Theories with NIP. List 14.

We work in a monster model $\mathfrak{C}$ of a complete theory $T$.

**Problem 1.** Let $E$ be a bounded, invariant (over $\emptyset$) equivalence relation. Show that the quotient map $\pi_E: \mathfrak{C} \to \mathfrak{C}/E$ factors through the map $\rho: \mathfrak{C} \to S(M)$ given by $\rho(a) := \text{tp}(a/M)$ via a continuous map $h: S(M) \to \mathfrak{C}/E$ (i.e. $\pi_E = h \circ \rho$).

**Problem 2.** Let $G$ by a $\emptyset$-type-definable group and $H < G$ be an invariant subgroup of bounded index. Prove that $G/H$ is a topological group (i.e. the group operation and inversion are continuous).

*Hint. First, prove it for $H$ type-definable, and then use it in general.*

**Problem 3.** Prove that (where $\cong$ denotes topological isomorphisms):

(i) for any type-definable group $G$, $G/G^0_0$ is always profinite;
(ii) for $M := (\mathbb{Z}, +)$ and $G(M) := \mathbb{Z}$, $G/G^{00} \cong G^0 \cong \mathbb{Z}$;
(iii) for $M := (\mathbb{R}, +, \cdot)$ and $G(M) := S^1$, $G/G^{00} \cong S^1$ and $G/G^0$ is trivial;
(iv) more generally, in the context of the fact on p. 74, $G^*/G^{*00}_G \cong G$.

*Comment. The fact on p. 74 says that they are abstractly isomorphic which is the content of Problem 9 from list 13. Here, one has to show that the isomorphism is topological.*

**Problem 4.** Let $M \prec \mathfrak{C}$ be small. Assume that $\mu$ is a left invariant Keisler measure on $G(M)$ (i.e. a $(G(M))$-invariant, finitely additive probability measure on Def($G$)). Prove that if $\mu$ has a unique extension to a left invariant Keisler measure on $G = G(\mathfrak{C})$, then for every $N \succ M$ the measure $\mu$ has a unique extension to a left invariant Keisler measure on $G(N)$. (In particular, the uniqueness of the extension does no depend on the choice of the monster model $\mathfrak{C}$.)

**Problem 5.** Prove Beth’s theorem for types.

*Comment. For the precise statement see Fact 2.12 in: J. Gismatullin, K. Krupiński, On model-theoretic connected components in some group extensions, Journal of Mathematical Logic (15), 1550009 (51 pages), 2015.*

**Problem 6.** Let $G$ be a group definable in a structure $M$. Let $N = (M, X, \cdot)$ be $M$ expanded by the “affine copy” $X$ of $G$. Prove that:

(i) the definable subsets of $M^n$ computed in the structure $M$ coincide with the definable subsets of $M^n$ computed in $N$ (equivalently, the restriction map from $S_{M^n}(N)$ computed in the language of $N$ to $S_n(M)$ computed in the language of $M$ is a homeomorphism);
(ii) $S_G(M) \approx S_X(N)$ (the map is given in the proof of Lemma 6 on p. 84; here check the details);
(iii) a type $q \in S_1(M)$ does not fork over $A \subseteq M$ if and only if $q$ treated (by (i)) as a type in $S_M(N)$ does not fork over $A$ if and only if it does not fork over $A\beta$ for every/some $\beta \in X$.

**Problem 7.** Let $D$ be $\emptyset$-definable and $p \in S_D(\mathfrak{C})$. Let $f: D \to D'$ be an $A$-definable bijection. Then we get a well-defined $f(p) \in S_{D'}(\mathfrak{C})$. 1
(i) Prove that $p$ does not fork over $A$ if and only if $f(p)$ does not fork over $A$.

(ii) Choose any $b \in \mathcal{C}$. Prove that $p$ does not fork over $A$ if and only if it does not fork over $Ab'$ for every $b' \models tp(b/A)$. 