Moreover there is a* unique *left invariant Keisler measure on G (over M) which is also the unique right invariant Keisler measure.*

Explanation. This is well-known but spelled out in detail for the more general case of "generically stable" groups in Corollary 6.10 of [15]. Also it is done explicitly in the local (formula-by-formula) case in [8].

It was asked by several people, including the sixth author, whether groups definable in models of simple theories are definably amenable. Note that this *is* the case for groups definable in pseudofinite fields (or arbitrary pseudofinite theories). Nevertheless, our second main result is:

Theorem 1.8. *There is a simple theory (of SU-rank 1) and a definable group G in it which is NOT definably amenable.*

The usual move of expanding a theory by a new sort for a principal homogeneous space (PHS) for a definable group yields:

Corollary 1.9. *There is a simple theory which is NOT amenable. In fact there is a sort S with a unique* 1*-type over* $\emptyset$*, such that there is no global invariant Keisler measure on sort S.*

We recall briefly the situation for definable groups in NIP theories. First there DO exist non definably amenable groups; such as $SL(2,\mathbb{R})$ as a group definable in the real field. Nevertheless there is a very nice theory of definably amenable groups, beginning in [15], continued in [16, 6] and brought to a fairly comprehensive conclusion in [7]. The latter paper includes a *classification* of the translation invariant Keisler measures on definably amenable groups in NIP theories.

Theorem 1.5 will be proved in Section 2. Theorem 1.8 and Corollary 1.9 will be proved in Section 3. The constructions of the theories and structures which give these (counter-)examples are a bit complicated from the combinatorial point of view.

There is a general theory of "definable paradoxical decompositions" from [14], which gives obstructions to definable amenability of groups. A general problem is to determine which interesting model-theoretic properties are inconsistent with the existence of a definable paradoxical decomposition. In Section 4, we show directly that smallness of T (as well as stability) are such properties, yielding the definable amenability of groups definable in small theories and in stable theories (although the latter was given earlier in the paper). We also give a "simpler" witness to Theorem 1.8, in terms of certain invariants related to definable paradoxical decompositions. Finally we discuss Grothendieck rings of structures, and show the non-triviality of the *graded Grothendieck ring* of any structure with small theory.

2. A simple theory where forking is not detected by measures

Here we prove Theorem 1.5. We first give an overview and then the technical details. Recall first that for any group G and a free action of G on a set P we can consider P as a structure in a language with function symbols f_g for each $g \in G$. When G is infinite, all such structures are elementarily equivalent, the theory is strongly minimal and there is a unique 1-type over $\emptyset$. We will choose G to be the free group F_5 on 5 generators. We will add another sort O to the picture and a relation $R \subseteq O \times P$ and find $a_1, \dots, a_5$ in P such that $R(x, a_1)$, $R(x, a_2)$, $R(x, a_3)$ are disjoint infinite sets, which are contained in the union of $R(x, a_4)$ and $R(x, a_5)$. It will be done sufficiently generically such that there is still a unique 1-type realized in P, and the theory of the structure is simple (of SU-rank 1). As all of the a_i have the same type, any automorphism invariant Keisler measure (on the sort O) will assign the same measure to each of the $R(x, a_i)$, which will have to be 0. But $R(x, a_i)$ (being infinite) does not fork over $\emptyset$.

2.1. The universal theory.

As usual we mix up notation for symbols of the language and their interpretations. As above we have two sorts O, P, and relation $R \subseteq O \times P$. And it is convenient to only have function symbols for 5 free generators of F_5 and their inverses, which we will call $f_1^{\pm}$, $f_2^{\pm}$, $f_3^{\pm}$, $g_1^{\pm}$, and $g_2^{\pm}$. We get a language L. Terms corresponds to elements of the free group F_5, which will act on the sort P, via the function symbols. For $a \in P$, let R_a denote the subset of O defined by $R(x, a)$.

Then we can express by a collection of universal sentences in L that

(i) the map taking $(t, a) \in G \times P$ to $ta \in P$ is a free action of G on P,

(ii) for all $a \in P$, the sets (subsets of O), $R_{f_1(a)}$, $R_{f_2(a)}$, $R_{f_3(a)}$ are pairwise disjoint and each is contained in the union of $R_{g_1(a)}$ and $R_{g_2(a)}$.

We will call this universal L-theory T.

We will define a theory T^* in L which extends T and has quantifier elimination, so will be the model companion of T. As usual to show the existence of model companions one needs to describe, in the parameters, when a quantifier-free formula $\phi(x)$ over a model M of T has a solution in a larger model N of T. The key issue is Axiom (ii) above. So some combinatorics is required which will be done in the next section.

2.2. **Colourings and free actions.** We fix a free action of F_5 on a set X. As above, we will denote by $\{f_1, f_2, f_3, g_1, g_2\}$ a system of free generators for the free group F_5. There is an induced "Cayley graph" metric on X, where $d(u, v) = 1$ if one can get from one of u, v to the other by multiplying by one of the distinguished generators. So $d(x, y)$ is finite if x, y are in the same orbit of F_5 and ∞ otherwise. We should clarify here that a "path" from u to v will be represented by a reduced word w in the $f_i^{\pm}$ and $g_j^{\pm}$ such that $wu = v$. There is at most one such path (as F_5 is free on these generators, and the action is free). And $d(u, v)$ is precisely the length of w.

For $v \in X$, let $B_n(v)$, the ball around v of radius n, be $\{u \in X : d(v, u) \leq n\}$ and for V a subset of X, $B_n(V) = \bigcup_{v \in V} B_n(v)$.

Definition 2.1. (i) Define $\leq^*$ on X by $u \leq^* v$ if there exist $i \in [3]$ and $j \in [2]$ such that $v = g_j f_i^{-1} u$.

(ii) Let $\leq$ be the reflexive and transitive closure of $\leq^*$, and for $v \in X$, let $U_v = \{u \in X : v \leq u\}$.

(iii) The *nth* level of U_v is $\{u \in U_v : d(v, u) = 2n\}$.

(iv) By a complete tree for $v \in X$ we mean a subset T of X containing v such that for all $u \in T$, and $i \in [3]$ there is $j \in [2]$ such that $g_j f_i^{-1}(u) \in T$.

(v) By a depth n tree for $v \in X$, we restrict (iv) to $T \subseteq B_{2n}(v)$ and require the second clause of (iv) only for $u \in B_{2n-2}(v) \cap T$.

Remark 2.2. (a) Explanation of (v): Note that if $d(v, u) = 2n - 2$ then for any $i \in [3]$ and $j \in [2]$, $g_j f_i^{-1} u$ has distance at most $2n$ from v.

(b) Any product of words of the form $g_j f_i^{-1}$ for $i \in [3]$ and $j \in [2]$ will be a reduced word. Hence if w, w' are distinct such reduced words, and $u, v \in X$ then we could not have that both $wu = v$ and $w'u = v$.

(c) Note that U_v is a maximal complete tree for v.

Lemma 2.3. *Suppose $v \in X$, and $Y \subset X$ with $|Y| \leq n + 1$. Suppose there is a depth n tree T for v with $T \cap Y = \emptyset$. Then there is a complete tree T' for v which is disjoint from Y.*

Proof. The proof is by induction on n. When $n = 0$, we may assume Y is a singleton $\{x\}$, and $T = \{v\}$ with $v \neq x$.

For $i \in [3]$ and $j \in [2]$ let $v_{i,j} = g_j f_i^{-1} v$. By Remark 2.2(b), there will be at most one $v_{i,j}$ such that $v_{i,j} \leq x$. Hence for each $i \in [3]$ there is $j(i) \in [2]$ such that $v_{i,j(i)} \nleq x$. Hence also for each $i \in [3]$, $x \notin U_{v_{i,j(i)}}$. Hence $\{v\} \cup \bigcup_{i \in [3]} U_{v_{i,j(i)}}$ is a complete tree for v which is disjoint from $Y = \{x\}$.

The inductive step: Suppose $|Y| = n + 1$ and T is a depth n tree for v such that $T \cap Y = \emptyset$ (and $n > 0$). As above denote by $v_{i,j}$, $g_j f_i^{-1} v$. Fix $i \in [3]$ and one of

the j's $\in [2]$ such that $v_{i,j} \in T$. Then clearly $T \cap U_{v_{i,j}}$ is a depth $n-1$ tree for $v_{i,j}$ which is disjoint from Y.
Case 1. $|Y \cap U_{v_{i,j}}| \leq n$. Then by induction hypothesis, there is complete tree T_i for $v_{i,j}$ which is disjoint from $Y \cap U_{v_{i,j}}$. As $T_i \subseteq U_{v_{i,j}}$ it follows that T_i is also disjoint from Y.
Case 2. $|Y \cap U_{v_{i,j}}| = n+1$. Namely $Y \subseteq U_{v_{i,j}}$. Let $j' \neq j$, $j' \in [2]$. So clearly $U_{v_{i,j'}}$ is disjoint from $U_{v_{i,j}}$ (again by freeness of the action of F_5) and so disjoint from Y. In this case define T_i to be $U_{v_{i,j'}}$, a complete tree for $v_{i,j'}$ which is disjoint from Y.

Now let $T' = \{v\} \cup \bigcup_{i \in [3]} T_i$. Then T is disjoint from Y and is a complete tree for v. □

The motivation for part (1) of the next definition is to use colourings to describe quantifier-free 1 types over P realized in O in models of T. That is, a colouring c of P with colours $+, -$ will correspond to the quantifier-free type $p(x)$ on O where $R(x, a) \in p(x)$ iff $c(a) = +$. Conditions (a) and (b) below correspond to Axiom (ii) from the universal theory T.

Definition 2.4. (1) Suppose $D \subseteq X$. By a *good colouring* of D we mean a function $c : D \to \{+, -\}$, such that if $v \in D$ and $c(v) = +$ then
(a) for all $i \in [3]$ there is $j \in [2]$ such that $c(g_j f_i^{-1}(v)) = +$ if $g_j f_i^{-1}(v) \in D$.
(b) and for all $i \neq j \in [3]$, $c(f_j f_i^{-1} v) = -$, if $f_j f_i^{-1} v \in D$.
Moreover if $D = X$ we call c a *total* good colouring.
(2) We say that $v_1, v_2 \in X$ are a *conflicting pair*, if there are $w_1 \in U_{v_1}$ and $w_2 \in U_{v_2}$ such that $w_2 = f_j f_i^{-1} w_1$ for some $i \neq j \in [3]$.

Lemma 2.5. *(i) Being a conflicting pair is symmetric.*
(ii) If v_1 and v_2 are a conflicting pair, then there are unique $w_1 \in U_{v_1}$ *and* $w_2 \in U_{v_2}$ *such that $w_2 = f_j f_i^{-1} w_1$ for some $i \neq j \in [3]$. We call w_1, w_2* the *conflict points.*

Proof. (i) is obvious.
(ii) Let $w_1 \in U_{v_1}$, $w_2 \in U_{v_2}$ witness that v_1 and v_2 are a conflicting pair, namely $w_2 = f_j f_i^{-1} w_1$ for some $i \neq j \in [3]$. Let $w_1 = x v_1$ and $w_2 = y v_2$, where x and y are products (maybe empty) of pairs of free generators of the form $g_k f_\ell^{-1}$ (as $w_1 \in U_{v_1}$ and $w_2 \in U_{v_2}$). Then $v_2 = y^{-1} f_j f_i^{-1} x v_1$. The product $y^{-1} f_j f_i^{-1} x$ is already reduced (as y^{-1} ends and x begins with a g-generator). Thus x and y are uniquely determined, hence w_1 and w_2 too. □

Proposition 2.6. *Let V and W be disjoint finite subsets of X, both of which have cardinality at most n. Let $c : V \cup W \to \{+, -\}$ be a good colouring of $V \cup W$ given by c is $+$ on V and $-$ on W. Let $N = n(n+1) - 2$. Then there is total good colouring (i.e. of X) extending c if and only if there is good colouring of $B_N(V)$ extending the restriction of c to $B_N(V) \cap (V \cup W)$.*

Proof. One direction is obvious: if c' is a total good colouring then its restriction to $B_N(V)$ of course extends its further restriction to $B_N(V) \cap (V \cup W)$.

For the other direction: suppose c' is a good colouring of $B_N(V)$ extending the restriction of c to $B_N(V) \cap (V \cup W)$.

Note in passing that $V \subseteq B_N(V)$. We will define a set Y which consists of W together with one element from each pair (w, w') of conflict points which come from

a conflicting pair (v_1, v_2) of elements of V. So given such $v_1, v_2 \in V$ and conflict points w_1, w_2:
Case 1. Both $w_1, w_2 \in B_N(V)$. Then by the good colouring condition 1(b) (from Definition 2.4), not both $c'(w_1)$ and $c'(w_2)$ equal $+$. So choose one of them, without loss w_1 such that $c'(w_1) = -$ and put w_1 into Y.
Case 2. At least one of w_1, w_2, without loss w_1 is NOT in $B_N(V)$. Then add w_1 to Y.

There are at most $n(n-1)/2$ conflicting (unordered) pairs from V, and hence $|Y| \leq n + n(n-1)/2 = n(n+1)/2 = N/2 + 1$, and by construction $c'(x) = -$ for all $x \in Y \cap B_N(V)$.

Now for each $v \in V$, $T = \{u \in B_N(v) : c'(u) = +\}$ is a depth $N/2$ tree for v which is disjoint from Y (by definition of a good colouring and the construction of Y). By Lemma 2.3 (as $|Y| \leq N/2 + 1$) there is, for each $v \in V$, a complete tree T_v for v which is disjoint from Y. Let us then define a (total) colouring c'' of X which has value $+$ on T_v for each $v \in V$ and $-$ otherwise.

As c is $+$ on V, and $-$ on W which is contained in Y which is disjoint from each T_v, c'' extends c.

Claim. *c'' is good.*

Proof of Claim. Suppose $c''(u) = +$. So $u \in T_v$ for some $v \in V$. But T_v is a complete tree for v, so for each $i \in [3]$ there is $j \in [2]$ such that $g_j f_i^{-1} u \in T_v$, whereby $c''(g_j f_i^{-1} u) = +$. This gives 1(a) in the definition (Definition 2.4) of a good colouring.

For 1(b): suppose for a contradiction that $c''(w_1) = +$ and $c''(w_2) = +$ for w_1, w_2 in X such that $w_2 = f_j f_i^{-1} w_1$ for some $i \neq j \in [3]$. But then $w_1 \in T_{v_1}$ and $w_2 \in T_{v_2}$ for some $v_1, v_2 \in V$, and we see that w_1, w_2 are conflict points for the conflicting pair $v_1, v_2 \in V$. But by the definition of Y, one of w_1, w_2 is in Y and so gets c'' colour $-$. A contradiction. □

Corollary 2.7. *For each $v \in X$ there are good colourings c, c' of X such that $c(v) = +$ and $c'(v) = -$.*

2.3. **The model companion T^*.** We return to the context of Section 2.1, namely the language L and universal theory T. To any element h of F_5 expressed in terms of the generators and their inverses in reduced form we have a term t_h of L. Note that if t is a term in nonreduced form then there will be some h such that $t = t_h$ is true in all models of T.

We will give two axiom schema, which in addition to T give a theory T^* in the given language. We will check subsequently that $(T^*)_\forall = T$, and that T^* has quantifier elimination (and is complete), so is the model companion of T.

We want to describe which quantifier-free 1 types over a model M of T can be realized in some extension N of M to a model of T, by expressing the existence of solutions of appropriate approximations. There are two kinds of 1-types: realized by an element of P, and realized by an element of O. We introduce some notation to deal with each of these cases.

Let $p_i(z, x)$ for $i \in I$ be a list of all (complete) quantifier-free types (over $\emptyset$) of pairs (a, b) in models M of T where $a \in O(M)$ and $b \in P(M)$. So $p_i(z, x)$ will be a maximal consistent (with T) set of formulas of the form $R(z, t_h(x))$, $\neg R(z, t_h(x))$ for h ranging over F_5. (The inequalities between x and the $t_h(x)$ for $h \neq 1$ will come free from T).

For each n, let $\gamma_n(x_1,\dots,x_n,y_1,\dots,y_n)$ be a quantifier-free L-formula expressing the existence of a good colouring c of $B_N(\{x_1,\dots,x_n\})$ such that $c(x_i)=+$ for $i=1,\dots,n$ and $c(y_i)=-$ for each y_i which happens to be in $B_N(\{x_1,\dots,x_n\})$ (where $N=n(n+1)-2$).

Axiom Schema I. All sentences of the form

$$(\forall x_1,\dots,x_n\in P)(\forall z_1,\dots,z_n\in O)\Big(\bigwedge_{i\neq j} z_i\neq z_j\to$$
$$(\exists x\in P)\big(\bigwedge_{j=1,\dots,n}\phi_{i_j}(z_j,x)\wedge\bigwedge_{i=1,\dots,n} x\neq x_i\big)\Big),$$

where $n\geq 1$, $i_1,\dots,i_n\in I$ and each $\phi_{i_j}(z,x)$ is a finite conjunction of formulas in $p_{i_j}(z,x)$.

Axiom Schema II. All sentences of the form

$$(\forall x_1,\dots,x_n,y_1,\dots,y_n\in P)(\forall z_1,\dots,z_n\in O)\Big(\gamma_n(x_1,\dots,x_n,y_1,\dots,y_n)\to$$
$$(\exists z\in O)\big(\bigwedge_{i=1,\dots,n}(R(z,x_i)\wedge\neg R(z,y_i))\wedge\bigwedge_{i=1,\dots,n} z\neq z_i\big)\Big)$$

for $n\geq 1$.

We define T^* to be (the theory axiomatized by) T together with Axiom Schemas I and II.

Lemma 2.8. *Any existentially closed model of T is a model of T^*. In particular T^* is consistent and $(T^*)_\forall = T$.*

Proof. Let M be an existentially closed model of T. Consider an axiom

$$(\forall x_1,\dots,x_n\in P)(\forall z_1,\dots,z_n\in O)\Big(\bigwedge_{i\neq j} z_i\neq z_j\to$$
$$(\exists x\in P)\big(\bigwedge_{j=1,\dots,n}\phi_{i_j}(z_j,x)\wedge\bigwedge_{i=1,\dots,n} x\neq x_i\big)\Big)$$

belonging to Axiom Schema I.

Choose $a_1,\dots,a_n\in O(M)$, which we may assume to be distinct. We will build a certain model M' of T containing M. Let X be a principal homogeneous space for F_5 (disjoint from $P(M)$) with a distinguished point b. Let $P(M')=P(M)\cup X$ with the natural action of F_5. For $h\in F_5$, we put $(a_j,hb)\in R$ iff $R(z,t_h(x))\in p_{i_j}(z,x)$. And for any other $a\in O(M)$, we put $\neg R(a,c)$ for any $c\in X$. We also define $O(M')$ to be $O(M)$. Then it can be checked that M' is a model of T. Now b witnesses that the formula $(\exists x\in P)(\bigwedge_{j=1,\dots,n}\phi_{i_j}(a_j,x)\wedge\bigwedge_{i=1,\dots,n}x\neq b_i))$ for any $b_1,\dots,b_n\in P(M)$ holds in M'. As M is existentially closed in M', this formula also holds in M. We have shown that M is a model of Axiom Schema I.

Now let

$$(\forall x_1,\dots,x_n,y_1,\dots,y_n\in P)(\forall z_1,\dots,z_n\in O)\Big(\gamma_n(x_1,\dots,x_n,y_1,\dots,y_n)\to$$
$$(\exists z\in O)\big(\bigwedge_{i=1,\dots,n}(R(z,x_i)\wedge\neg R(z,y_i))\wedge\bigwedge_{i=1,\dots,n} z\neq z_i\big)\Big)$$

be a sentence in Axiom Schema II.

Choose $b_1, \ldots, b_n, c_1, \ldots, c_n \in P(M)$. We will add a new point $\star$ to the O sort to get a structure M' extending M. Let us assume that $M \models \gamma_n(b_1, \ldots, b_n, c_1, \ldots, c_n)$. By Proposition 2.6, there is a good colouring c of $P(M)$ such that $c(b_i) = +$ and $c(c_i) = -$ for $i = 1, \ldots, n$. For $d \in P(M) = P(M')$ we define $R(\star, d)$ iff $c(d) = +$. Then M' is a model of T, and again as M is existentially closed in M', $(\exists z \in O)(\bigwedge_{i=1,\ldots,n}(R(z, b_i) \wedge \neg R(z, c_i)) \wedge \bigwedge_{i=1,\ldots,n} z \neq a_i))$ is true in M, for any $a_1, \ldots, a_n \in O(M)$. So M is a model of Axiom Schema II. $\square$

Proposition 2.9. *(i) T^* is complete with quantifier elimination,*
(ii) T^* is the model companion of T,
(iii) for any model M of T^* and $A \subseteq M$, the algebraic closure of A in M (in the sense of the structure M) is precisely $\langle A \rangle$, the substructure of M generated by A.

Proof. For (i) we use the well-known criterion that for M, N ω-saturated models of T^*, the collection of partial isomorphisms between finitely generated substructures of M and N is nonempty and has the back-and-forth property.

First to show nonemptiness: Let $a \in O(M)$ and $b \in O(N)$. Then $\{a\}$, $\{b\}$ are isomorphic substructures of M and N.

Now suppose f is an isomorphism between finitely generated substructures M_0 and N_0 of M and N respectively. Let $a \in M$. We want to extend f to g with $a \in dom(g)$. We may assume $a \notin M_0$.

Case 1. $a \in P(M)$.
Let $p(x) = qftp(a/M_0)$ (quantifier-free type of a over M_0). For each $b \in O(M_0)$, let $p_b(z, x) = qftp(b, a/\emptyset)$. Then $p(x)$ is axiomatized by $\{x \neq c : c \in P(M_0)\} \cup \bigcup_{b \in O(M_0)} p_b(b, x)$. Now $f(p)$ is precisely $\{x \neq d : d \in P(N_0)\} \cup \bigcup_{b \in O(M_0)} p_b(f(b), x)$.

By Axiom Schema I and ω-saturation, $f(p)$ is realized in N.

Case 2. $a \in O(M)$.
Let $q(z) = qftp(a/M_0)$. Then $f(q) = \{z \neq d : d \in O(N_0)\} \cup \{R(z, f(b)) : b \in P(M_0),\ M \models R(a, b)\} \cup \{\neg R(z, f(b)) : b \in P(M_0),\ M \models \neg R(a, b)\}$. Choose $b_1, \ldots, b_n \in P(M_0)$ such that $M \models R(a, b_i)$, and $c_1, \ldots, c_n \in P(M_0)$ such that $M \models \neg R(a, c_i)$ (if such exist). Then as M is a model of T^* (and so of T) we have $M \models \gamma_n(b_1, \ldots, b_n, c_1, \ldots, c_n)$, whereby

$$N \models \gamma_n(f(b_1), \ldots, f(b_n), f(c_1), \ldots, f(c_n)).$$

So by Axiom Schema II and the ω-saturation of N, $f(q)$ is realized in N.
(ii) follows immediately as T^* is model-complete (by (i)) and $(T^*)_\forall = T$ (by Lemma 2.8).
(iii) By quantifier-elimination, we have to show that for any small substructure M_0 of a (saturated) model of T^*, and $a \in M \setminus M_0$, $qftp(a/M_0)$ has infinitely many realizations. For $a \in P(M)$ this is by Axiom Schema I and saturation. And for $a \in O(M)$ this is by Axiom Schema II and saturation. $\square$

2.4. **Simplicity and the proof of Theorem 1.5.** We now work in a saturated model $\bar{M}$ of the complete theory T^* defined earlier.

Proposition 2.10. *Let a be an element (so an element of $O(\bar{M})$ or of $P(\bar{M})$), and B a (small) subset. Then $a \notin acl(B)$ implies that $tp(a/B)$ does not divide over $\emptyset$.*

Proof. We may assume that B is a substructure, enumerated by an infinite tuple b_0. Let $I = (b_0, b_1, b_2, \ldots)$ be an indiscernible sequence. Note that $\bigcup I$ is a substructure, say M_0, of $\bar{M}$.

Let $p(x, b_0) = tp(a/b_0)$ with $a \notin B$.
Case 1. $a \in P(\bar{M})$.
Define a new structure M_1 extending M_0, by adjoining new elements $\{\star_g : g \in F_5\}$ satisfing P, and for any element c in some b_n such that $O(c)$, define R to hold of $(c, \star_g)$ iff the corresponding element of b_0 is in the relation R with $t_g(a)$. Also define the $f_i^{\pm}$ and $g_j^{\pm}$ tautologically on $\{\star_g : g \in F_5\}$. Then check that M_1 is a model of T, so by quantifier elimination and saturation of $\bar{M}$ we may assume that M_1 is an extension of M_0 inside $\bar{M}$. And we see that $\star_e$ realizes $p(x, b_i)$ for all i.

Hence $p(x, b_0)$ does not divide over $\emptyset$.
Case 2. $a \in O(\bar{M})$.
Do the analogous thing: define an L-structure extending M_0 with a single new element $\star$ which is in O and with $R(\star, c)$ for c in some b_n (such that $P(c)$) iff a is R-related to the corresponding element of b_0. Again check that we get a model of T, so can be assumed to live in $\bar{M}$ over M_0 and $\star$ realises $p(x, b_i)$ for all i. □

Corollary 2.11. *(i) T^* is simple and of SU-rank 1 (each of the sorts O, P has SU-rank 1).*
(ii) For all tuples a, b and subset A (of $\bar{M}$), a is independent from b over A iff $\langle aA\rangle \cap \langle bA\rangle = \langle A\rangle$.
(iii) Each of the sorts has a unique 1-type over $\emptyset$.

Proof. By Proposition 2.10, every complete 1-type (over any set) is either algebraic or does not divide over $\emptyset$, which implies that T is simple. In particular forking equals dividing and is symmetric. And so the proposition says that the only forking extensions of any complete 1-type are algebraic, namely that each of the sorts has SU-rank 1.

(ii) follows from Proposition 2.10 (and Proposition 2.9 (iii)) by forking calculus, using also the fact for any set B, $\langle B\rangle = \bigcup_{b\in B}\langle b\rangle$, which follows from there being only unary function symbols in the language.

And (iii) is a consequence of quantifier elimination. □

The proof of Theorem 1.5 is completed by the following results:

Proposition 2.12. *For any $a \in P$, the formula $R(z, a)$ does not fork over $\emptyset$ but has measure 0 for any (automorphism) invariant Keisler measure μ (on the sort O).*

Proof. Let μ be an invariant Keisler measure on the sort O. As there is a unique 1-type over $\emptyset$ realized in P, $\mu(R(x, a)) = \mu(R(x, b))$ for all $a, b \in P$. But for any given a, and $i \in [3]$, $R(x, f_i(a)) \to (R(x, g_1(a)) \vee R(x, g_2(a)))$, and $R(x, f_1(a))$, $R(x, f_2(a))$, $R(x, f_3(a))$ are pairwise inconsistent. So this forces $\mu(R(x, a)) = 0$ for all $a \in P$. On the other hand $R(x, a)$ has infinitely many realizations, so as O has SU-rank 1, $R(x, a)$ does not fork over $\emptyset$. □

Proposition 2.13. *The theory T^* is* extremely amenable*: every complete type over $\emptyset$ has a global (automorphism) invariant extension.*

Proof. We just give a sketch, leaving details to the interested reader. Let $p(\bar{x}, \bar{z}) = tp(\bar{a}, \bar{b}/\emptyset)$ where $\bar{a}$ is a tuple from P and $\bar{b}$ a tuple from O. Let M be a saturated model. Then we can find a realization $(\bar{a}', \bar{b}')$ of p in some elementary extension N of M such that all the elements from the tuple $(\bar{a}', \bar{b}')$ are in $N \setminus M$, $N \models \neg R(d, a)$

for each $a \in \bar{a}'$, $d \in O(M)$, and $N \models \neg R(b, d)$ for each $b \in \bar{b}'$ and $d \in P(M)$. Then $tp((\bar{a}', \bar{b}')/M)$ is clearly $Aut(M)$-invariant (using quantifier elimination). $\square$

3. A non definably amenable group definable in a simple theory

In this section we will prove Theorem 1.8. Again we start with an overview. Our theory T^* will be a certain expansion of ACF_0, and the group G which is not definably amenable will be $SL_2(K)$, where K is the underlying algebraically closed field. Of course working just in ACF_0, $SL_2(K)$ will be definably (extremely) amenable. The additional structure we will add will be a partition of $SL_2(K)$ into 4 sets C_1, C_2, C_3, C_4. We will choose matrices $a(i, j) \in SL_2(\mathbb{Z})$ for $i \in [4]$, $j \in [3]$, which freely generate F_{12}, and require that for each $i \in [4]$, $\bigcup_{j \in [3]} a(i, j)^{-1} C_i = SL_2(K)$. The C_i will be chosen sufficiently generically so that the theory T^* of the structure $(K, +, \times, C_1, C_2, C_3, C_4)$ is simple of SU-rank 1. If by way of contradiction $G = SL_2(K)$ were definably amenable, witnessed by (left) invariant Keisler measure μ, then the requirement above implies that $\mu(C_i) \geq 1/3$ for each $i \in [4]$ but then by disjointness, $\mu(G) \geq 4/3$ a contradiction.

In Section 4, we will mention a closely related example with F_6 in place of F_{12} but with a partition of $SL_2(K)$ into six sets rather than four. In terms of certain invariants related to "definable paradoxical decompositions", this other example could be considered "better". The general theory of paradoxical decompositions in both the abstract or discrete groups setting and the definable setting will also be discussed.

As in Section 2, we will describe a universal theory T, and T^* will be its model companion, but no longer complete.

3.1. The universal theory.

The language L will be that of unital rings, together with four 4-ary predicate symbols C_1, C_2, C_3, C_4.

It is well-known that

$$a = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

generate a free group in $\mathrm{SL}_2(\mathbb{Z})$. Hence so do the matrices

$$a^{-k} b a^k = \begin{pmatrix} 1 - 4k & -8k^2 \\ 2 & 4k + 1 \end{pmatrix},$$

for $k = 0, \ldots, 11$. We number these 12 matrices in some way as $a(i, j)$, for $i \in [4]$, $j \in [3]$. We will refer to the group generated by these matrices as F_{12}. Note that the entries of each $a(i, j)$ are terms of the language.

For an integral domain R of characteristic 0, $SL_2(R)$ is the collection of 2×2 matrices over R of determinant 1. The (universal) theory T in the language L will be the theory of integral domains R of characteristic 0 together with axioms:

(i) The 4-ary predicates $C_1, \ldots, C_4$ partition $SL_2(R)$, and
(ii) For each $x \in SL_2(R)$ and each $i \in [4]$, there is $j \in [3]$ such that $a(i, j) \cdot x \in C_i$.

3.2. Combinatorics and colourings.

We prove some lemmas needed for defining T^*. The context in this section is simply the free group $G = F_{12}$ on 12 generators numbered as $a(i, j)$ for $i \in [4]$ and $j \in [3]$ together with a free action of G on a set X.

Definition 3.1. Let $X_0 \subseteq X$. A colouring $c : X_0 \to [4]$ is *good* if for all $x \in X_0$ and $i \in [4]$, IF $a(i,j) \cdot x \in X_0$ for all $j \in [3]$, THEN $c(a(i,j) \cdot x) = i$ for some $j \in [3]$. We call this condition the *ith* colouring axiom at x.

Also we may call the (good) colouring total if $X_0 = X$.

We will use similar notation to that in Section 2.2, regarding the graph structure on X, distance, connectedness, and uniqueness of paths etc. For example, a subset X_0 is *connected* if for any $x, y \in X$, $d(x,y) < \infty$ and all points on the unique path from x to y are in X_0. For $X_0 \subseteq X$, $B_n(X_0)$ denotes the set of $x \in X$ such that there is $y \in X_0$, $d(x,y) \leq n$.

For C_0, C_1 subsets of X, $d(C_0, C_1)$ is the length of a shortest path between an element of C_0 and an element of C_1, if there is such a path, and ∞ otherwise.

We now give some lemmas about extending good colourings.

Lemma 3.2. *Suppose that $X_0 \subseteq X$ is connected. Then any good colouring $c_0 : X_0 \to [4]$ extends to a total good colouring.*

Proof. We may assume that $X_0 \neq \emptyset$, otherwise replace it by a singleton coloured with any colour. As good colourings can be defined independently on connected components of X, we may assume that X is connected, so equals $\bigcup_n B_n(X_0)$. And note that each $B_n(X_0)$ is connected. We extend c_0 to X by induction. Assume that we already have a good colouring $c_n : B_n(X_0) \to [4]$ extending c_0. We extend to c_{n+1}. Suppose $y = a(i,j) \cdot x \in B_{n+1}(X_0) \setminus B_n(X_0)$ for some $x \in B_n(X_0)$, and some i, j, then define $c_{n+1}(y) = i$. Note that this is well defined, for if y also equals $a(i',j') \cdot x'$ for some $x \neq x' \in B_n(X_0)$, then y is on the unique path between x and x' so $y \in B_n(X_0)$ as it is connected.

If $y \in B_{n+1}(X_0) \setminus B_n(X_0)$ is not of this form, define $c_{n+1}(y) \in [4]$ arbitrarily.

We have to check that c_{n+1} is a good colouring of $B_{n+1}(X_0)$. Suppose $x \in B_{n+1}(X_0)$ and $i \in [4]$, and $a(i,j) \cdot x \in B_{n+1}(X_0)$ for all $j \in [3]$. Now if $a(i,j) \cdot x \in B_n(X_0)$ for all $j \in [3]$ then by connectedness of $B_n(X_0)$ also $x \in B_n(X_0)$ and so as c_n is a good colouring and c_{n+1} extends c_n, the *ith* colouring axiom at x is satisfied. Otherwise $a(i,j) \cdot x \in B_{n+1}(X_0) \setminus B_n(X_0)$ for some $j \in [3]$. Then $x \in B_n(X_0)$, for if not, there are $x_0, y_0 \in X_0$ such that $d(x_0, x) = n+1$ and $d(y_0, y) = n+1$ and we get that x, y lie on the unique path connecting x_0 and y_0, whereby $x, y \in X_0$, a contradiction. Hence $c_{n+1}(y) = i$ by definition. □

Lemma 3.3. *Let C_0, C_1 be disjoint connected subsets of X with $3 \leq d(C_0, C_1) < \infty$. Let C be the smallest connected subset of X containing $C_0 \cup C_1$. Then any good colouring c_0 of $C_0 \cup C_1$ extends to a good colouring of C.*

Proof. Note that C is the union of C_0, C_1 and the points on the unique shortest path I connecting them. By assumption the length of I is ≥ 3, namely $|I| \geq 4$. Now extending, if necessary, C_0 to a suitable $B_n(C_0)$ and extending $c_0|C_0$ to a good colouring of $B_n(C_0)$ we may assume that $I = (u, v, y, z)$ with $u \in C_0$, $z \in C_1$ and $v, y \notin C_0 \cup C_1$.

If $v = a(i,j) \cdot u$ for some i, j put $c(v) = i$. Otherwise define it arbitrarily. Likewise if $y = a(i,j) \cdot z$ for some i, j define $c(y) = i$. Note that this is well-defined. We have to check that c is a good colouring. And for this it is clear that we only need to check the *ith* colouring axioms at u, v, y, z (for all i). For u, z it is clear by construction. And for v, y it is also clear vacuously, because it cannot be the case that all of $a(i,1) \cdot v$, $a(i,2) \cdot v$ and $a(i,3) \cdot v$ lie in C, and similarly for y. □

Lemma 3.4. *Suppose $X_0 \subseteq X$ has n connected components, any two of which are of distance $\geq 2^n$ apart. Then any good colouring c_0 of X_0 extends to a good colouring of X.*

Proof. By induction on n. The case $n = 1$ is Lemma 3.2. The case $n = 2$ is Lemma 3.3, noting that $2^2 = 4 \geq 3$.

So let us assume $n \geq 2$ and the lemma holds for n and we want to prove it for $n+1$. Let X_0 have $n+1$ connected components $C_0, \ldots, C_n$ and let c_0 be a good colouring of X_0. As the connected components of X can be coloured separately, we may assume that the C_i lie on a common connected component of X. We may also assume that the distance l between C_0 and C_1 is the minimal distance between distinct pairs C_i, C_j. Let C_1' be the smallest connected subset of X containing C_0 and C_1 (as mentioned earlier C_1' is the union of C_0, C_1 and the points on the unique shortest path between C_0 and C_1). Using Lemma 3.3, let c_1' be a good colouring of C_1' extending $c_0|(C_0 \cup C_1)$.

Claim. *For each $i > 1$, the distance between C_1' and C_i is at least 2^n.*

Proof of Claim. Fix $i > 1$ and let $d = d(C_1', C_i)$ and suppose for a contradiction that $d < 2^n$. As $d(C_0, C_i)$ and $d(C_1, C_i)$ are both $\geq 2^{n+1}$, then d has to be witnessed by $d(x, C_i)$, where x is a point on the unique shortest path I between C_0 and C_1 which we know has length l. So $d(x, C_i) < 2^n$, $d(x, C_0) = l_0$ say, and $d(x, C_1) = l_1$ say with $l_0 + l_1 = l$. Moreover $d(C_0, C_i) \leq l_0 + d$ and $d(C_1, C_i) \leq l_1 + d$, both of which are $\geq l$ by choice of C_0 and C_1. But then $l + 2d = l_0 + d + l_1 + d \geq 2l$ which implies $2d \geq l \geq 2^{n+1}$, which implies $d \geq 2^n$, a contradiction.

Let $X_0' = C_1' \cup C_2 \cup \ldots \cup C_n$, and let c_0' be c_0 on $C_2 \cup \ldots \cup C_n$ and c_1' on C_1'. Note that c_0' is a good colouring on X_0' as it is good on each connected component of X_0'. Then by the claim, and the induction hypothesis, c_0' extends to a good colouring c of X, and as c_0' extends c_0, c extends c_0 too. □

Lemma 3.5. *Suppose $X_0 \subseteq X$ has size n. Let $\alpha(n) = 2^{n+1} - 1$, and let $c_0 : X_0 \to [4]$ be a good colouring which extends to a good colouring $c' : B_{\alpha(n)}(X_0) \to [4]$. Then c_0 extends to a good colouring $c : X \to [4]$ of X.*

Proof. Let k_0 be the number of connected components of X_0. So $k_0 \leq n$.
Case 1. Either $k_0 = 1$ (X_0 is connected) or $k_0 > 1$ and the k_0 connected components of X_0 are at distance $\geq 2^{k_0}$ apart.
Then by Lemma 3.4, c_0 extends to a good colouring of X. And we are finished.
Case 2. Otherwise. Then define $X_1 = B_{2^{k_0}}(X_0)$, and k_1 to be the number of connected components of X_1. And note that $k_1 < k_0$ and $X_1 \subseteq B_{\alpha(n)}(X_0)$.
Again if either X_1 is connected or the k_1 connected components of X_1 are of distance $\geq 2^{k_1}$ apart, then the good colouring $c'|X_1$ extends to a good colouring of X, and we finish.

Otherwise define $X_2 = B_{2^{k_1}}(X_1)$ and k_2 to be the number of connected components of X_2. So $k_2 < k_1$.

We continue this way to produce $k_0 > k_1 > \ldots > k_l \geq 1$ and $X_0 \subseteq X_1 \subseteq \ldots \subseteq X_l$ where X_i has k_i connected components, until we get that X_l is connected or its k_l connected components are at distance $\geq 2^{k_l}$ apart, and we extend $c'|X_l$ to a good colouring of X.

We have to check why the process can be continued, in particular why each $X_i \subseteq B_{\alpha(n)}(X_0)$. It is because, $k_i \leq n - i$ for each i, and so $\sum_{i=0,\ldots,l} 2^{k_i} \leq$

$\sum_{i=0,\dots,n} 2^{n-i} = \sum_{i=0,\dots,n} 2^i = 2^{n+1} - 1 = \alpha(n)$. Whereby $X_i \subseteq B_{\alpha(n)}(X_0)$ for all $i = 1, \dots, l$. $\square$

3.3. **The theory T^*.** Here we will obtain the model companion T^* of the universal theory T introduced in Section 3.1. In terms of compatibility with notation in the previous section, we will write a model of T as $M = (R, c)$, where R is an integral domain of characteristic 0 and c is the colouring $SL_2(R) \to [4]$ such that $C_i(M) = c^{-1}(i)$ for $i = 1, \dots, 4$. So as F_{12} is acting freely on $SL_2(R)$ by left multiplication, the axioms from Section 3.1 say precisely that c is a good colouring. In this context we will use freely the colouring notation from the previous section.

Lemma 3.6. *For any model (R, c) of T and integral domain $S \supseteq R$, c extends to a good colouring c' of $SL_2(S)$, whereby (S, c') is also a model of T.*

Proof. Note that $SL_2(S) \setminus SL_2(R)$ is a union of F_{12} orbits. On each such orbit one can define a good colouring, by Lemma 3.2 for example, and these good colourings together with c give a good colouring of $SL_2(S)$. $\square$

So if (R, c) is an existentially closed model of T, then R is an algebraically closed field. So from now on we will assume that $R = K$ is an algebraically closed field, and we situate K in a larger saturated algebraically closed field $\tilde{K}$ from which we can choose generic points of algebraic varieties over K (and write SL_2 for $SL_2(\tilde{K})$).

For technical reasons related to a subsequent relative quantifier elimination proof by a back and forth argument we will be concerned with extending the colouring c of $SL_2(K)$ to generic points of curves on $(SL_2)^n$. By a *good curve* over K, or good K-curve, we mean an absolutely irreducible curve $C \subseteq (SL_2)^n$ for some n, defined over K, with the property that if $d = (d_1, \dots, d_n)$ is a generic point of C over K, then each "coordinate" $d_i \notin SL_2(K)$. In the following $\alpha(n) = 2^{n+1} - 1$ as in Lemma 3.5.

Definition 3.7. Let K be an algebraically closed field. Let $n \geq 1$, let $C \subseteq (SL_2)^n$ be a good K-curve, and let $c_0 : [n] \to [4]$. We will say that C is *safe for c_0 over K* if for $d = (d_1, \dots, d_n)$ a generic point of C over K, the colouring $\tilde{c} : \{d_1, \dots, d_n\} \to [4]$ defined by $\tilde{c}(d_i) = c_0(i)$, extends to a good colouring $\tilde{c}'$ of $B_{\alpha(n)}(\{d_1, \dots, d_n\}) \subseteq SL_2(\tilde{K})$.

Fix n. Let us now fix a (quantifier-free) formula $\phi(\bar{x}, \bar{y})$ in the language L_r of rings such that for any algebraically closed field F and tuple $\bar{a}$ from F (of length that of $\bar{y}$), $\phi(\bar{x}, \bar{a})$ if consistent, defines a good F-curve $D_{\bar{a}} \subseteq SL_2(F)^n$. We call such $\phi(\bar{x}, \bar{y})$ a "*good formula*".

Remark 3.8. Note that for any algebraically closed field F and good curve $D \subseteq SL_2(F)^n$, there is a good formula $\phi(\bar{x}, \bar{y})$ and $\bar{a} \in F$ such that $D = D_{\bar{a}}$. This is because we can express dimension and irreducibility of algebraic varieties, and we can also express that the projection of a curve onto each coordinate has infinite image.

Lemma 3.9. *Given n, good formula $\phi(\bar{x}, \bar{y})$ as above, and a function $c_0 : [n] \to [4]$, there is a formula $\psi(\bar{y})$ in L_r, such that for every algebraically closed field K and $\bar{a} \in K$, $K \models \psi(\bar{a})$ iff the curve $D_{\bar{a}}$ is safe for c_0 over K.*

Proof. Note that we are working completely in the language of rings, even though we mention colourings. First note that for a curve $C \subseteq SL_2(K)^n$ and any $(d_1, \dots, d_n) \in$

$C(K)$, there is a bound κ_n on the cardinality of $B_{\alpha(n)}(\{d_1, \ldots, d_n\})$, and moreover by a case analysis we can identify definably, from properties of the d_i the precise cardinality. There is a formula $\chi(z_1, \ldots, z_n)$ in L_r expressing that c is a good coloring of $B_{\alpha(n)}(\{z_1, \ldots, z_n\})$ into 4 colours $\{1, 2, 3, 4\}$ such that $c(z_i) = c_0(i)$

We now bring in the good formula $\phi(\bar{x}, \bar{y})$. Let $\psi(\bar{y})$ express that for infinitely many $\bar{x}$ such that $\phi(\bar{x}, \bar{y})$ holds, $\chi(\bar{x})$ holds. Then for K algebraically closed, and $\bar{a} \in K$, $K \models \psi(\bar{a})$ iff for generic $\bar{d}$ on $D_{\bar{a}}$ over K, there is a good colouring c of $B_{\alpha(n)}(\{d_1, \ldots, d_n\})$ such that $c(d_i) = c_0(i)$ for $i = 1, \ldots, n$, namely that $D_{\bar{a}}$ is safe for c_0 over K. □

We can now define T^*.

Definition 3.10. T^* is the L-theory expressing of (K, c), that:

- (i) K is algebraically closed and $(K, c) \models T$;
- (ii) whenever $C \subseteq (SL_2)^n$ is a good curve over K, $c_0 : [n] \to [4]$ and C is safe for c_0, then there are infinitely many $d = (d_1, \ldots, d_n) \in C(K)$ such that $c_0(i) = c(d_i)$ for $i = 1, \ldots, n$.

Remark 3.11. By Remark 3.8 and Lemma 3.9, the property (ii) in the definition of T^* above is expressed by an axiom schema, ranging over n and good formulas $\phi(\bar{x}, \bar{y}) \in L_r$.

Lemma 3.12. *Any model (R, c) of T extends to a model (F, c') of T^*. In particular $(T^*)_\forall = T$ and T^* is consistent.*

Proof. Fix $(R, c) \models T$ and as mentioned after Lemma 3.6 we may assume $R = K$ to be an algebraically closed field. We will fix a good curve $C \subset (SL_2)^n$ over K and $c_0 : [n] \to [4]$, such that C is safe for c_0, and find an extension (F, c') of (K, c) and $d = (d_1, \ldots, d_n) \in C(F)$ such that $c'(d_i) = c_0(i)$ for $i = 1, \ldots, n$. We will also choose F algebraically closed. So in (F, c') we satisfy Axiom Schema (i) as well as a weaker form of one instance of the Axiom Schema (ii) for T^* (namely that there is at least one, rather than infinitely many, d satisfying the required conditions). Extending (K, c) to a model of T^* is then a routine union of chains argument, including finding the infinitely many d as above. Details are left to the reader.

Simply choose $d = (d_1, \ldots, d_n)$ to be a point of C in $\tilde{K}$ generic over K. By goodness of C, each $d_i \in SL_2(\tilde{K}) \backslash SL_2(K)$. By assumption there is a good colouring c'' of $B_{\alpha(n)}(\{d_1, \ldots, d_n\})$ such that $c''(d_i) = c_0(i)$ for $i = 1, \ldots, n$. Let F be the algebraic closure of the field generated by K and d. And let $X = SL_2(F) \backslash SL_2(K)$. Then X is a union of F_{12}-orbits and $B_{\alpha(n)}(\{d_1, \ldots, d_n\}) \subset X$. Hence, by Lemma 3.5, there is a good colouring c''' of X with $c'''(d_i) = c_0(i)$ for $i = 1, \ldots, n$. As X and $SL_2(K)$ are both unions of F_{12}-orbits, $c \cup c'''$ will be a good colouring of $SL_2(F)$ extending c. Denote $c \cup c'''$ by c', and we have produced our required extension (F, c') of (K, c). □

Lemma 3.13. *Let (F_1, c_1), (F_2, c_2) be $\aleph_1$-saturated models of T^*. Let I be the collection of partial isomorphisms between (nonempty) countable substructures of F_1, F_2 respectively which are of the form $(K_1, c_1|K_1)$, $(K_2, c_2|K_2)$ where K_1, K_2 are algebraically closed fields. Then I has the back-and-forth property.*

Proof. So suppose that we are given an isomorphism f between $(K_1, c_1|K_1)$ and $(K_2, c_2|K_2)$. It is enough to extend f to g with domain $L_1 \supseteq K_1$ where L_1 is algebraically closed and of transcendence degree 1 over K_1. By compactness, it suffices to prove the following.

Claim. *For every finite tuple $d_1, \ldots, d_n$ from $SL_2(L_1)$ there are $e_1, \ldots, e_n$ in $SL_2(F_2)$ such that the map g which extends f and takes d_i to e_i for $i = 1, \ldots, n$ preserves quantifier-free L_r-types, as well as satisfying $c_2(e_i) = c_1(d_i)$ for $i = 1, \ldots, n$.*

Proof of Claim. We may clearly assume that $d_1, \ldots, d_n \notin SL_2(K_1)$ for $i = 1, \ldots, n$. It follows that $(d_1, \ldots, d_n)$ is a generic over K_1 point of a good curve $C_1 \subset SL_2^n$ over K_1. Let $c_0 : [n] \to [4]$ be defined by $c_0(i) = c_1(d_i)$. Hence C_1 is safe for c_0 over K_1. As f is an isomorphism of algebraically closed fields, the curve $C_2 = f(C_1)$ is safe for c_0 over K_2. In particular C_2 is safe for c_0 over F_2. However (F_2, c_2) is a model of T^*, so Axiom Schema (ii) implies that there are infinitely many $e = (e_1, \ldots, e_n) \in C_2(F_2)$ such that $c_0(i) = c_2(e_i)$ for $i = 1, \ldots, n$. By $\aleph_1$-saturation of (F_2, c_2) (and countability of K_2) we can find $e = (e_1, \ldots, e_n) \in SL_2(F_2)$ a generic over K_2 point of C_2 such that $c_0(i) = c_2(e_i)$ for $i = 1, \ldots, n$. As the quantifier-free L_r-type of e over K_2 is the image under f of the quantifiier-free L_r-type of d over K_1, and $c_1(d_i) = c_0(i) = c_2(e_i)$ for $i = 1, \ldots, n$ we have proved the claim, and hence the lemma. ☐

Corollary 3.14. *(i) Let $\bar{a} = (a_\alpha : \alpha < \gamma)$, $\bar{b} = (b_\alpha : \alpha < \gamma)$ be tuples of the same length γ in models M, N of T^*, where γ is an ordinal. Then $tp_M(\bar{a}) = tp_N(\bar{b})$ iff the map taking a_α to b_α for $\alpha < \gamma$ extends to an isomorphism between the substructures (K, c) of M and (K', c') of N where K is the algebraic closure of the field generated by $\bar{a}$, and likewise for K' and $\bar{b}$.*

(ii) In particular the completions of T^* are determined by the isomorphism types of the algebraic closure of $\mathbb{Q}$ equipped with an L-structure.

(iii) T^* is the model companion of T.

(iv) In a model M of T^*, the model theoretic algebraic closure of a subset A of M coincides with the (field theoretic) algebraic closure of the field generated by A.

Proof. (i) is an immediate consequence of Lemma 3.13. And as said above (ii) is a special case (for the empty tuples).

(iii) is another special case: let $M \subseteq N$ be models of T^*. Then the identity map $M \to N$ is an isomorphism of L-structures whose underlying field is algebraically closed, hence an elementary map by (i). So T^* is model complete, hence by Lemma 3.12 is the model companion of T.

(iv) In the light of (i) we have to check that if M is a saturated model of T^* and (K, c) is a (small) substructure of M where K is algebraically closed as a field, then for any $a \in M \setminus K$, there are infinitely many realizations of the type of a over K in the sense of the ambient model M of T^*. Let K' be the (field-theoretic) algebraic closure in M of the field $K(a)$. Then $(K', c|K')$ is an L-structure whose isomorphism type determines its type by (i). Now we build abstractly another "algebraically closed" model of T, as follows. Let $\tilde{K}$ be a large algebraically closed field containing K and let $(a_i : i < \omega)$ in $\tilde{K}$ be algebraically independent over K. Let K_i' be the (field-theoretic) algebraic closure of $K(a_i)$. Fix field isomorphisms f_i of K' with K_i' over K which take a to a_i, and use these to copy the additional structure (the colouring) to the K_i'. So each K_i' is equipped with a good colouring c_i extending c on K. Let F be the field generated by $\bigcup_i K_i'$. Notice that $\bigcup_i SL_2(K_i')$ is a union of F_{12}-orbits inside F and $\bigcup_i c_i$ gives a good colouring of this union.

Hence (by Lemma 3.2 for example) we can extend $\bigcup_i c_i$ to a good colouring c' of $SL_2(F)$ to get $(F, c') \models T$. Embed (F, c') in a model N of T^*, and we see by (i), that each a_i has the same type over K in N, which also equals $tp(a/K)$ in M. $\square$

3.4. **Simplicity and the proof of Theorem 1.8.**

Proposition 3.15. *Every completion of T^* is simple, and nonforking independence coincides with independence in the sense of the reduct to ACF_0. In particular the SU-rank of $x = x$ is* 1.

Proof. Fix a saturated model $\bar{M}$ of T^*. We let c denote the colouring on $\bar{M}$.

Types will refer to types in $\bar{M}$ and tp_{ACF} to types in the reduct of $\bar{M}$ to the field language. We will use Theorem 4.2 from [22] which says that it suffices to prove that ACF-independence is a "notion of independence" which satisfies the Independence Theorem over a model. The only nontrivial thing to check in terms of being a notion of independence is the extension property, but it follows easily from Corollary 3.14(iv), or by our method of proof below of the Independence Theorem. So it remains to prove that ACF-independence in $\bar{M}$ satisfies the "Independence Theorem over a model": namely suppose M is a small elementary substructure of $\bar{M}$ and a, b, d_0, d_1 are tuples such that a and b are ACF independent over M, d_0 and a are ACF independent over M, d_1 and b are ACF-independent over M, and $tp(d_0/M) = tp(d_1/M)$, THEN there is d realizing $tp(d_0/M)$ such that d is ACF-independent from M, a, b over M. Let $D_0 = acl(d_0 M)$, $D_1 = acl(d_1 M)$, $A = acl(aM)$ and $B = acl(bM)$. In spite of the notation we will enumerate D_0, D_1, A and B (and other sets introduced below) in a consistent fashion (vis-a-vis, d_0, d_1, a, b) as tuples and treat them as such. In particular D_0 and D_1 will have the same type over M in the structure $\bar{M}$ so also in the ACF reduct. By stationarity of this type in the ACF-reduct, if D realizes this ACF-type, ACF-independently from $A \cup B$ over M then D realizes $tp_{ACF}(D_0/A)$ as well as $tp_{ACF}(D_1/B)$.

Let σ_0 be a (field) isomorphism between $acl(D_0 A)$ and $acl(DA)$ over A (again treating these consistently as tuples), and likewise σ_1 an isomorphism between $acl(D_1 B)$ and $acl(DB)$. Use the isomorphisms σ_0 and σ_1 to transport the colourings of $SL_2(acl(D_0 A))$ and $SL_2(acl(D_1 B))$ (coming from the structure $\bar{M}$) to $SL_2(acl(DA))$ and $SL_2(acl(DB))$, which we call c_0 and c_1. Let F be the subfield of $\bar{M}$ generated by $acl(AB)$, $acl(DA)$ and $acl(DB)$. Note that the colouring c' obtained by taking the union of $c|acl(AB)$, c_0 and c_1 is well-defined, as we have that D is ACF-independent from AB over each of A, B, and A is ACF-independent from B over D. Moreover this colouring c' is good, being defined on a union of connected components on each of which it is good. Hence, as usual we can extend c' to a good colouring c'' of $SL_2(F)$. By Lemma 3.13 we can embed (F, c'') into $\bar{M}$ over $acl(AB)$, by a map σ. Let $D' = \sigma(D)$. So D' is ACF-independent from AB over M.

Claim. *D' realizes $tp(D_0/A) \cup tp(D_1/B)$.*

Proof of Claim. We let $alg(C)$ denote the field-theoretic algebraic closure of the subfield of $\bar{M}$ generated by C.

Then $\sigma \circ \sigma_0(alg(D_0 A)) = alg(D'A)$, and for every $e \in SL_2(alg(D_0 A))$ we have that $c(e) = c_0(\sigma_0(e)) = c'(\sigma_0(e)) = c(\sigma \circ \sigma_0(e))$. Thus we have an isomorphism over A between

$$(alg(D_0 A), c|SL_2(alg(D_0 A))) \text{ and } (alg(D'A), c|SL_2(alg(D'A))).$$

Hence by Corollary 3.14(i), D' realizes $tp(D_0/A)$. By a similar proof, D' realizes $tp(D_1/B)$.

This proves the claim as well as the proposition. □

Proof of Theorem 1.8: existence of a non definably amenable group definable in a simple theory. This is precisely as mentioned in the introduction to Section 3: Fix a model $M = (K, c)$ of T^* and let T^{**} be the complete theory of M. Proposition 3.15 says that T^{**} is simple of SU-rank 1. Let $G = SL_2(K)$ as a group definable in M and we use notation as in Section 3.1. Assume for the sake of contradiction that μ is a left invariant Keisler measure on G. Fix an arbitrary $a \in G$ and $i \in [4]$. Then by Axiom (ii) (of T), $G = a(i,1)^{-1}C_i \cup a(i,2)^{-1}C_i \cup a(i,3)^{-1}C_i$. So by invariance of μ, $\mu(C_i) \geq 1/3$. On the other hand the C_i for $i \in [4]$ partition G, whereby $\mu(G) \geq 4/3$, a contradiction. □

Proof of Corollary 1.9. Let $M = (K, c)$ be a saturated model of T^*. Adjoin an "affine copy" of $SL_2(K)$ as a new sort. Namely add a new sort S together with a regular action of $SL_2(K)$ on S, to get a (saturated) structure M'. Then there is a unique 1-type over $\emptyset$ realized in S. Any automorphism invariant Keisler measure on the sort S would yield a translation invariant Keisler measure on $SL_2(K)$, contradicting Theorem 1.8. □

Remark 3.16. Combining the proof of Theorem 1.8 with the setting of [2], it should be possible to obtain the following generalization of Theorem 1.8. Let T be a simple model complete theory eliminating $\exists^\infty$ quantifier, and G a definable group containing a non-abelian free subgroup (as an abstract group, not necessarily definable). Then there exists a simple theory T^* expanding T so that forking in T^* coincides with forking in the reduct T (in particular, T^* has the same SU-rank as T) and G is not definably amenable in T^*.

4. Paradoxical decompositions and additional results.

Lying behind the second example (and also in a sense the first example) is the theory of "definable paradoxical decompositions" from [14], giving necessary and sufficient conditions for a group G definable in a structure M to be definably amenable. When the structure M is a model of set theory and G is just a group, or just when *all* subsets of G are definable, then we are in the context of amenability of a discrete group G, and where there are classical results giving equivalent conditions. In any case the theory of definable paradoxical decompositions gives some interesting invariants of non definably amenable groups and we can ask about the invariants of the example in Section 3. This and various other things are discussed in this final section.

4.1. Definable paradoxical decompositions.

Let us first recall the (classical) notion of a *paradoxical decomposition* of a discrete or abstract group G. We will abbreviate this notion as *cpd* for "classical paradoxical decomposition". A *cpd* for G consists of pairwise disjoint subsets $X_1, \ldots, X_m, Y_1, \ldots, Y_n$ of G and $g_1, \ldots, g_m, h_1, \ldots, h_n \in G$ such that G is the union of the g_iX_i and is also the union of the h_jY_j. Recall that the discrete group G is said to be *amenable* if there is a (left) translation invariant finitely additive probability measure on the collection (Boolean algebra) of *all* subsets of G. The well-known theorem of Tarski is:

Fact 4.1. *Let G a be group. Then G is amenable if and only if G has no paradoxical decomposition.*

Remark 4.2. Clearly after replacing the X_i, Y_j by suitable subsets, we can assume that each of the $(g_i X_i)_i$ and $(h_j Y_j)_j$ form *partitions* of G.

One could ask whether for a definable group G (essentially a group equipped with a certain Boolean algebra of subsets, closed under left translation), we have the identical result: G is definably amenable iff G has a *definable cpd*, namely where the X_i, Y_j are definable? We expect the answer is no. In any case Tarski's proof of "nonamenability implies the existence of a *cpd*" is nonconstructive and does not go over immediately to a definable version.

So in [14] there is another version of paradoxical decomposition which does give a characterization of definable amenability, remaining in the Boolean algebra of definable sets.

We will briefly describe this here. We fix a definable group G in a structure M. Definable will mean *with parameters*.

By a *(m-)cycle* (for $m \geq 0$) we mean a formal sum $\sum_{i=1,\ldots,m} X_i$ of definable subsets X_i of G. If all the X_i are the same we could write this formal sum as mX_i. We can add such cycles in the obvious way to get the "free abelian monoid" generated by the definable subsets of G. And any definable subset X of G (including G itself) is of course a (1-)cycle.

If $X = \sum_{i=1,\ldots,m} X_i$ and $Y = \sum_{j=1,\ldots,n} Y_j$ are two cycles, then by a *definable piecewise* translation f from X to Y we mean a map f from the formal disjoint union $X_1 \sqcup \ldots \sqcup X_m$ to the formal disjoint union $Y_1 \sqcup \ldots \sqcup Y_n$ for which there is a partition of each X_i into definable subsets $X_{i1}, \ldots, X_{in_i}$, and for each i and $t \leq n_i$, an element g_{it} of G such that the restriction $f|X_{it}$ of f to X_{it} is just left translation by g_{it}, and $g_{it}X_{it}$ is a subset of one of the Y_j's. By a *definable* map from X to Y we mean just the same thing except that translation by g_{it} on X_{it} is replaced by a definable function with domain X_{it} and image contained in some Y_j.

Such a definable piecewise translation (or definable map) f is said to be *injective* if it is injective as a map between formal disjoint unions. So for example, in the case of definable piecewise translations this would mean that for each $i, i' \leq m$ and $t \leq n_i, t' \leq n_{i'}$ if f takes both X_{it} and $X_{i't'}$ into the same Y_j, then for $x \in X_{it}$ and $x' \in X_{i't'}$, $f(x) = f(x')$ implies that $i = i'$, $t = t'$ and $x = x'$.

We write $X \leq Y$ if there is an injective piecewise definable translation f from X to Y. Note that $\leq$ is reflexive and transitive. Also $X \leq W$ and $Y \leq Z$ implies $X + Y \leq W + Z$.

Definition 4.3. By a *definable paradoxical decomposition* (*dpd*) of the definable group G we mean an injective definable piecewise translation from $G + Y$ to Y for some cycle Y.

The following is proved in [14] (Proposition 5.4).

Fact 4.4. *G is definably amenable if and only if G does not have a dpd.*

Lemma 4.5. *Suppose $G + Y \leq Y$ where $Y = \sum_{i=1,\ldots,n} Y_i$ (with the Y_i definable). Then*

(i) $mG + Y \leq Y$ *for all* $m \geq 1$,
(ii) $2Y \leq Y$,
(iii) $(n+1)G \leq nG$.

Proof. (i) By induction: $G + Y \leq Y$ implies $(m+1)G + Y = mG + G + Y \leq mG + Y \leq Y$.

(ii) $Y + Y \leq nG + Y \leq Y$ (by taking $n = m$ in part (i)).

(iii) $(n+1)G \leq (n+1)G + Y \leq Y$ (by (i)) $= \sum_{i=1,\ldots,n} Y_i \leq \sum_{i=1,\ldots,n} G = nG$. □

Corollary 4.6. *G has a dpd iff $(n+1)G \leq nG$ for some $n \geq 1$.*

On the other hand:

Lemma 4.7. *G has a definable cpd (a cpd where the X_i and Y_j are definable) if and only if $2G \leq G$ (if and only if $(n+1)G \leq nG$ for all n).*

Proof. Suppose $G = \bigcup_{i=1,\ldots,m} g_i X_i = \bigcup_{j=1,\ldots,n} h_i Y_j$ witnesses a definable *cpd*. As mentioned in Remark 4.2, by replacing the X_i and Y_j by suitable subsets we can assume pairwise disjointness of the $g_i X_i$, as well as pairwise disjointness of the $h_j Y_j$, and we get that $G + G \leq \bigcup_i X_i \cup \bigcup_j Y_j \leq G$.

The converse works the same way: if $G + G \leq G$, then we have two partitions of G, as $\bigcup_i X_i$ and $\bigcup_j Y_j$ as well as $g_i, h_j \in G$, such that the sets $g_i^{-1} X_i$, $h_j^{-1} Y_j$ are *all* pairwise disjoint. □

Hence the question of whether a non definably amenable group G has a definable *cpd* is the same as asking whether $2G \leq G$. (Of course when G is equipped with predicates for all subsets then this has a positive answer, by Tarski's theorem.) We expect it has a negative answer in general.

Remark 4.8. Let G be the definable group produced in Section 3 above. Then $4G \leq 3G$.

Proof. We have (with notation as in Section 3), that $G = \bigcup_{j \in [3]} a(i,j)^{-1} C_i$ for each $i \in [4]$. By cutting down each C_i we may assume that for each i, the $a(i,j)^{-1} C_i$ are disjoint. (Of course the C_i's remain disjoint although their union may no longer equal G.)

Now we obtain an injective piecewise definable translation from $4G$ to $3G$ by taking $a(i,j)^{-1} C_i$ in the *ith* copy of G to C_i in the *jth* copy of G, for $i \in [4]$, $j \in [3]$ □

It is likely that the generic nature of the example from Section 3 implies that $n = 3$ is least such that $(n+1)G \leq nG$.

In the rest of this subsection we will explain how to modify the example so as to produce $2G \leq G$ also in an ambient SU-rank 1 theory. So this will be in a sense, a "better" example, with respect to the invariant "least possible n" where n is as in Corollary 4.6. Thus, in this modified example there is a definable *cpd* and we will see below that the "definable Tarski number" (the least sum $m + n$ that can appear in a definable *cpd* of G) equals 6 which is the least possible for non definably amenable groups definable in simple theories.

We do a similar thing to Section 3, but with F_6 in place of F_{12} and six colours in place of four colours. We choose a_i for $i = 1, \ldots, 6$ to be elements of $SL_2(\mathbb{Z})$ which are free generators of a copy of F_6 inside $SL_2(\mathbb{Z})$. For the universal theory T in Section 3.1 we work in the language of rings with 6 additional 4-ary predicates $C_1, \ldots, C_6$ and the axioms say that R is an integral domain of characteristic 0, that $C_1, \ldots, C_6$ partition $SL_2(R)$, and $SL_2(R) = a_1^{-1} C_1 \cup a_2^{-1} C_2 \cup a_3^{-1} C_3 = a_4^{-1} C_4 \cup a_5^{-1} C_5 \cup a_6^{-1} C_6$.

Note that this will already give a definable *cpd* of $G = SL_2(R)$ and so $2G \leq G$ by Lemma 4.7.

We have to construct again the model companion T^* of T and show it to be simple (of SU-rank 1).

The main thing is to modify the combinatorial lemmas in Section 3.2. So now we have a free action of F_6 on a set X, and for $X_0 \subseteq X$, by a good colouring $c : X_0 \to [6]$ we mean that for all $x \in X_0$

(i) if $a_i \cdot x \in X_0$ for all $i = 1, 2, 3$ then $c(a_i \cdot x) = i$ for some $i = 1, 2, 3$, and
(ii) if $a_j \cdot x \in X_0$ for all $j = 4, 5, 6$ then $c(a_j \cdot x) = j$ for some $j = 4, 5, 6$.

Again we have the notions of distance, connectedness etc., with respect to the relevant Cayley graph on X.

New version of Lemma 3.2: extending a good colouring of X_0 to a good colouring of X when X_0 is connected.

Proof. (By induction on n.) Suppose we have extended the good colouring c of X_0 to a good colouring c_n of $B_n(X_0)$. Suppose $i \in [6]$, and $y = a_i \cdot x$ is in $B_{n+1}(X_0) \setminus B_n(X_0)$ for some $x \in B_n(X_0)$, then define $c_{n+1}(y) = i$. This is well-defined by uniqueness of paths. And if $y \in B_{n+1}(X_0) \setminus B_n(X_0)$ is not of this form, define $c_{n+1}(y)$ arbitrarily.

Again we have to check that c_{n+1} is a good colouring of $B_{n+1}(X_0)$. Suppose $x \in B_{n+1}(X_0)$ and $a_i \cdot x \in B_{n+1}(X_0)$ for all $i = 1, 2, 3$. If $a_i \cdot x \in B_n(X_0)$ for all $i = 1, 2, 3$, then connectedness of $B_n(X_0)$ implies that also $x \in B_n(X_0)$. So as c_n is a good colouring of $B_n(X_0)$, and c_{n+1} extends c_n, Axiom (i) is satisfied at x. Otherwise $a_i \cdot x \in B_{n+1}(X_0) \setminus B_n(X_0)$ for some $i = 1, 2, 3$ and so $x \in B_n(X_0)$, hence $c_{n+1}(a_i \cdot x) = i$.

Exactly the same holds for $x \in B_{n+1}(X_0)$ for which $a_i \cdot x \in B_{n+1}(X_0)$ for all $i = 4, 5, 6$.

So, as in Lemma 3.2, we have extended the good colouring of X_0 to a good colouring of X. □

New version of Lemma 3.3.

Proof. We have C_0, C_1 connected subsets of X with $3 \leq d(C_0, C_1) < \infty$ and C is the smallest connected set containing $C_0 \cup C_1$. And we want to extend a good colouring c_0 of $C_0 \cup C_1$ to a good colouring c of C. As in Lemma 3.3 we reduce to the case of a path (u, v, y, z) from $u \in C_0$ to $z \in C_1$ where $v, y \notin C_0 \cup C_1$. If $v = a_i \cdot u$ for some $i \in [6]$, put $c(v) = i$, and define it arbitrarily otherwise. Likewise if $y = a_i \cdot z$ for some $i \in [6]$ put $c(y) = i$, and define it arbitrarily otherwise. Again we check that c is well-defined and that the good colouring axioms are satisfied. □

Lemmas 3.4 and 3.5 adapt (formally) word for word to the new context. As well as the definition of the model companion T^* in Section 3.3 and the simplicity of (all completions of) T^* in Section 3.4.

So the conclusion is:

Proposition 4.9. *There is a definable group G in a model of a simple theory such that non definable amenability of G is witnessed by a definable cpd, equivalently such that $2G \leq G$.*

A final remark in this section concerns the numbers m, n witnessing a definable *cpd*, namely the existence of pairwise disjoint definable subsets $X_1, \ldots, X_m$,

$Y_1, \ldots, Y_n$ of G and $g_1, \ldots, g_m, h_1, \ldots, h_n \in G$ such that $G = \cup g_i X_i = \cup h_j Y_j$. Following classical terminology, for a definable group G which is not definably amenable, a least possible value of $m+n$ that occurs in a definable *cpd* of G can be called the *definable Tarski number* of G. (And if G has *no* definable *cpd* we will say that its definable Tarski number is ∞.)

Proposition 4.10. *Suppose G is a definable group in a structure M and G has a definable cpd with attached numbers m, n. If either $m = 2$ or $n = 2$ then $Th(M)$ has the strict order property. In particular, the definable Tarski number of a non definably amenable group definable in a simple theory is at least* 6.

Proof. So we assume that G is the disjoint union of nonempty X_1, X_2, and Y and that $G = g_1 X_1 \cup g_2 X_2$. Then G is also the disjoint union of $g_1 X_1$, $g_1 X_2$ and $g_1 Y$. Replacing X_1, X_2, Y by their g_1-translates, and changing notation there is $g \in G$ such that $X_1 \cup g X_2 = G$. So X_2 is a proper subset of $g X_2$. Iterating we have a strictly increasing sequence $X_2 \subset g X_2 \subset g^2 X_2 \subset g^3 X_2 \subset \ldots$, yielding the strict order property. □

Thus in the modified example above, the definable Tarski number is 6 (as there is a definable *cpd* with $n = m = 3$). So in terms of definable Tarski number, this is the simplest possible example of a non definably amenable goup definable in a simple theory.

4.2. **Small theories.** The aim here is to give some positive results concerning definable amenability for groups definable in (models of) small theories T, as well as some related results around amenability of small theories.

Recall that a complete countable theory T is said to be *small* if for all $n \in \mathbb{N}_{\geq 1}$ the type space $S_n(T)$ is countable. This is equivalent to saying that for any model M of T and finite subset A of M, the type space $S_1(A)$ is countable.

We could prove the definable amenability of definable groups in small theories directly from Fact 4.4. But we can slightly generalize the set-up so as to obtain other corollaries.

Our general context consists of a group G acting on a set S and where we are given a Boolean algebra $\mathcal{B}$ of subsets of S which is closed under the action of G (in particular $\emptyset$ and S are elements of $\mathcal{B}$). We will call $\mathcal{B}$ a *G-invariant Boolean algebra* of subsets of S.

Replacing "definable" by "in $\mathcal{B}$", we can copy the notions of (m-)cycles and $\mathcal{B}$-piecewise translations from Section 4.1 to the present context. We can also introduce the notion of *$\mathcal{B}$-map* f from a cycle $\sum_i X_i$ to a cycle $\sum_j Y_j$. This will be a map from the formal disjoint union $\bigsqcup_i X_i$ to the formal disjoint union $\bigsqcup_j Y_j$ such that for every i and $B \in \mathcal{B}$ with $B \subseteq X_i$, and for every j, $f(B) \cap Y_j \in \mathcal{B}$. Note that this makes sense without any G-action. Note also that any $\mathcal{B}$-piecewise translation is a $\mathcal{B}$-map, and that injectivity makes sense for $\mathcal{B}$-maps.

Observe that both the class of $\mathcal{B}$-piecewise translations and the class of $\mathcal{B}$-maps are closed under composition . As before we write $X \leq Y$ if there is an injective $\mathcal{B}$-piecewise translation from X to Y.

By a *$\mathcal{B}$-paradoxical decomposition* (*$\mathcal{B}$pd*) we mean an injective $\mathcal{B}$-piecewise translation from $S+Y$ to Y for some cycle Y. Also we say that the G-set S is *$\mathcal{B}$-amenable* if there is a G-invariant finitely additive probability measure on $\mathcal{B}$. The proof of Fact 4.4 in [14] adapts to yield:

Proposition 4.11. *The G-set S is $\mathcal{B}$-amenable if and only if S has no $\mathcal{B}$-paradoxical decomposition.*

Lemma 4.5 and Corollary 4.6 remain valid in the more general context of $\mathcal{B}$-piecewise translations. Actually, we use this generalization of Lemma 4.5(ii) in the proof of Proposition 4.13 below.

We will call a Boolean algebra $\mathcal{B}$ *small* if its Stone space is countable, in other words there are only countably many ultrafilters on $\mathcal{B}$.

Let us introduce some notation for cycles which will be used in a proof below. The context here and in the next lemma is simply a Boolean algebra $\mathcal{B}$ of subsets of a set S. Let $X = \sum_{i=1,\dots,n} X_i$ and $Z = \sum_{i=1,\dots,n} Z_i$ be cycles (so n is the same in both). We also fix the ordering of the X_i and Z_i.

(1) $X \sqsubseteq Z$ means that $X_i \subseteq Z_i$ for each i,
(2) $X \cap Z = \emptyset$ means that $X_i \cap Z_i = \emptyset$ for each i, and
(3) $X \neq \emptyset$ means that some X_i is nonempty.

If moreover f is a $\mathcal{B}$-map from X to Z, then by the *image* $f(X)$ of X under f we mean the cycle $\sum_i W_i$ where W_i is $f(X_1 \sqcup \dots \sqcup X_n) \cap Z_i$ which we note is in $\mathcal{B}$.

Lemma 4.12. *Suppose that Y is a nonempty cycle and f_0, f_1 are injective $\mathcal{B}$-maps from Y to Y such that $f_0(Y) \cap f_1(Y) = \emptyset$. Then $\mathcal{B}$ is not small.*

Proof. The proof goes by induction on the length of the cycle Y. First suppose that Y is a 1-cycle, i.e. Y is in $\mathcal{B}$. For $\eta \in 2^{<\omega}$, let $f_\eta : Y \to Y$ be given by: $f_\emptyset$ is the identity, and $f_\eta = f_{\eta(0)} \circ f_{\eta(1)} \circ \dots \circ f_{\eta(k-1)}$ when $dom(\eta) = \{0, \dots, k-1\}$ with $k > 0$. And let $Y^\eta = f_\eta(Y)$. Then the Y^η are nonempty subsets of Y which are in $\mathcal{B}$ (as the f_i are $\mathcal{B}$ maps), $Y^\eta \supset Y^\tau$ when τ extends η, and $Y^{\eta 0} \cap Y^{\eta 1} = \emptyset$ for all η. For $\eta \in 2^\omega$ let $\Sigma_\eta = \{Y^{\eta|n} : n < \omega\}$. Then each Σ_η extends to an ultrafilter p_η on $\mathcal{B}$ and $p_\eta \neq p_\tau$ for $\eta \neq \tau \in 2^\omega$.

When Y is an n-cycle $\sum_{i=1,\dots,n} Y_i$ for $n > 1$ it is a bit more complicated. With the notation introduced above, define the $f_\eta : Y \to Y$ in the same way for $\eta \in 2^{<\omega}$, and define $Y^\eta = f_\eta(Y)$, and now define Y_i^η to be $f_\eta(Y_1 \sqcup \dots \sqcup Y_n) \cap Y_i$.

Again the sets Y_i^η are in $\mathcal{B}$ and we have, for all $\eta \in 2^{<\omega}$:

(1) $Y^\eta = \sum_{i=1,\dots,n} Y_i^\eta$,
(2) $Y^\eta \neq \emptyset$,
(3) $Y^\eta \sqsubseteq Y^\tau$ whenever η extends τ, and
(4) $Y^{\eta 0} \cap Y^{\eta 1} = \emptyset$.

Note that in particular the Y_n^η satisfy both (3) and (4). If they also satisfy (2) then we get continuum many ultrafilters on $\mathcal{B}$ as in the $n = 1$ case. Otherwise there is η such that $Y_n^\eta = \emptyset$ and therefore so is $Y_n^{\eta'}$ for all η' extending η. Consider the tree of cycles $(Y'_\tau)_\tau$ where $Y'_\tau = \sum_{i=1,\dots,n-1} Y_i^{\eta\tau}$. These are $(n-1)$-cycles and so by the inductive hypothesis, we obtain continuum many ultrafilters on $\mathcal{B}$. $\square$

Proposition 4.13. *Let S be a G-set with a G-invariant Boolean algebra $\mathcal{B}$ of subsets. Suppose that for every finitely generated subgroup G_0 of G and finite subset $\mathcal{B}_0$ of $\mathcal{B}$ the Boolean algebra $\langle G_0 \mathcal{B}_0 \rangle$ generated by all translates of elements of B_0 by elements of G_0 is small. Then S is $\mathcal{B}$-amenable. In particular if $\mathcal{B}$ is small, S is $\mathcal{B}$-amenable.*

Proof. If S is not $\mathcal{B}$-amenable then it is witnessed by $S + Y \leq Y$ for some nonempty cycle Y. By the obvious generalization of Lemma 4.5(ii) mentioned above, we get

$2Y \leq Y$, so we have injective $\mathcal{B}$-piecewise translations $f_0 : Y \to Y$ and $f_1 : Y \to Y$ such that $f_0(Y) \cap f_1(Y) = \emptyset$. Let G_0 be the subgroup of G generated by the finitely many elements of G appearing in the translations in f_0, f_1, and let $\mathcal{B}_0$ be the finite collection of elements of $\mathcal{B}$ which appear as the subsets of the elements of the cycle Y which are translated in the maps f_0, f_1. Then f_0 and f_1 are $\langle G_0\mathcal{B}_0\rangle$-maps, so $\langle G_0\mathcal{B}_0\rangle$ is not small by Lemma 4.12. Hence, $\mathcal{B}$ is not small. □

Here are some applications:

Corollary 4.14. *Suppose that G is a definable group in a model M of a small theory T. Then G is definably amenable.*

Proof. First as T remains small after naming finitely many parameters we may assume G is $\emptyset$-definable. Remember that definable amenability of G refers to there being a translation invariant Keisler measure on the family of all definable, with parameters in M, subsets of G. We apply Proposition 4.13 to the case $S = G$ and $\mathcal{B}$ the collection of definable subsets of G. If $\mathcal{B}_0$ is a finite subset of $\mathcal{B}$ and G_0 is a finitely generated subgroup of G then the elements of the Boolean algebra $\langle G_0\mathcal{B}_0\rangle$ are all definable over a fixed finite set A of parameters. So by smallness of T this Boolean algebra is small, and we can apply Proposition 4.13. □

Proposition 4.13 also gives another proof of Fact 1.7 above:

Corollary 4.15. *Any group G definable in a model M of a stable theory is definably amenable.*

Proof. By Fact 4.4 we may assume that T is countable and M is countable. Then for any finite collection $X_1, \ldots, X_n$ of definable subsets of G, the Boolean algebra generated by the set of all left G-translates of the X_i is small. (For any finite collection of L-formulas $\phi_1(x, y_1), \ldots, \phi_n(x, y_n)$ where x ranges over G and the y_i are arbitrary tuples, the Boolean algebra generated by instances $\phi_i(x, a_i)$ of the ϕ_i, with a_i in a given countable model is small.) So we can apply Proposition 4.13 again. □

One could unify the two previous corollaries as follows. Let G be a definable group. Suppose that for every finite set $\Delta = \{\phi_1(x, y_1), \ldots, \phi_n(x, y_n)\}$ of L-formulas, and finite set A of parameters, the Boolean algebra of subsets of G which are both Δ-definable and A-definable, is small. Then G is definably amenable.

It is natural to ask whether every complete countable small theory T is *amenable*, as defined in the introduction. However the theory of the dense circular ordering is ω-categorical, with a unique 1-type over $\emptyset$, but there is no automorphism invariant Keisler measure on the universe $x = x$, as explained in Remark 2.2 of [11], as $\emptyset$ is not an extension base.

But we point out that a rather weaker property follows from Proposition 4.13
$(*)$ For every $\emptyset$-definable set D there is a global Keisler measure concentrated on D which is invariant under *definable* automorphisms.

We may want to call a complete theory T *weakly amenable* if it satisfies $(*)$, but this would be an unnecessary introduction of new terminology. In any case:

Corollary 4.16. *Suppose that the countable complete theory T is small. Then T satisfies $(*)$.*

Proof. Let D be a $\emptyset$-definable set in a saturated model $\bar{M}$ of T. Let $Aut_{def}(\bar{M})$ be the group of automorphisms of $\bar{M}$ which are definable (with parameters) in $\bar{M}$. Apply Proposition 4.13 to the situation where $G = Aut_{def}(\bar{M})$, $S = D$, and $\mathcal{B}$ is the Boolean algebra of all definable (with parameters) subsets of D. Then by smallness the assumption of Proposition 4.13 is satisfied, so we get $(*)$. □

Remark 4.17. (i) Corollary 4.16 implies Corollary 4.14 via the usual trick of adding a new affine sort.

(ii) We obtain a characterization of when an $\emptyset$-definable set D satisfies $(*)$, by the nonexistence of an appropriate paradoxical decomposition. This is by Proposition 4.11 applied to $G = Aut_{def}(\bar{M})$, $S = D$ and $\mathcal{B}$ the Boolean algebra of definable subsets of D.

(iii) Similarly taking $G = Aut(\bar{M})$, $S = D$ and $\mathcal{B}$ as in (ii) we obtain a characterization of when there is an automorphism invariant global Keisler measure on D.

Finally we give an application of the material above (Lemma 4.12) to prove the nontriviality of *graded* Grothendieck rings of small theories. We first recall the usual Grothendieck rings ([23]) attached to a structure M which may be many sorted, although we give a slightly different presentation. We will assume that some sort has at least 2 elements. Let $Def(M)$ be the collection of all definable (with parameters from M) subsets of products of the basic sorts of M. Let $K(M)$ be the free abelian monoid generated by $Def(M)$. We can view the elements of $K(M)$ as cycles $\sum_{i=1,\ldots,n} X_i$ where the X_i are definable sets. As earlier we have the notion of a definable map between cycles and in particular a definable bijection between cycles. Let $\sim$ be the equivalence relation on $K(M)$ of being in definable bijection, for a cycle D let $[D]$ be its $\sim$-equivalence class, and let $K_{semi}(M)$ be the quotient $K(M)/\sim$. In this context one sees that every cycle is $\sim$-equivalent to a definable set (in an appropriate product of sorts), whereby $K_{semi}(M) = \{[D] : D \in Def(M)\}$, and is moreover an abelian monoid with $0 = [\emptyset]$. It also has a unital commutative semiring structure by defining $[D_1] \cdot [D_2] = [D_1 \times D_2]$ and taking the multiplicative identity to be $[\{a\}]$ for any singleton in any sort. Finally we put an equivalence relation $\sim_0$ on $K_{semi}(M)$: $[D_1] \sim_0 [D_2]$ if there is $[D]$ such that $[D_1] + [D] = [D_2] + [D]$. We denote the quotient by $K'_{semi}(M)$, a cancellative, unital, commutative semiring. We let $[D]_0$ denote the $\sim_0$-class of $[D]$. Then adding formal additive inverses yields a canonical unital commutative ring $K_0(M)$ extending $K'_{semi}(M)$, called the *Grothendieck ring* of the structure M. The elements of $K_0(M)$ can be written in the form $[D_1]_0 - [D_2]_0$, for $D_1, D_2 \in Def(M)$.

Example 4.18. Let $M = (\mathbb{N}, s)$, where s is the successor function. Then $Th(M)$ is small, but $K_0(M)$ is trivial.

Proof. The function s gives a definable bijection from $\mathbb{N}$ to $\mathbb{N} \setminus \{0\}$ whereby $[\{0\}]_0 + [\mathbb{N}]_0 = [\mathbb{N}]_0$ in $K_0(M)$, hence $[\{0\}]_0$ is the zero element of $K_0(M)$. As it is also the 1 of $K_0(M)$, $K_0(M)$ is trivial. □

However by working with *graded* Grothendieck rings we get a rather different situation. Again we fix a structure M, maybe many-sorted, but we define $K_0(S)$ for S a sort, or product of sorts, and define $K_0^{grad}(M)$ to be $\bigoplus_S K_0(S)$. Here are the details. First fix a sort S (or product of sorts). Start with $Def(S)$ the Boolean algebra of definable (with parameters) subsets of S. Again define $K(S)$ to be collection of cycles of formal sums of elements of $Def(S)$, and $\sim$ the equivalence relation on $K(S)$ of being in definable bijection. Let $K_{semi}(S)$ be the quotient $K(S)/\sim$. It is no longer true that every element of $K_{semi}(S)$ is of the form $[D]$ for $D \in Def(S)$. Again form $K'_{semi}(S)$, and $K_0(S)$ whose elements are of the form $[D_1]_0 - [D_2]_0$ for $D_1, D_2 \in K_{semi}(S)$. Now $K_0(S)$ is just a commutative group with no ring structure. We define $K_0^{grad}(M)$ to be the direct sum $\bigoplus_S K_0(S)$ with its abelian group structure, but also with a commutative ring structure obtained as follows: for $D_1 \in Def(S_1)$ and $D_2 \in Def(S_2)$, let $[D_1]_0 \cdot [D_2]_0 = [D_1 \times D_2]_0$ in $K_0(S_1 \times S_2)$. And extend it bilinearly to $\cdot$ from $K_0(S_1) \times K_0(S_2)$ to $K_0(S_1 \times S_2)$.

Proposition 4.19. *Let M be any structure in a countable language such that $Th(M)$ is small. Then*

- *(i) For every sort S, the group $(\mathbb{Z}, +)$ embeds into the group $K_0(S)$, in particular $K_0(S)$ is nontrivial,*
- *(ii) The ideal $t\mathbb{Z}[t]$ in the polynomial ring $\mathbb{Z}[t]$ embeds in $K_0^{grad}(M)$.*

Proof. (i) Consider the homomorphism from $(\mathbb{Z}, +)$ to $K_0(S)$ which takes 1 to $[S]_0$. To show it is an embedding we have only to show that for each $n \geq 1$, $n[S]_0 \neq 0$. Otherwise there is a cycle $Y \in K(S)$ such that $n[S]+[Y] = [Y]$, yielding a definable injection from $S + Y$ to Y, and thus from $Y + Y$ to Y by the obvious variant of Lemma 4.5(ii). So we have definable injections $f_0 : Y \to Y$ and $f_1 : Y \to Y$ with $f_0(Y) \cap f_1(Y) = \emptyset$. Let $\mathcal{B}$ be the Boolean algebra of subsets of S which are definable over the fixed finite set of parameters over which f_0, f_1 and the summands of Y are defined. Then f_0, f_1 are $\mathcal{B}$-maps. By Lemma 4.12, $\mathcal{B}$ is not small, hence $Th(M)$ is not small, a contradiction.

(ii) Given a sort S, we see from the proof of (i) that $[S]_0$ generates a subring isomorphic to $t\mathbb{Z}[t]$. □

Note that we obtain a characterization of when $K_0^{grad}(M)$ is trivial. It is precisely that for every sort S and definable subset D of S there is a cycle $Y \in K(S)$ such that $D + Y \sim Y$.

On the other hand, triviality of the Grothendieck ring $K_0(M)$ is equivalent to there being a definable set D and $d \in D$ such that D and $D \setminus \{d\}$ are in definable bijection.

A final remark is that the definition of $K_0^{grad}(M)$ depends on the choice of sorts S. We could rechoose all definable sets to be sorts, in which case the new graded Grothendieck ring will be bigger and nontrivial, because for a singleton sort S, all cycles on S are finite sums of singletons and two such cycles are $\sim$-equivalent iff they have the same cardinality.

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