Theories with NIP. List 10.

We work in a monster model $\mathfrak{C}$ of a complete theory T.

Problem 1. Let M_n, $n \in \omega$, be finite L-structures, and μ_n be the normalized counting measure on M_n. Let $M := \prod_n M_n/\mathcal{D}$ be an ultraproduct. For $b \in M$ take any representative $(b_0, b_1, \dots) \in \prod_n M_n$. Define $\mu(\varphi(x, b)) := \lim_{\mathcal{D}} \mu_n(\varphi(x, b_n))$.
(i) Prove that this is a well-defined Keisler measure over M.
(ii) Let L^+ be the original language L expanded by the new sort S carrying $+$ and $\leq$, as well as function symbols $f_\varphi(y)$ from the original sort to S for all $\varphi(x, y) \in L$ (where $|x| = 1$). Each M_n can be naturally expanded to an L^+-structure M_n^+, by interpreting S as $[0, 1]$, $+$ as addition modulo 1, $\leq$ as the usual order, and $f_\varphi(b) := \mu_n(\varphi(M_n, b))$. Define $M^+ := \prod M_n^+/\mathcal{D}$ (so M^+ has sorts M and a nonstandard interval $[0, 1]^*$). Check that $\mu(\varphi(x, b)) = \mathrm{st}(f_\varphi(b))$.

Problem 2. Prove Claim 1 from page 55 of the notes.

Problem 3. Prove Claim 2 from page 55 of the notes.

Problem 4. Show that the function μ'_x obtained in Construction $(*)$ on page 56 is a Keisler measure extending μ_x.

Problem 5. Let μ_x be a Keisler measure over A. Let $U \subseteq S_x(A)$ be open. Show that $\inf\{\mu_x([\psi(x)] \cap U) : [\neg\psi(x)] \subseteq U\} = 0$.

Problem 6. Assume T has NIP. Let $\mu \in M_x(M)$. Prove that there does not exist $\varphi(x, y) \in L$, $(b_i)_{i\in\omega} \subseteq M$, and $\epsilon > 0$ such that for all distinct $i, j < \omega$, $\mu(\varphi(x, b_i) \wedge \varphi(x, b_j)) \geq \epsilon$.
Hint. Use Construction $(*)$, Lemma 1 from p. 58, and NIP.

Problem 7. Assume T has NIP. Let $\mu \in M_x(\mathfrak{C})$.
(i) Prove that $\mathcal{L}_x(\mathfrak{C})/\sim_\mu$ is of bounded size, and find a bound in terms of $|T|$.
(ii) Prove that $S(\mu)$ is of bounded size, and find a bound in terms of $|T|$.