We work in a monster model $\mathfrak{C}$ of a complete NIP theory $T$.

**Problem 1.** Prove that each global $M$-invariant Keisler measure is Borel definable over $M$.

*Hint. Use Proposition 1 from p. 66 and Proposition 4 from p. 62 (from my notes).*

**Problem 2.** Here, for a type $p \in S(\mathfrak{C})$ by $\mu_p$ we will mean $p$ treated as Keisler measure. Let $p \in S_x(\mathfrak{C})$ be $M$-invariant and $q \in S_y(\mathfrak{C})$. Prove that $\mu_{p \otimes q} = \mu_p \otimes \mu_q$.

**Problem 3.** Let $\mu_x \in M_x(\mathfrak{C})$ and $\nu_y \in S_y(\mathfrak{C})$. Prove that:
(i) if $\mu_x$ and $\nu_y$ are both invariant over $M$, then so is $\mu_x \otimes \nu_y$;
(ii) if $\mu_x$ and $\mu_y$ are both finitely satisfiable in $M$, then so is $\mu_x \otimes \mu_y$;
(iii) if $\mu_x$ and $\mu_y$ are both definable in $M$, then so is $\mu_x \otimes \mu_y$.

**Problem 4.** Let $\mu_x \in M_x(\mathfrak{C})$ be $M$-invariant and $\nu_y \in M_x(\mathfrak{C})$. Prove that for every Borel set $B(x,y)$ over $M$ (i.e. $B(x,y)$ is a Borel subset of $S_{xy}(M)$ or the associated set of realizations in $\mathfrak{C}^{[x,y]}$ or the associated Borel subset of $S_{xy}(\mathfrak{C})$ contained in the set of types invariant over $M$), we have $\mu_x \otimes \nu_y(B(x,y)) = \int_{q \in S_y(M)} f(q) d\nu_y|_M$, where $f(q) := \mu_x(B(x,c))$ for some/every $c \in q(\mathfrak{C})$ (assuming that $f$ is Borel measurable).

**Problem 5.** Let $\pi: S_{x,y}(\mathfrak{C}) \to S_x(\mathfrak{C}) \times S_y(\mathfrak{C})$ be the obvious map, and $\mu_x$ and $\nu_y$ be global Keisler measures such that $\mu_x$ is invariant over $M$. Let $B \subseteq B(S_x(\mathfrak{C})) \otimes B(S_y(\mathfrak{C}))$ (the product $\sigma$-alegbra of the $\sigma$-algebras of Borel sets). Prove that $\mu_x \times \nu_y(B) = \mu_x \otimes \nu_y(\pi^{-1}[B])$ (where $\mu_x \times \nu_y$ is the product measure).

**Problem 6.** Assume $\mu_x \in M_x(\mathfrak{C})$ is invariant over $M$.
(i) Prove that if $\mu_x$ is definable (over some small set), then it is definable over $M$.
(ii) Prove that if $\mu_x$ is finitely satisfiable (in some small model), then it is finitely satisfiable in $M$.

**Problem 7.** Let $\mathcal{F}$ be a family of measurable functions from a probability space $(X, \Omega, \mu)$ to $[0,1]$. Prove that $\mathcal{F}$ has an essential supremum which is the usual supremum of some countable subfamily of $\mathcal{F}$.

**Problem 8.** Deduce VC-theorem$^*$ from VC-theorem and Problem 7.

**Problem 9.** Prove Proposition 4 from page 62 using VC-theorem$^*$. 

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