# Theories with NIP. List 13.

We work in a monster model $\mathfrak{C}$ of a complete theory $T$; $A \subseteq \mathfrak{C}$ is small; $G = G(\mathfrak{C})$ will denote a type-definable (over $\emptyset$) group.

**Problem 1.** Prove that $G^0_A$, $G^{00}_A$, and $G^{000}_A$ are all normal subgroups of $G$.

**Problem 2.** Prove that $G^{000}_A = \langle ab^{-1} : a \equiv^L_A b \rangle$.

**Problem 3.** Let $H$ be an $A$-invariant, bounded index, normal subgroup of $G$. Prove that for any other monster model $\mathfrak{C}'$ which contains $A$ as a small set we have $G(\mathfrak{C}')/H(\mathfrak{C}') \cong G/H$.

**Problem 4.** Let $\mathfrak{C}'$ be a monster model which contains $A$ as a small. Prove that $G^*_A(\mathfrak{C}') = G(\mathfrak{C}')^*_A$, where $* \in \{0, 00, 000\}$. Conclude that $G/G^*_A$ does not depend on the choice of the monster model.

**Problem 5.** For $* \in \{0, 00, 000\}$ prove that TFAE:
(i) $G^*$ exists;
(ii) $\bigcap_{A \subseteq \mathfrak{C} \text{ small}} G^*_A$ has bounded index;
(iii) $G^*_A = G^*_\emptyset$ for all small $A \subseteq \mathfrak{C}$.

Deduce that the existence of $G^{000}$ implies the existence of $G^{00}$ implies the existence of $G^0$

**Problem 6.** Prove that an $A$-type-definable subgroup $H$ of $G$ is the intersection of some family of groups type-definable over countable sets.

**Problem 7.** Let $* \in \{0, 00, 000\}$. Prove that if $G^*$ exists, then it exists in any other monster model $\mathfrak{C}'$ and $G^*(\mathfrak{C}') = G(\mathfrak{C}')^*$.

**Problem 8.** Prove that if $T$ is stable, then $G^0 = G^{00} = G^{000}$.
*Comment. The existence of $G^0$ was proved in the "Stable groups" course; on List 1 of this course we generalized it to NIP theories. To prove the above equalities, use stabilizers.*

**Problem 9.** Prove the fact stated on p.74 of the notes.
*Hint. Use Pettis theorem.*

**Problem 10.** Recall that $X_{\equiv_A} := \{ab^{-1} : a \equiv_A b\}$. Prove that for any $c \in G$ we have $cX_{\equiv_A}c^{-1} \subseteq (X_{\equiv_A})^2$.