Theories with NIP. List 11.

We work in a monster model $\mathfrak{C}$ of a complete theory T.

Problem 1. Let μ_x be a Keisler measure over B treated as a regular, Borel probability measure on $S_x(B)$. Let $A \subseteq B$. Define μ_A on Borel subsets of $S_x(A)$ by $\mu_A(X) := \mu(\pi^{-1}[X])$, where $\pi\colon S_x(B) \to S_x(A)$ is the restriction map. Check that μ_A is a regular, Borel probability measure on $S_x(A)$, and so it coincides with $\mu\restriction_A$.

Problem 2. (i) Let $p \in S_x(A)$ and $X \subseteq S_x(A)$ be Borel. Check that

$$p(X) := \mu_p(X) = \begin{cases} 1 & \text{if} \quad p \in X \\ 0 & \text{if} \quad p \notin X. \end{cases}$$

(ii) Let $a_0, \dots, a_{n-1} \in \mathfrak{C}^{|x|}$, $p_0 := \mathrm{tp}(a_0/A), \dots, p_{n-1} := \mathrm{tp}(a_{n-1}/A)$, and $X \subseteq S_x(A)$ be Borel. Conclude that $\mathrm{Av}(p_0, \dots, p_{n-1}; X) = \mathrm{Av}(a_0, \dots, a_{n-1}; X)$.

Problem 3. Let $\mu \in M_x(A)$.
(i) Check that $\mu(S(\mu)) = 1$.
(ii) Let $\epsilon > 0$. Prove that there is $\delta > 0$ such that whenever $p_0, \dots, p_{n-1} \in S_x(A)$ satisfy $|\mu(S(\mu)) - \mathrm{Av}(p_0, \dots, p_{n-1}; S(\mu))| < \delta$, then for every Borel $X \subseteq S_x(A)$ we have $|\mathrm{Av}(p_0, \dots, p_{n-1}; X) - \mathrm{Av}(q_0, \dots, q_{l-1}; X)| < \epsilon$, where $\{q_0, \dots, q_{l-1}\} = S(\mu) \cap \{p_0, \dots, p_{n-1}\}$.

Problem 4. Assume NIP. Let $\mu \in M_x(\mathfrak{C})$ and $A \subseteq \mathfrak{C}$. Prove that μ does not fork over A iff μ is Lstp_A-invariant.

Definition. (i) We say that a function $f\colon M^n \to C$ (where C is compact, Hausdorff) is *definable* if for every disjoint closed subsets F_1 and F_2 of $\mathfrak{C}$ the preimages $f^{-1}[F_1]$ and $f^{-1}[F_2]$ can be separated by a definable set.
(ii) We say that a function $f\colon \mathfrak{C}^n \to C$ (where C is compact, Hausdorff) is *definable over M* (or *M-definable*) if the preimage under f of any closed set is type-definable over M.

Problem 5. Let C be a compact, Hausdorff space. Prove that:
i) if $f\colon M^n \to C$ is definable, then it extends uniquely to an M-definable function $f^*\colon \mathfrak{C}^n \to C$. Moreover, f^* is given by $\{f^*(a)\} = \bigcap_{\varphi \in \mathrm{tp}(a/M)} \mathrm{cl}(f[\varphi(M)])$;
ii) conversely, if $f^*\colon \mathfrak{C}^n \to C$ is an M-definable function, then $f^*|_M\colon M^n \to C$ is definable;
iii) a function $f^*\colon \mathfrak{C}^n \to C$ is definable over M iff there is a continuous function $h\colon S_n(M) \to C$ such that $f^* = h \circ r$, where $r\colon \mathfrak{C}^n \to S_n(M)$ is given by $r(a) := \mathrm{tp}(a/M)$.

Definition. A Keisler measure $\mu \in M_x(\mathfrak{C})$ is said to be *definable over M* if for every formula $\varphi(x,y) \in L$ the function $f_\varphi\colon \mathfrak{C}^{|y|} \to [0,1]$ given by $f_\varphi(b) := \mu(\varphi(x,b))$ is definable over M in the above sense.

Problem 6. Let $\mu \in M_x(\mathfrak{C})$. Prove that μ is definable over M iff it is invariant over

M and for every $r > 0$ the set $\{q \in S_y(M) :$ for any/some $b \models q,\ \mu(\varphi(x,b)) < r\}$ is open in $S_y(M)$.

Problem 7. Assume NIP. Let $\mu \in M_x(\mathfrak{C})$. Prove that:
(i) if μ is finitely satisfiable in M, then μ is invariant over M;
(ii) μ is finitely satisfiable in M iff each type in $S(\mu)$ is finitely satisfiable in M;
(iii) μ is invariant over M iff each type in $S(\mu)$ is invariant over M.