Theories with NIP. List 11.

We work in a monster model $\mathfrak{C}$ of a complete theory $T$.

**Problem 1.** Let $\mu_x$ be a Keisler measure over $B$ treated as a regular, Borel probability measure on $S_x(B)$. Let $A \subseteq B$. Define $\mu_A$ on Borel subsets of $S_x(A)$ by $\mu_A(X) := \mu(\pi^{-1}(X))$, where $\pi : S_x(B) \to S_x(A)$ is the restriction map. Check that $\mu_A$ is a regular, Borel probability measure on $S_x(A)$, and so it coincides with $\mu|_A$.

**Problem 2.** (i) Let $p \in S_x(A)$ and $X \subseteq S_x(A)$ be Borel. Check that

$$p(X) := \mu_p(X) = \begin{cases} 1 & \text{if } p \in X \\ 0 & \text{if } p \notin X. \end{cases}$$

(ii) Let $a_0, \ldots, a_{n-1} \in \mathfrak{C}[x^i]$; $p_0 := \text{tp}(a_0/A), \ldots, p_{n-1} := \text{tp}(a_{n-1}/A)$, and $X \subseteq S_x(A)$ be Borel. Conclude that $\text{Av}(p_0, \ldots, p_{n-1}; X) = \text{Av}(a_0, \ldots, a_{n-1}; X)$.

**Problem 3.** Let $\mu \in M_x(A)$.

(i) Check that $\mu(S(\mu)) = 1$.
(ii) Let $\epsilon > 0$. Prove that there is $\delta > 0$ such that whenever $p_0, \ldots, p_{n-1} \in S_x(A)$ satisfy $|\mu(S(\mu)) - \text{Av}(p_0, \ldots, p_{n-1}; S(\mu))| < \delta$, then for every Borel $X \subseteq S_x(A)$ we have $|\text{Av}(p_0, \ldots, p_{n-1}; X) - \text{Av}(q_0, \ldots, q_{l-1}; X)| < \epsilon$, where $\{q_0, \ldots, q_{l-1}\} = S(\mu) \cap \{p_0, \ldots, p_{n-1}\}$.

**Problem 4.** Assume NIP. Let $\mu \in M_x(\mathfrak{C})$ and $A \subseteq \mathfrak{C}$. Prove that $\mu$ does not fork over $A$ iff $\mu$ is $\text{Lstp}_A$-invariant.

**Definition.** (i) We say that a function $f : M^n \to C$ (where $C$ is compact, Hausdorff) is definable if for every disjoint closed subsets $F_1$ and $F_2$ of $\mathfrak{C}$ the preimages $f^{-1}[F_1]$ and $f^{-1}[F_2]$ can be separated by a definable set.

(ii) We say that a function $f : \mathfrak{C}^n \to C$ (where $C$ is compact, Hausdorff) is definable over $M$ (or $M$-definable) if the preimage under $f$ of any closed set is type-definable over $M$.

**Problem 5.** Let $C$ be a compact, Hausdorff space. Prove that:

(i) if $f : M^n \to C$ is definable, then it extends uniquely to an $M$-definable function $f^* : \mathfrak{C}^n \to C$. Moreover, $f^*$ is given by \( \{f^*(a)\} = \bigcap_{\varphi \in \text{tp}(a/M)} \text{cl}(f[\varphi(M)]) \);

(ii) conversely, if $f^* : \mathfrak{C}^n \to C$ is an $M$-definable function, then $f^*|_M : M^n \to C$ is definable;

(iii) a function $f^* : \mathfrak{C}^n \to C$ is definable over $M$ iff there is a continuous function $h : S_n(M) \to C$ such that $f^* = h \circ r$, where $r : \mathfrak{C}^n \to S_n(M)$ is given by $r(a) := \text{tp}(a/M)$.

**Definition.** A Keisler measure $\mu \in M_x(\mathfrak{C})$ is said to be definable over $M$ if for every formula $\varphi(x, y) \in L$ the function $f_\varphi : \mathfrak{C}^{[n]} \to [0, 1]$ given by $f_\varphi(b) := \mu(\varphi(x, b))$ is definable over $M$ in the above sense.

**Problem 6.** Let $\mu \in M_x(\mathfrak{C})$. Prove that $\mu$ is definable over $M$ iff it is invariant over
and for every $r > 0$ the set $\{q \in S_y(M) : \text{for any/some } b \models q, \mu(\varphi(x, b)) < r\}$ is open in $S_y(M)$.

**Problem 7.** Assume NIP. Let $\mu \in M_x(\mathfrak{C})$. Prove that:

(i) if $\mu$ is finitely satisifiable in $M$, then $\mu$ is invariant over $M$;
(ii) $\mu$ is finitely satisifiable in $M$ iff each type in $S(\mu)$ is finitely satisifiable in $M$;
(iii) $\mu$ is invariant over $M$ iff each type in $S(\mu)$ is invariant over $M$. 